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VECTORIAL MECHANICS

VECTORIAL MECHANICS

BY

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PREFACE

This book has been designed as an introductory textbook in mechanics for students of engineering and of physics. It is hoped, moreover, that it will serve as a book of reference to those who, not content with merely "passing the course," wish to gain a fundamental understanding of a fundamental science. Also those students have been kept in mind who, freed from the necessity of making grades, wish to review mechanics thoughtfully and thoroughly, beyond the point of having a few pat rules for solving special types of problems.

The entire subject has been developed from three general principles, and no pains have been spared to show how they support the superstructure. I have aimed at an exposition as simple and direct as is consistent with a respectable standard of rigor. Moreover *every* part of the theory is fully illustrated by examples and accompanied by a large and varied collection of problems. Each chapter is followed by a concise summary of the principal results; this should enable the student to see the woods in spite of the trees.

The subject matter has been chosen with a view to its applications, especially in engineering. A narrow utilitarianism, however, has been avoided; for as John Dewey has said, "It does not pay to tether one's thought to the post of usefulness with too short a rope."

Finally, while fitting the beams and columns into the structure of mechanics, my first concern has been their rigidity and strength; but I have not been totally unmindful of the architecture.

The order of the book is: Statics, Kinematics, Kinetics. This is roughly the historical order of development. Possibly the individual learns mechanics in the same way the race has acquired it. At any rate this order, though not the best on purely logical grounds, leads the student by easy stages into the more difficult parts of the subject.

Statics is founded upon four basic principles, kinetics upon three. Chapter XIV closes by showing that the principles of statics are contained in the principles of kinetics: Force and Acceleration, Vector Addition of Forces, Action and Reaction.

There are some departures from tradition in the material included in the text. The method of "index stresses" is developed in Chapter IV. Flexible cables are treated from a uniform point of view in Chapter VII. Chapter X in plane kinematics is fairly complete and could form the substance of a course on the kinematics of machinery. Attention is called to the simple proof of the Theorem of Coriolis, a proof so worded that it also applies to the most general case. In Chapter XII a thorough treatment of free, damped, and forced vibrations is given without presupposing a knowledge of differential equations. The importance of these topics in engineering led to their inclusion. Here also Newton's induction of the Law of Universal Gravitation finds a place, together with a very brief deduction of Kepler's Laws. As this deduction is perhaps the greatest single achievement in classical mechanics, it is hoped that its inclusion will not be taken amiss. In Chapter XIV, on rigid dynamics, the essential facts on the balancing of both revolving and reciprocating masses are simply obtained. A brief discussion of the kinematics of a rigid body is then followed by a treatment of gyroscopic motion, leading to the result of greatest technical importance.

Since some of the greatest minds of all time have contributed to the development of mechanics, it is hoped that this book shadows forth a little of the beauty and profound imagination in their work. In the graceful words of Professor F. G. Donnan:

> "The power of rigorous deductive logic in the hands of a mathematician of insight and imagination has always been one of the greatest aids in man's effort to understand that mysterious universe in which he lives. Without the presence of this power, the experimental discoverer might wander in the fields and pick the wild flowers of knowledge, but there would be no beautiful garden of understanding wherein the mind of man can find a serene delight."

The book contains more than can be given in a single course; but it is easy to arrange a course of any length from its subject matter. The introductory chapter in vector algebra may be studied as the course progresses; the summary of this chapter shows how little is actually needed for a working knowledge — a knowledge which will prove of service in other parts of physics.

Although considerations of space have necessitated a brief

treatment (or even the omission) of some important topics in mechanics, the author hopes that this book will also prove of service to the student in the years following his formal education.

The author wishes, finally, to express his thanks to Professors R. E. Hundley and H. K. Justice for their advice and assistance during the preparation of this book, and to Professor L. R. Culver for a number of excellent problems.

LOUIS BRAND

THE UNIVERSITY OF CINCINNATI.
February 20, 1930.

INTRODUCTION

Mechanics is the science which deals with the motion of bodies, including the special circumstances in which they remain at rest. The motion of a body is always referred to a *frame of reference* fixed in another body. Thus we would refer the motion of the connecting rod in a locomotive to the frame of the locomotive, the motion of a bullet to the earth, the motion of the earth to the sun, and the motion of the sun to a hypothetical rigid body supposed to be absolutely at rest. As no body has ever been found in a state of "absolute rest," this term indeed being probably without any physical meaning, it is seen that mechanics is really concerned with the *relative motions* of bodies.

The motion of a body, consisting of a certain portion of *matter*, is described in terms of *space* and *time*, and regarded as being the result of certain *forces* acting on the body. *Matter, space, time* and *force* are the primitive and essentially undefinable concepts upon which the science of mechanics is based. Our experience and intuition form the source of our knowledge as to the nature of these concepts, and definitions and explanations only serve to give our ideas about them a greater precision.

Mechanics may be divided into two great branches: *Kinematics* and *Dynamics.*

That branch of mechanics which deals with the motion of bodies without reference to the forces acting on them is called *Kinematics.* Thus kinematics deals only with space and time and the concepts, such as velocity and acceleration, which are derived from them. Kinematics is geometry with the added element of time, and may therefore be described as the *geometry of motion.*

In Dynamics, however, the motion of bodies is considered in relation to the forces acting on them. Dynamics in turn is subdivided into *Statics* and *Kinetics.* Statics deals with the relation between the forces when the bodies considered remain at rest (relative to the frame of reference). If the bodies are moving or are set in motion by the forces acting on them the problem falls in the province of *Kinetics.*

The subdivisions of Mechanics are shown in the following scheme:

$$\text{Mechanics}\begin{cases}\text{Kinematics}\\ \text{Dynamics}\begin{cases}\text{Statics}\\ \text{Kinetics}\end{cases}\end{cases}$$

Statics, although but a special case of kinetics, will be considered first in this book, as it forms, perhaps, the simplest part of mechanics, and leads the student by easy stages into the subject. Then kinematics, and finally kinetics, will be dealt with.

Both in kinematics and dynamics we have constantly to consider quantities that vary in time and space. Thus *rates of change* are of fundamental importance in mechanics, and require for their computation that branch of mathematics which is specially devoted to the study of continuous change — the *Calculus*.

Finally the quantities that occur in mechanics are of two kinds, known as *scalars* and *vectors*. Scalar quantities are measured by the ordinary positive and negative numbers of arithmetic and algebra. Vector quantities, however, involve the idea of direction as well as magnitude; and in order to treat them simply and directly we shall begin with a chapter in the elements of vector algebra.

As very full cross references are given in this book, an article as well as a page number is given at the top of each page. Equations are referred to by article and number; thus (§ 17, 2) means equation (2) in § 17. Figures are given the number of the article in which they first appear, followed by a letter in case there is more than one in the article in question; Fig. 17*b*, for example, is the second figure in § 17. In the text all letters in **heavy type** denote vector quantities. In a figure, a heavy letter denotes the vector adjacent; a light letter near a vector denotes its length or magnitude.

After reading an article and carefully studying the solved examples, the student should next attack the problems as a test of his mastery of the subject-matter. Nearly all the answers of the problems are given in a list at the end of the book; this should be consulted *after* the solution is complete.

The following directions should be observed in solving problems.

1. *Read the problem through carefully.* If no figure is given, draw one, paying strict attention to the wording of the problem

statement and lettering the drawing accordingly. (Draw, for example, the figure for Problem 2, § 53.) If the lettering is not given, supply letters so that (*a*) the figure may be briefly referred to, (*b*) all unknown quantities have designations. It is frequently advisable to denote also the known quantities with letters and to solve the problem in general terms.

2. *Plan a method of solution.* If there are alternative methods, try to choose the shortest and most direct. Remember, however, that solving a problem in two ways affords a valuable check for correctness.

3. *Draw free-body diagrams for all problems in dynamics.* All forces acting on the body should be indicated by arrows pointing in the right directions.* Do not forget the reactions of other bodies on the body in question. Frictional forces oppose impending slippage. In this connection the words *smooth* and *rough* mean respectively that friction is neglected or taken into account.

4. *State briefly the principles of mechanics involved in the solution.* The examples solved in the text show how this may be simply done. If several equations are involved it is well to number them for convenient reference.

5. *If the problem consists of several parts, label each part of the solution.* See the solved examples of § 62.

6. *Make a practice of solving problems in general terms.* All known and unknown quantities are then designated by letters and the solution is effected algebraically. The solution should then be tested by the *check of dimensions* (§ 167). Besides affording this valuable check, this method shows the structure of the final results and facilitates the numerical calculations — effected by substituting numerical values in the formulas of the general solution. The arithmetic is thus postponed to the end of the problem, cancellations are readily made, and the final calculations carried out by slide-rule or four-place logarithms. The student may thus concentrate on mechanical principles in solving the problem, unhampered by the details of arithmetic. Although the student will not believe this at the outset, a general problem is often more readily solved than a specific one. To quote a saying of a great American scientist, J. Willard Gibbs — *The whole is simpler than its parts.*

* When the direction is unknown the force must nevertheless be drawn; the solution will determine the unknown angle.

7. *Carry out the solution neatly and systematically to the end.* Careless or badly arranged work frequently leads to serious errors and is difficult to retrace in checking.

8. *Compute all the required quantities in plain numbers, using a consistent set of units.* Do not give an answer such as $9\sqrt{5}$ or $3\,e^{-\frac{2}{3}}$; evaluate such expressions. Be particularly careful in matters of sign, for a mistake in sign is often disastrous. Finally, *give the units after all numerical results.*

CONTENTS

CHAPTER VI

Vector Calculus

CHAPTER VII

Flexible Cables

CHAPTER VIII

Kinematics of a Particle

CHAPTER XIII

Dynamics of a System of Particles

CHAPTER XIV

Dynamics of Rigid Bodies

CHAPTER XV

Impact

VECTORIAL MECHANICS

CHAPTER I

VECTOR ALGEBRA

1. Scalars and Vectors. There are certain physical magnitudes, such as length, time, mass, temperature, electric charge, that may be represented by a single real number. A mass, for example, may be represented by a certain positive number on a certain scale; and an electric charge may be represented by a certain positive or negative number. Magnitudes belonging to this class, and the numbers that represent them, are called *scalars*. Scalars do not involve the idea of direction.

On the other hand, there are physical magnitudes that involve the idea of direction as well as magnitude. Thus a rectilinear displacement in space, besides having a definite magnitude, has also a definite direction. To represent a displacement we must therefore give a number denoting its magnitude, and also indicate in some manner its direction in space. Graphically, a displacement may be represented by a segment of a straight line having a definite length and direction. Any physical magnitude that involves the idea of direction as well as magnitude, and which may be represented by a directed segment of a straight line, is called a *vector*. Velocity, acceleration, force, and torque are examples of vectors.

For the sake of brevity, the directed segments, as well as the magnitudes they represent, are usually called vectors. However, to avoid ambiguity, we shall reserve the term *vector* for a directed segment or arrow, and call the physical magnitude represented by it a *vector quantity*. We therefore lay down the following definition:

A vector is a segment of a straight line regarded as having a definite length and direction.

We shall represent a vector directed from the point A to the point B by the symbol $\overrightarrow{AB}$. With this notation $\overrightarrow{AB}$ and $\overrightarrow{BA}$ denote different vectors; they have the same length but opposite directions. We shall also employ single letters in heavy type (**A**, **a**, . . .) to denote vectors.

In a figure, the direction of a vector is denoted by an arrowhead.

A vector symbol between two vertical bars, as $|\overrightarrow{AB}|$ or $|\mathbf{a}|$, denotes the length of the vector.*

2. Equality of Vectors. *Two vectors are said to be equal when they have the same length and direction.*

According to this definition, equal vectors are necessarily parallel or segments of the same straight line. In Fig. 3*b*,† for example, the vectors forming the opposite sides of the parallelogram are equal and may be represented by the same symbol:

$$\overrightarrow{AB} = \overrightarrow{DC} = \mathbf{u}, \qquad \overrightarrow{AD} = \overrightarrow{BC} = \mathbf{v}.$$

If $\mathbf{u}$ represents a physical quantity which is not localized in space, $\mathbf{u}$ is called a *free* vector; then $\mathbf{u}$ may be shifted at pleasure parallel to itself. But if the quantity is associated with a given line or a given point, $\mathbf{u}$ is said to be localized in that line or at that point.

3. Addition of Vectors. To obtain a rule for combining vectors we shall regard them, for the moment, as representing rectilinear displacements in space. If a particle is given two rectilinear displacements, one from A to B, and a second from B to C, the result is the same as if the particle were given a single displacement from A to C. This equivalence may be represented by the notation

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}. \tag{1}$$

We shall regard this equation as *defining* the sum of any two vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$. The sum of two vectors, $\mathbf{u}$, $\mathbf{v}$, is therefore defined as follows:

* If a is a real number, $|a|$ denotes its numerical value. Thus $|-3| = 3$, $|3| = 3$.

† The number of a figure indicates the article in which it appears. Figures in the same article are distinguished by letters.

Draw $\mathbf{v}$ *from the end of* $\mathbf{u}$; *then the vector directed from the beginning of* $\mathbf{u}$ *to the end of* $\mathbf{v}$ *is the sum of* $\mathbf{u}$ *and* $\mathbf{v}$, *and is written* $\mathbf{u} + \mathbf{v}$.

From this construction (Fig. 3*a*) it appears that $\mathbf{u} + \mathbf{v}$ is the diagonal of the parallelogram formed with $\mathbf{u}$ and $\mathbf{v}$ as sides. For this reason the above rule for vector addition is called the *parallelogram law.*

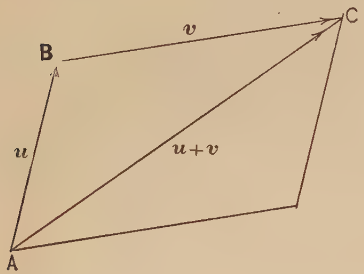

FIG. 3*a*.

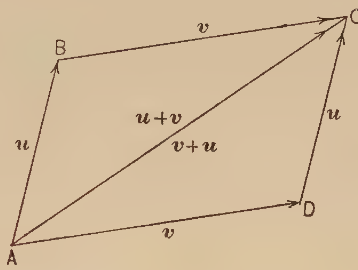

FIG. 3*b*.

The length of the sum of two vectors having different directions is evidently less than the sum of their lengths. Hence

$$|\mathbf{u} + \mathbf{v}| \leqq |\mathbf{u}| + |\mathbf{v}|, \tag{2}$$

the equal sign applying only when $\mathbf{u}$ and $\mathbf{v}$ have the same direction.

In the parallelogram formed with $\mathbf{u}$ and $\mathbf{v}$ as sides (Fig. 3*b*), we have

$$\mathbf{u} + \mathbf{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}, \qquad \mathbf{v} + \mathbf{u} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}.$$

Thus the sum of two vectors is independent of the order in which they are added; in brief, *vector addition is commutative:*

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}. \tag{3}$$

For three vectors, $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{CD}$ (Fig. 3*c*), we have from (1):

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD},$$
$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CD}) = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}.$$

Thus the sum of three vectors is not affected by the manner in which they are grouped when performing the addition; in other words, *vector addition is associative:*

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}). \tag{4}$$

Since the grouping of the vectors is immaterial, the above sum is simply written $\mathbf{u} + \mathbf{v} + \mathbf{w}$.

From the commutative and associative laws, (3), (4), we may deduce the following general result: *The sum of any number of*

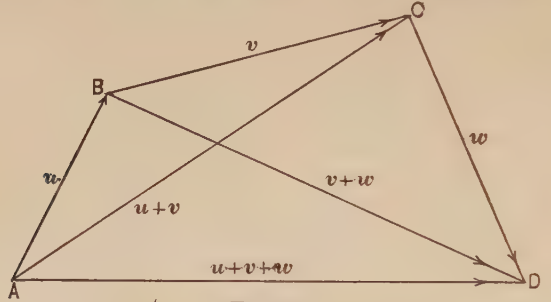

FIG. 3c.

vectors is independent of the order in which they are added, and of their grouping to form partial sums. For example:

$$[(\mathbf{u} + \mathbf{v}) + \mathbf{w}] + \mathbf{x} = (\mathbf{u} + \mathbf{v}) + (\mathbf{w} + \mathbf{x}) = (\mathbf{u} + \mathbf{w}) + (\mathbf{v} + \mathbf{x}).$$

To construct the sum of any number of vectors, form a broken line whose segments, in length and direction, are these vectors

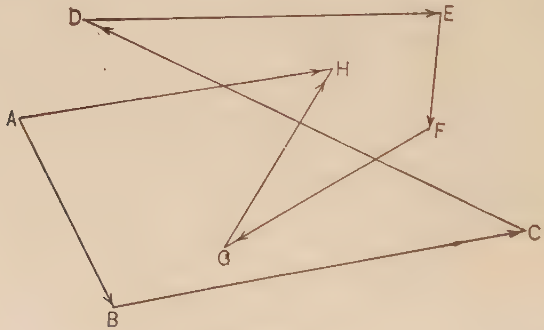

FIG. 3d.

placed beginning to end in any order whatever; then the vector directed from the beginning to the end of the broken line will be the sum required. The figure formed by the vectors and their sum is called a *vector polygon*. If $A, B, C, \ldots, G, H$ are the successive vertices of a vector polygon, then (Fig. 3d)

$$\overrightarrow{AB} + \overrightarrow{BC} + \cdots + \overrightarrow{GH} = \overrightarrow{AH}. \tag{5}$$

When the vectors to be added are all parallel, the vector "polygon" becomes a portion of a straight line described twice.

In constructing the sum of a number of vectors, it may happen that the end of the last vector coincides with the beginning of the first. In this case we say that the sum of the vectors is zero. Thus, if in (5) the point H coincides with the point A, we write:

$$\overrightarrow{AB} + \overrightarrow{BC} + \cdots + \overrightarrow{GA} = 0. \tag{6}$$

This equation may be regarded as a special case of (5) if we agree that $\overrightarrow{AA} = 0$.

Using the symbol $\overrightarrow{AA}$ (or $\overrightarrow{BB}$, etc.) to denote a vector of zero length,* we have from (1),

$$\overrightarrow{AB} + \overrightarrow{BB} = \overrightarrow{AB}, \qquad \overrightarrow{AA} + \overrightarrow{AB} = \overrightarrow{AB};$$

or writing $\overrightarrow{AB} = \mathbf{u}$,

$$\mathbf{u} + 0 = 0 + \mathbf{u} = \mathbf{u}. \tag{7}$$

PROBLEMS

1. Let $ABCDEF$ be a regular hexagon inscribed in a circle whose center is O. Construct the sum of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$, $\overrightarrow{AF}$ and compare it with $\overrightarrow{AO}$.

2. Draw any triangle, ABC, find the middle points, P, Q, R, of its sides, and choose any point O in its plane. Beginning at O, construct the vector sums

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}, \qquad \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}.$$

3. Draw any quadrilateral, $ABCD$, with unequal sides. Let P, Q be the middle points of the diagonals AC, BD; and M the middle point of PQ. Construct the vector sum

$$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD}.$$

4. The Negative of a Vector. The sum of two vectors is zero when, and only when, they have the same length and opposite directions. If, in the equation $\overrightarrow{AB} + \overrightarrow{BA} = 0$, we write $\overrightarrow{AB} = \mathbf{u}$, it is natural to write $\overrightarrow{BA} = -\mathbf{u}$ in order that the

*Since a segment of zero length has no definite direction, $\overrightarrow{AA}$ is not a vector in the proper sense.

characteristic equation for negatives,

$$(1) \qquad \mathbf{u} + (-\mathbf{u}) = 0,$$

will hold for vectors as well as for numbers. *The negative of a vector is therefore defined as a vector of the same length but opposite direction:* hence

$$(2) \qquad -\overrightarrow{AB} = \overrightarrow{BA}, \qquad -(-\overrightarrow{AB}) = \overrightarrow{AB}.$$

5. Subtraction of Vectors. To subtract the vector $\mathbf{v}$ from the vector $\mathbf{u}$ consists in finding the vector from which $\mathbf{u}$ can be obtained by adding $\mathbf{v}$: this vector, denoted by $\mathbf{u} - \mathbf{v}$, is therefore defined by the equation

$$(1) \qquad (\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u}.$$

Adding $-\mathbf{v}$ to both sides of (1), we have

$$(2) \qquad \mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v});$$

that is, *subtracting a vector is the same as adding its negative.* The construction of $\mathbf{u} - \mathbf{v}$ is shown in Fig. 5*a*.

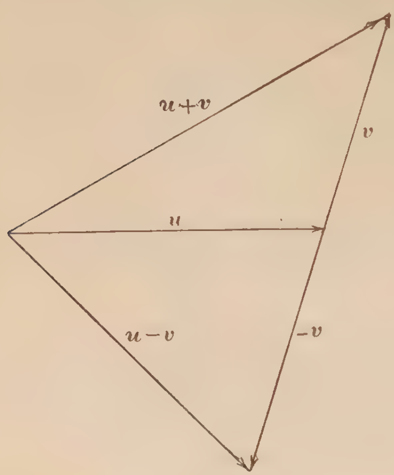

FIG. 5*a*.

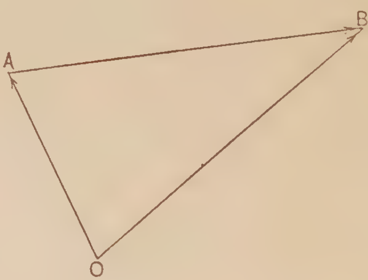

FIG. 5*b*.

If O is any fixed point of reference, any vector $\overrightarrow{AB}$ may be expressed as the difference of two vectors issuing from O (Fig. 5*b*): for

$$(3) \qquad \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} + (-\overrightarrow{OA}) = \overrightarrow{OB} - \overrightarrow{OA}.$$

The vector $\overrightarrow{OA}$ is called the *position vector* of the point A.

6. Multiplication of Vectors by Real Numbers. The vector $\mathbf{u} + \mathbf{u}$ is naturally denoted by $2\mathbf{u}$; moreover we write $-\mathbf{u} + (-\mathbf{u}) = -2\mathbf{u}$. Thus both $2\mathbf{u}$ and $-2\mathbf{u}$ denote vectors twice as long as $\mathbf{u}$; the former has the same direction as $\mathbf{u}$, the latter the opposite direction. This notation is generalized as follows:

The product $a\mathbf{u}$ *or* $\mathbf{u}a$ *of a vector* $\mathbf{u}$ *and a real number* a *is defined as a vector* $|a|$ *times as long as* $\mathbf{u}$, *and having the same direction as* $\mathbf{u}$, *or the opposite, according as* a *is positive or negative.*

In accordance with this definition we have

$$|a\mathbf{u}| = |a|\,|\mathbf{u}|, \tag{1}$$

$$0 \cdot \mathbf{u} = \mathbf{u} \cdot 0 = 0, \tag{2}$$

$$a(-\mathbf{u}) = (-a)\mathbf{u} = -a\mathbf{u}, \tag{3}$$

$$(-a)(-\mathbf{u}) = a\mathbf{u}. \tag{4}$$

These relations are the same in form as the rules for the multiplication of numbers. Moreover the multiplication of a vector by numbers is commutative (by definition), associative, and distributive:

$$a\mathbf{u} = \mathbf{u}a, \tag{5}$$

$$(ab)\mathbf{u} = a(b\mathbf{u}), \tag{6}$$

$$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}. \tag{7}$$

The product of the sum of two vectors by a number is also distributive:

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}. \tag{8}$$

The proof of (8) follows immediately from the theorem that the corresponding sides of similar triangles are proportional.

If $\mathbf{u}$ and $\mathbf{v}$ are two vectors having the same or opposite directions, we may write

$$\mathbf{u} = k\mathbf{v},$$

the number k being positive or negative in the respective cases. Thus if two vectors are parallel, each may be expressed as a scalar multiple of the other.

From this and the preceding articles it is clear that as far as addition, subtraction, and multiplication by real numbers are concerned, vectors may be treated formally in accordance with the rules of ordinary algebra.

The quotient $\mathbf{u}/a$ of a vector $\mathbf{u}$ by a number a (not zero) is defined as the product of $\mathbf{u}$ by $1/a$. Thus $\mathbf{u}/3$ is a vector in the direction of $\mathbf{u}$ and one-third as long.

7. Point of Division. *If A, B and P are points of a straight line, P is said to divide the segment AB in the ratio a/b if*

$$\overrightarrow{AP} = \frac{a}{b}\overrightarrow{PB}. \tag{1}$$

From Fig. 7*a* we see that a/b is positive or negative according as P lies within or without the segment AB. To find the position vector of P relative to an origin O, we write (1) in the form

$$b\,(\overrightarrow{OP} - \overrightarrow{OA}) = a\,(\overrightarrow{OB} - \overrightarrow{OP}),$$

hence

$$\overrightarrow{OP} = \frac{b\,\overrightarrow{OA} + a\,\overrightarrow{OB}}{a+b}. \tag{2}$$

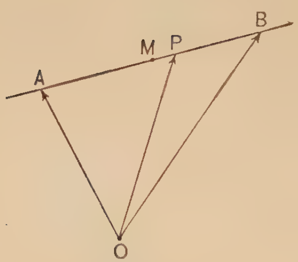

FIG. 7*a*.

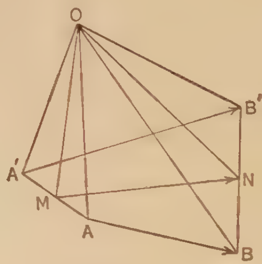

FIG. 7*b*.

The middle point M of AB divides AB in the ratio of $1/1$; hence from (2)

$$\overrightarrow{OM} = \tfrac{1}{2}\,(\overrightarrow{OA} + \overrightarrow{OB}). \tag{3}$$

This equation states that the diagonals of the parallelogram on OA, OB bisect each other.

Example 1. If $\overrightarrow{AB}$, $\overrightarrow{A'B'}$ are any two vectors, the vector from the middle point of AA' to that of BB' is equal to one-half of their sum.

Proof. Let M and N be the middle points of AA' and BB' respectively (Fig. 7*b*); then if O is any point of reference, we have, from (3),

$$\begin{aligned}\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} &= \tfrac{1}{2}\,(\overrightarrow{OB} + \overrightarrow{OB'}) - \tfrac{1}{2}\,(\overrightarrow{OA} + \overrightarrow{OA'})\\ &= \tfrac{1}{2}\,(\overrightarrow{OB} - \overrightarrow{OA}) + \tfrac{1}{2}\,(\overrightarrow{OB'} - \overrightarrow{OA'}) = \tfrac{1}{2}\,(\overrightarrow{AB} + \overrightarrow{A'B'}).\end{aligned}$$

If, in particular, the given vectors are parallel, they are also parallel to $\overrightarrow{MN}$. We thus have proved a familiar theorem on the trapezoid.

Example 2. If the numbers a, b, c are not zero and

$$a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} = 0, \qquad a + b + c = 0,$$

the points A, B, C lie in a straight line.

Proof. Since $a = -(b + c)$ we have from the first equation

$$\overrightarrow{OA} = \frac{b\,\overrightarrow{OB} + c\,\overrightarrow{OC}}{b + c}.$$

In view of (2), this states that A divides the segment BC in the ratio of c/b. Thus A lies on the line through B and C.

Example 3. If the numbers a, b, c, d are not zero and

$$a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} + d\,\overrightarrow{OD} = 0, \qquad a + b + c + d = 0,$$

the points A, B, C, D lie in a plane.

Proof. Since $a + b = -(c + d)$ we have from the first equation

$$\frac{a\,\overrightarrow{OA} + b\,\overrightarrow{OB}}{a + b} = \frac{c\,\overrightarrow{OC} + d\,\overrightarrow{OD}}{c + d}.$$

If we draw a vector $\overrightarrow{OP}$ equal to these vectors, we see from (2) that P divides AB in the ratio of b/a, and also CD in the ratio of d/c. Thus P is a point common to the lines AB and CD. As these lines either coincide or intersect at P they must lie in a plane.

Example 4. In Fig. 7c N divides AB in the ratio 1/2 and L divides BC in the ratio 2/3. Let us find the ratios in which P divides CN and AL.

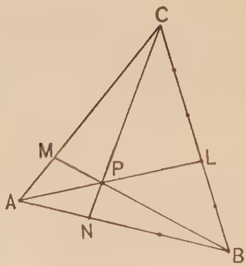

FIG. 7c.

Denote the position vectors of the points A, B, . . . relative to any origin by $\mathbf{a}$, $\mathbf{b}$, etc. Since P is at the intersection of CN and AL we seek a linear relation between $\mathbf{c}$, $\mathbf{n}$, $\mathbf{a}$, $\mathbf{l}$. Using the given ratios, we have from (2)

$$3\,\mathbf{n} = 2\,\mathbf{a} + \mathbf{b}, \qquad 5\,\mathbf{l} = 3\,\mathbf{b} + 2\,\mathbf{c};$$

hence, on eliminating $\mathbf{b}$,

$$9\,\mathbf{n} - 5\,\mathbf{l} = 6\,\mathbf{a} - 2\,\mathbf{c}.$$

From this we have

$$\frac{2\,\mathbf{c} + 9\,\mathbf{n}}{11} = \frac{6\,\mathbf{a} + 5\,\mathbf{l}}{11} = \mathbf{p};$$

for in view of (2) the first member represents a point on CN, the second member a point on AL. The point is therefore P: thus P divides CN in the ratio 9/2, AL in the ratio 5/6.

Let us next compute the ratios in which M divides CA and PB. For this purpose we seek a linear relation between $\mathbf{c}$, $\mathbf{a}$, $\mathbf{p}$, $\mathbf{b}$. From the above results

$$5\,\mathbf{l} = 3\,\mathbf{b} + 2\,\mathbf{c} = 11\,\mathbf{p} - 6\,\mathbf{a}.$$

Hence

$$\frac{2\,\mathbf{c} + 6\,\mathbf{a}}{8} = \frac{11\,\mathbf{p} - 3\,\mathbf{b}}{8} = \mathbf{m};$$

for the first member represents a point on CA, the second member a point on PB. Thus M divides CA in the ratio 3/1 and PB in the ratio $-3/11$.

The student should now prove the general theorem: If L divides BC in the ratio γ/β and M divides CA in the ratio α/γ, then N will divide AB in the ratio β/α. The product of these division ratios is therefore 1. This is the *Theorem of Ceva.* Show also that

$$(\alpha + \beta + \gamma)\,\mathbf{p} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

and hence that P divides AL, BM, CN in the ratios

$$(\beta + \gamma)/\alpha, \qquad (\gamma + \alpha)/\beta, \qquad (\alpha + \beta)/\gamma.$$

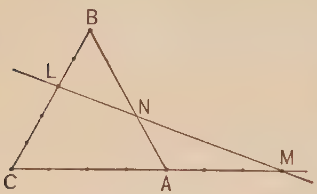

FIG. 7*d*.

Example 5. In Fig. 7*d*, L divides BC in the ratio 2/3 and M divides CA in the ratio $-7/3$. Let us find the ratios in which N divides AB and LM.

Using the notation of Ex. 4, we have from (2)

$$5\,\mathbf{l} = 3\,\mathbf{b} + 2\,\mathbf{c}, \qquad -4\,\mathbf{m} = 3\,\mathbf{c} - 7\,\mathbf{a}.$$

Since N is at the intersection of AB and LM we seek a linear relation between $\mathbf{a}$, $\mathbf{b}$, $\mathbf{l}$, $\mathbf{m}$. Eliminating $\mathbf{c}$ from the above equations we find

$$15\,\mathbf{l} + 8\,\mathbf{m} = 14\,\mathbf{a} + 9\,\mathbf{b};$$

hence

$$\mathbf{n} = \frac{15\,\mathbf{l} + 8\,\mathbf{m}}{23} = \frac{14\,\mathbf{a} + 9\,\mathbf{b}}{23}.$$

Thus N divides AB in the ratio 9/14, LM in the ratio 8/15.

The student should now prove the general theorem: If L divides BC in the ratio γ/β, and M divides CA in the ratio α/γ, then N will divide AB in the ratio $-\beta/\alpha$. The product of the three division ratios is therefore -1. This is the *Theorem of Menelaus.*

PROBLEMS

1. ABC is any triangle, and P, Q, R are the middle points of its sides. If O is any point (not necessarily in the plane ABC), show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}.$$

2. $ABCD$ is any quadrilateral, P, Q, the middle points of its diagonals AC, BD, and M the middle point of PQ. Prove that

(a) $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\,\overrightarrow{PQ}$;

(b) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4\,\overrightarrow{OM}$ for any point O.

3. If L, M, N are the middle points of the sides BC, CA, AB of the triangle ABC (Fig. 7e), and the medians AL, BM intersect at G, show that

(a) G divides both medians in the ratio of $2/1$;

(b) CN passes through G;

(c) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3\,\overrightarrow{OG}$ for any point O.

G is called the *mean center* of the points A, B, C.

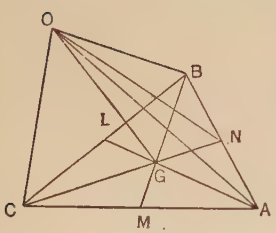

FIG. 7e.

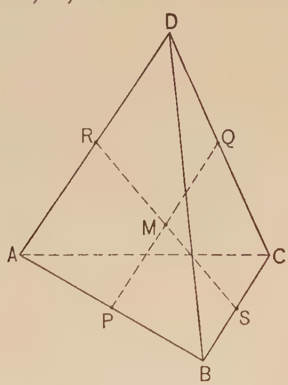

FIG. 7f.

4. If G and G' are the mean centers of A, B, C, and A', B', C', respectively, prove that

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = 3\,\overrightarrow{GG'}.$$

5. E, F are the middle points of the sides AB, BC, of the parallelogram $ABCD$. Show that the lines DE, DF divide the diagonal AC into thirds and that AC cuts off a third of each line.

6. If A, B, C, D are the middle points of the sides of any space quadrilateral, taken in order, prove that $\overrightarrow{AB} = \overrightarrow{DC}$, $\quad \overrightarrow{AD} = \overrightarrow{BC}$.

7. In a tetrahedron the lines joining the middle points of the three pairs of opposite edges meet in a point and bisect each other (Fig. 7f).

If M, M' are the middle points of PQ, RS, and O an arbitrary origin, prove that

$$\overrightarrow{OM} = \tfrac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}) = \overrightarrow{OM'}.$$

8. In Fig. 7f let G be the mean center of A, B, C (see Ex. 3). Show that the point M in Ex. 7 divides DG in the ratio 3/1.

9. Prove *Desargues' Theorem:* If the triangles ABC, $A'B'C'$ are in perspective (the lines AA', BB', CC' meet in a point P), the points of intersection of the corresponding sides are collinear. [Let A', B', C' divide PA, PB, PC in the ratios $\alpha/1$, $\beta/1$, $\gamma/1$; then

$$(1+\alpha)\,\mathbf{a}' = \mathbf{p} + \alpha\mathbf{a}, \quad (1+\beta)\,\mathbf{b}' = \mathbf{p} + \beta\mathbf{b}, \quad (1+\gamma)\,\mathbf{c}' = \mathbf{p} + \gamma\mathbf{c}.$$

If AA' and BB' meet at L, $(\alpha - \beta)\,\mathbf{l} = \alpha\mathbf{a} - \beta\mathbf{b}$, etc. Apply Ex. 2.]

8. Vectors in a Plane. Three vectors are said to be *coplanar* when they are parallel to the same plane. Coplanar vectors that are *free* may be shifted so that they actually lie in the same plane.

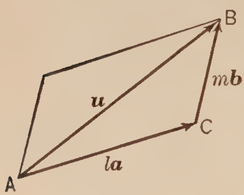

Fig. 8.

Let $\mathbf{u}$ denote any vector coplanar with the non-parallel vectors $\mathbf{a}$, $\mathbf{b}$. Then $\mathbf{u}$ may be expressed as the sum of two vectors parallel to $\mathbf{a}$ and $\mathbf{b}$ respectively. For if we construct a parallelogram on $\mathbf{u} = \overrightarrow{AB}$ as diagonal with sides parallel to $\mathbf{a}$ and $\mathbf{b}$ (Fig. 8) we have

$$\mathbf{u} = \overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = l\mathbf{a} + m\mathbf{b}$$

where l and m are numbers.

Since $l\mathbf{a}$ and $m\mathbf{b}$ are not parallel, the equation

$$l\mathbf{a} + m\mathbf{b} = 0 \quad \text{implies that} \quad l = m = 0.$$

More generally, if

$$l\mathbf{a} + m\mathbf{b} = l'\mathbf{a} + m'\mathbf{b}, \quad \text{then} \quad l = l', \quad m = m'.$$

To prove this, write the equation as $(l - l')\,\mathbf{a} + (m - m')\,\mathbf{b} = 0$; then $l - l' = 0$, $m - m' = 0$.

Example. If the points A, B, C are not collinear, the position vector of any point P in their plane may be expressed as

$$\overrightarrow{OP} = a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} \quad \text{where} \quad a + b + c = 1.$$

Proof. Since the vectors $\overrightarrow{CA}$, $\overrightarrow{CB}$ are not collinear we can write

$$\overrightarrow{CP} = a\,\overrightarrow{CA} + b\,\overrightarrow{CB}.$$

Hence from (§ 5, 3)

$$\overrightarrow{OP} - \overrightarrow{OC} = a\,(\overrightarrow{OA} - \overrightarrow{OC}) + b\,(\overrightarrow{OB} - \overrightarrow{OC}),$$
$$\overrightarrow{OP} = a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + (1 - a - b)\,\overrightarrow{OC}.$$

This gives the desired expression on putting $c = 1 - a - b$.

$\overrightarrow{OP}$ may be expressed in this form in one way only. For if

$$\overrightarrow{OP} = a'\,\overrightarrow{OA} + b'\,\overrightarrow{OB} + c'\,\overrightarrow{OC} \quad \text{where} \quad a' + b' + c' = 1,$$

then

$$(a - a')\,\overrightarrow{OA} + (b - b')\,\overrightarrow{OB} + (c - c')\,\overrightarrow{OC} = 0,$$
$$a - a' + b - b' + c - c' = 0,$$

and the points A, B, C would be collinear (§ 7, Ex. 2), contrary to our hypothesis.

9. Vectors in Space. Let **a**, **b**, **c** be any three vectors which are *not coplanar*. Then any vector **u** may be expressed as the sum of three vectors parallel to **a**, **b**, **c** respectively. For if we construct a parallelopiped on $\mathbf{u} = \overrightarrow{AB}$ as diagonal by passing planes through A and B parallel to **a** and **b**, **b** and **c**, **c** and **a** (Fig. 9), the edges of the parallelopiped will be parallel to **a**, **b** and **c**, and

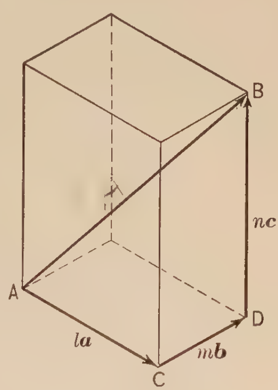

FIG. 9.

(1) $$\mathbf{u} = \overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DB} = l\mathbf{a} + m\mathbf{b} + n\mathbf{c},$$

where l, m, n are numbers.

The equation

$$l\mathbf{a} + m\mathbf{b} + n\mathbf{c} = 0 \quad \text{implies that} \quad l = m = n = 0.$$

For since $l\mathbf{a}$ and $m\mathbf{b} + n\mathbf{c}$ are not coplanar these vectors are not parallel; hence $l = 0$ (§ 8). Now $m\mathbf{b} + n\mathbf{c} = 0$, where **b** and **c** are not parallel; hence $m = n = 0$.

We may now state the theorem: *If* a, b, c *are not coplanar, the equation*

$$l\mathbf{a} + m\mathbf{b} + n\mathbf{c} = l'\mathbf{a} + m'\mathbf{b} + n'\mathbf{c}$$

implies that $l = l'$, $m = m'$, $n = n'$.

Example. If the points A, B, C, D are not coplanar, the position vector of any point P may be expressed as

$$\overrightarrow{OP} = a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} + d\,\overrightarrow{OD} \quad \text{where} \quad a + b + c + d = 1.$$

Proof. Since the vectors $\overrightarrow{DA}$, $\overrightarrow{DB}$, $\overrightarrow{DC}$ are not coplanar we can write

$$\overrightarrow{DP} = a\,\overrightarrow{DA} + b\,\overrightarrow{DB} + c\,\overrightarrow{DC}.$$

Hence from (§ 5, 3)

$$\overrightarrow{OP} - \overrightarrow{OD} = a\,(\overrightarrow{OA} - \overrightarrow{OD}) + b\,(\overrightarrow{OB} - \overrightarrow{OD}) + c\,(\overrightarrow{OC} - \overrightarrow{OD}),$$

$$\overrightarrow{OP} = a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} + (1 - a - b - c)\,\overrightarrow{OD}.$$

This gives the expression above on putting $d = 1 - a - b - c$. We may now prove as in § 8 that $\overrightarrow{OP}$ can be expressed in this form in one way only.

PROBLEM

1. Lines drawn through the point P and the vertices A, B, C, D of a tetrahedron cut the planes of the opposite faces at K, L, M, N. Show that the sum of the ratios in which these points divide the segments PA, PB, PC, PD is -1. [Apply the equations of the above Example.]

10. Component of a Vector on an Axis. A straight line upon which two directions are distinguished is called a *directed line* or an *axis*. To specify the directions, one is called *positive*, the other *negative*. In a figure the positive direction on an axis is indicated by an arrowhead.

Let s denote an axis and $\mathbf{e}$ a *unit vector* (a vector of unit length) in its positive direction. The *projection* of a vector $\overrightarrow{AB}$ on s is $\overrightarrow{A'B'}$ (Fig. 10) where A', B' are the feet of the perpendiculars from A, B on s. The *component* of $\overrightarrow{AB}$ on s is the number giving the length of $\overrightarrow{A'B'}$, taken with a positive or negative sign accord-

ing as $\overrightarrow{A'B'}$ has the positive or negative direction on s. Thus if $\overrightarrow{A'B'} = k\mathbf{e}$, the number k is the component of $\overrightarrow{AB}$ on s.

We shall denote the component of a vector $\mathbf{u}$ on s by $\text{comp}_s\,\mathbf{u}$ or, more briefly, by u_s. With this notation, the definition of u_s is given by the equation:

(1) $$\text{Projection of } \mathbf{u} \text{ on } s = u_s\mathbf{e}.$$

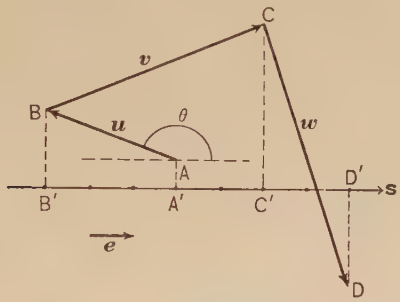

Fig. 10.

Thus in the figure

$$\mathbf{u} = \overrightarrow{AB}, \qquad \mathbf{v} = \overrightarrow{BC}, \qquad \mathbf{w} = \overrightarrow{CD};$$

$$\overrightarrow{A'B'} = -3\,\mathbf{e}, \quad \overrightarrow{B'C'} = 5\,\mathbf{e}, \quad \overrightarrow{C'D'} = 2\,\mathbf{e};$$

$$u_s = -3, \qquad v_s = 5, \qquad w_s = 2.$$

The projection of $(\mathbf{u} + \mathbf{v} + \mathbf{w})$ on s is

$$\overrightarrow{A'D'} = \overrightarrow{A'B'} + \overrightarrow{B'C'} + \overrightarrow{C'D'} = u_s\mathbf{e} + v_s\mathbf{e} + w_s\mathbf{e} = (u_s + v_s + w_s)\,\mathbf{e};$$

hence from (1),

$$\text{comp}_s\,(\mathbf{u} + \mathbf{v} + \mathbf{w}) = u_s + v_s + w_s.$$

The component of the sum of a number of vectors on an axis is equal to the sum of their components on this axis.

If the segment $A'B'$ is given its proper sign,

$$\cos\theta = \frac{A'B'}{AB} = \frac{u_s}{|\mathbf{u}|}, \qquad u_s = |\mathbf{u}|\cos\theta.$$

We denote the angle θ between the positive direction of s and $\mathbf{u}$ by $(s, \mathbf{u})$; then

(2) $$u_s = |\mathbf{u}|\cos(s, \mathbf{u}).$$

11. Rectangular Axes. A vector **u** is determined by giving its components u_x, u_y, u_z on the axes of a fixed system of rectangular coördinates. We shall always choose a "right-handed" (r-h) system of axes; that is, a system in which the rotation of a right-handed screw (in a fixed nut) from $+x$ toward $+y$ would cause it to move along $+z$. In a r-h system a rotation from $+x$ toward $+y$ is counterclockwise when viewed from the $+z$ side of the xy-plane.

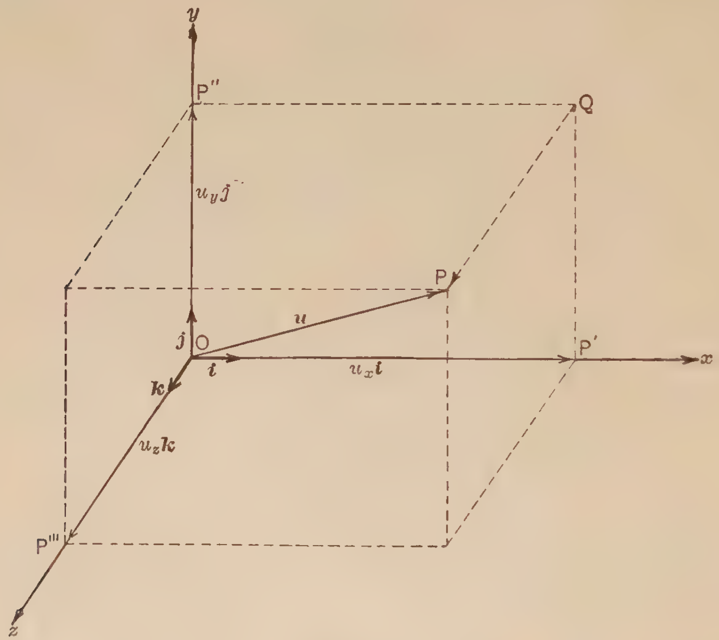

FIG. 11.

Draw the vector $\overrightarrow{OP} = \mathbf{u}$ from the origin (Fig. 11) and pass planes through P perpendicular to the axes of x, y, z, cutting them in the points P', P'', P'''. Then OP is the diagonal of a rectangular parallelopiped bounded by these planes and the three coördinate planes, and

$$\overrightarrow{OP} = \overrightarrow{OP'} + \overrightarrow{P'Q} + \overrightarrow{QP} = \overrightarrow{OP'} + \overrightarrow{OP''} + \overrightarrow{OP'''}.$$

Since $\overrightarrow{OP'}$, $\overrightarrow{OP''}$, $\overrightarrow{OP'''}$ are the projections of u on the axes, we have from (§ 10, 1)

(1) $$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the unit vectors in the positive directions of x, y, z. It is often convenient to denote the vector $\mathbf{u}$ by the symbol $[u_x, u_y, u_z]$.

The *coördinates* of a point P are defined as the components of its position vector $\overrightarrow{OP}$. Thus if P has the coördinates (x, y, z)

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = [x, y, z].$$

If the points P_1, P_2 have the coördinates (x_1, y_1, z_1), (x_2, y_2, z_2), then

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]. \tag{2}$$

We shall denote the position vectors of P, P_1, P_2 by $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$.

The angles between $+x$, $+y$, $+z$ and $\mathbf{u}$ are called the *direction angles* of $\mathbf{u}$; their cosines are called the *direction cosines* of $\mathbf{u}$. When the length and direction angles of $\mathbf{u}$ are given, its components may be calculated from (§ 10, 2):

$$u_x = |\mathbf{u}| \cos (x, \mathbf{u}), \quad u_y = |\mathbf{u}| \cos (y, \mathbf{u}), \quad u_z = |\mathbf{u}| \cos (z, \mathbf{u}). \tag{3}$$

Conversely, if the components of $\mathbf{u}$ are given we may obtain its length from

$$|\mathbf{u}|^2 = u_x^2 + u_y^2 + u_z^2 \tag{4}$$

and then compute its direction cosines from (3). To prove (4), draw $P'P$ in Fig. 11; then

$$|\mathbf{u}|^2 = OP^2 = OP'^2 + P'P^2 = OP'^2 + P'Q^2 + QP^2$$
$$= u_x^2 + u_y^2 + u_z^2.$$

If we substitute the values of u_x, u_y, u_z from (3) in (4) and divide the resulting equation by $|\mathbf{u}|^2$, we have

$$\cos^2 (x, \mathbf{u}) + \cos^2 (y, \mathbf{u}) + \cos^2 (z, \mathbf{u}) = 1.$$

The sum of the squares of the direction cosines of any vector is equal to unity.

From the distributive laws of § 6 we have

$$a\mathbf{u} = au_x\,\mathbf{i} + au_y\,\mathbf{j} + au_z\,\mathbf{k},$$
$$\mathbf{u} + \mathbf{v} = (u_x + v_x)\,\mathbf{i} + (u_y + v_y)\,\mathbf{j} + (u_z + v_z)\,\mathbf{k}.$$

In the bracket notation these equations become

$$a\,[u_x, u_y, u_z] = [au_x, au_y, au_z],$$
$$[u_x, u_y, u_z] + [v_x, v_y, v_z] = [u_x + v_x, u_y + v_y, u_z + v_z].$$

To multiply a vector by a number, multiply its components by that number; to add two vectors, add their corresponding components. In particular

$$-[u_x, u_y, u_z] = [-u_x, -u_y, -u_z].$$

Example 1. The vector $[-2, 1, 2]$ has the length $\sqrt{4+1+4} = 3$ and its direction cosines are $-2/3, 1/3, 2/3$. Its direction angles may now be found from a table of cosines.

If we are dealing with vectors $\mathbf{u}$ parallel to the xy-plane, $u_z = 0$ and we write $\mathbf{u} = [u_x, u_y]$. The angle $(x, \mathbf{u})$ will now suffice to give the direction of $\mathbf{u}$ *if it is given a sign.* The usual convention is to regard $(x, \mathbf{u})$ as positive when the rotation from $+x$ to $\mathbf{u}$ has the same sense as the rotation from $+x$ to $+y$. With this agreement

$$(y, \mathbf{u}) = (y, x) + (x, \mathbf{u}) = (x, \mathbf{u}) - 90°, \quad \cos(y, \mathbf{u}) = \sin(x, \mathbf{u}).$$

From (3) we have

$$(5) \qquad u_x = |\mathbf{u}| \cos(x, \mathbf{u}), \qquad u_y = |\mathbf{u}| \sin(x, \mathbf{u});$$

hence

$$(6), (7) \qquad |\mathbf{u}|^2 = u_x^2 + u_y^2, \qquad \tan(x, \mathbf{u}) = \frac{u_y}{u_x}.$$

When u_x, u_y are known, these equations serve to find $|\mathbf{u}|$ and $(x, \mathbf{u})$. The precise quadrant in which the angle $(x, \mathbf{u})$ lies is determined by the signs of u_x and u_y.

Example 2. Let $\mathbf{u} = [5, 2]$, $\mathbf{v} = [-3, -4]$; then if $\mathbf{w} = \mathbf{u} + \mathbf{v}$,

$$\mathbf{w} = [5 - 3, 2 - 4] = [2, -2], \qquad |\mathbf{w}| = \sqrt{4+4} = \sqrt{8};$$

$$\tan(x, \mathbf{w}) = \frac{-2}{2} = -1, \qquad (x, \mathbf{w}) = 45° + 270° = 315°.$$

The angle lies in the fourth quadrant since $w_x > 0$, $w_y < 0$.

Example 3. To find the equations of a line through the point P_1 (x_1, y_1, z_1) and parallel to the vector $\mathbf{l} = [a, b, c]$.

If $P(x, y, z)$ is any point on the line, $\overrightarrow{P_1P} = \mathbf{r} - \mathbf{r}_1$ is parallel to $\mathbf{l}$; hence (§ 6)

$$\mathbf{r} - \mathbf{r}_1 = \lambda \mathbf{l} \quad \text{where } \lambda \text{ is a variable scalar.}$$

The corresponding components of $\mathbf{r} - \mathbf{r}_1$ and $\mathbf{l}$ are therefore proportional:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

These are the Cartesian equations of the line.

A line in the xy-plane through P_1 $(x_1, y_1, 0)$ and parallel to $\mathbf{l} = [a, b, 0]$ has the equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}, \qquad z = 0.$$

PROBLEMS

1. Find the magnitude and direction of $\mathbf{u} + \mathbf{v} + \mathbf{w}$ graphically and analytically for the following vectors in the xy-plane:

(*a*) $|\mathbf{u}| = 6$, $(x, \mathbf{u}) = 60°$, $|\mathbf{v}| = 10$, $(x, \mathbf{v}) = 120°$, $|\mathbf{w}| = 8$, $(x, \mathbf{w}) = 270°$.

(*b*) $|\mathbf{u}| = 1$, $(x, \mathbf{u}) = 0°$, $|\mathbf{v}| = 1$, $(x, \mathbf{v}) = 120°$, $|\mathbf{w}| = 1$, $(x, \mathbf{w}) = 240°$.

(*c*) $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -5\mathbf{i} + 2\mathbf{j}$, $\mathbf{w} = -\mathbf{j}$.

(*d*) $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = -\mathbf{j}$, $\mathbf{w} = \mathbf{i} - \mathbf{j}$.

2. Find the sum of the vectors

$$2\mathbf{i} + 3\mathbf{j} - \mathbf{k}, \qquad \mathbf{i} + 2\mathbf{k}, \qquad \mathbf{i} + \mathbf{j} - 3\mathbf{k},$$

and determine its magnitude and direction cosines.

3. Find the length and direction cosines of the sum of the vectors $\overrightarrow{B}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, given the coördinates

$$A\,(-1, 2, -1), \qquad B\,(-3, 6, 6), \qquad C\,(4, 3, 1), \qquad D\,(0, 0, 2).$$

4. With the coördinates given above:

(*a*) Find the point dividing AB in the ratio 2/1.

(*b*) Find the mean center of A, B, C (§ 7, Prob. 3).

(*c*) Show that the middle points of the sides of the skew quadrilateral $ABCD$ are the vertices of a parallelogram.

5. Prove that

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\,|\mathbf{u}|\,|\mathbf{v}| \cos(\mathbf{u}, \mathbf{v}).$$

6. From the equation of Prob. 5 show that

$$\cos(\mathbf{u}, \mathbf{v}) = \frac{u_x v_x + u_y v_y + u_z v_z}{|\mathbf{u}|\,|\mathbf{v}|}.$$

7. Find the Cartesian equations of a line through the points A $(-2, 0, 2)$ and B $(2, 1, -3)$.

12. Centroid. We shall encounter problems in which each point of a given set is associated with a certain number. The points, for example, may represent particles of matter and the numbers their weights or electric charges. In the latter case the numbers may be positive or negative. We shall now define a

point called the *centroid* of the given set of "weighted" points, which, in a certain sense, represents their average position.

Consider a set of n points $P_1, P_2, \ldots, P_n$ associated with the numbers $m_1, m_2, \ldots, m_n$ respectively. We shall denote any one of these points by P_i and its associated number by m_i. Then *the centroid of these "weighted" points is defined as the point* P^* *for which the sum of all the vectors* $m_i \overrightarrow{P^*P_i}$ *is zero:*

$$(1) \qquad m_1 \overrightarrow{P^*P_1} + m_2 \overrightarrow{P^*P_2} + \cdots + m_n \overrightarrow{P^*P_n} = 0.$$

Let us first consider a particular case.

Example 1. The centroid of the points P_1, P_2, P_3, when associated with the numbers 3, -1, 2 respectively, is a point P^* which satisfies the equation

$$(a) \qquad 3\,\overrightarrow{P^*P_1} + (-1)\,\overrightarrow{P^*P_2} + 2\,\overrightarrow{P^*P_3} = 0.$$

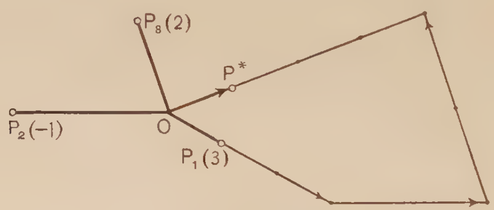

Fig. 12.

We first assure ourselves that there is only one point P^* which satisfies (a). For if the point Q also satisfies (a),

$$3\,\overrightarrow{QP_1} + (-1)\,\overrightarrow{QP_2} + 2\,\overrightarrow{QP_3} = 0.$$

On subtracting this equation from (a) we have

$$3\,(\overrightarrow{P^*P_1} + \overrightarrow{P_1Q}) + (-1)\,(\overrightarrow{P^*P_2} + \overrightarrow{P_2Q}) + 2\,(\overrightarrow{P^*P_3} + \overrightarrow{P_3Q}) = 0,$$

$$3\,\overrightarrow{P^*Q} - \overrightarrow{P^*Q} + 2\,\overrightarrow{P^*Q} = 4\,\overrightarrow{P^*Q} = 0;$$

hence $\overrightarrow{P^*Q} = 0$ and the points P^* and Q coincide.

To locate P^* we choose any point O as origin and write (a) in the form

$$3\,(\overrightarrow{OP_1} - \overrightarrow{OP^*}) + (-1)\,(\overrightarrow{OP_2} - \overrightarrow{OP^*}) + 2\,(\overrightarrow{OP_3} - \overrightarrow{OP^*}) = 0.$$

Hence

$$3\,\overrightarrow{OP_1} - \overrightarrow{OP_2} + 2\,\overrightarrow{OP_3} = (3 - 1 + 2)\,\overrightarrow{OP^*},$$

$$(b) \qquad \overrightarrow{OP^*} = \frac{3\,\overrightarrow{OP_1} - \overrightarrow{OP_2} + 2\,\overrightarrow{OP_3}}{4}.$$

The point P^* given by (b) is a solution of (a); for we may retrace our work step by step from (b) back to (a). The construction of the centroid from equation (b) is shown in Fig. 12.

Return now to the general case in which the equation defining P^* is

(1) $$\sum m_i \overrightarrow{P^*P_i} = 0.$$

We shall first show that if the sum of the numbers m_i is not zero, there can be only *one* point P^* which satisfies (1). To prove this, suppose that Q is a second point for which

$$\sum m_i \overrightarrow{QP_i} = 0.$$

On subtracting this equation from (1) we have

$$\sum m_i (\overrightarrow{P^*P_i} - \overrightarrow{QP_i}) = \sum m_i (\overrightarrow{P^*P_i} + \overrightarrow{P_iQ}) = \sum m_i \overrightarrow{P^*Q} = 0,$$

or since $\overrightarrow{P^*Q}$ appears in every term of the sum,

$$\left(\sum m_i\right) \overrightarrow{P^*Q} = 0.$$

If $\sum m_i \neq 0$ we must have $\overrightarrow{P^*Q} = 0$ and hence the points P^* and Q coincide.

We next prove that when $\sum m_i \neq 0$ there is always one point P^* which satisfies (1). Choose an origin O at pleasure; then (1) may be written

$$\sum m_i (\overrightarrow{OP_i} - \overrightarrow{OP^*}) = 0 \quad \text{or} \quad \sum m_i \overrightarrow{OP_i} = \left(\sum m_i\right) \overrightarrow{OP^*};$$

hence

(2) $$\overrightarrow{OP^*} = \frac{\sum m_i \overrightarrow{OP_i}}{\sum m_i}$$

gives the position vector of the centroid P^* relative to O. If the position vectors of P^* and P_i are written $\mathbf{r}^*$ and $\mathbf{r}_i$, this becomes

$$\mathbf{r}^* = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}.$$

The position of the centroid is not altered by replacing the associated numbers $m_1, \ldots, m_n$ by any set of numbers propor-

tional to them. For if we multiply all of the m's by the same constant c, we see from (2) that c may be canceled from the numerator and denominator. In particular if the m's are all equal we may replace them all by unity; the centroid is then given by

$$\text{(3)} \qquad \overrightarrow{OP^*} = \frac{1}{n}\sum \overrightarrow{OP_i} \quad \text{or} \quad \mathbf{r}^* = \frac{1}{n}\sum \mathbf{r}_i.$$

In the case of *equal* associated numbers the centroid is called the *mean center* of the set of points.

Example 2. If P^* is the centroid of the points A, B associated with the numbers a, b, we have from (1) and (2)

$$a\,\overrightarrow{P^*A} + b\,\overrightarrow{P^*B} = 0, \qquad \overrightarrow{OP^*} = \frac{a\,\overrightarrow{OA} + b\,\overrightarrow{OB}}{a+b}.$$

Either equation shows that P^* divides AB in the ratio of b/a (§ 7), that is, inversely in the ratio of the associated numbers. P^* is an internal or an external point of division according as b/a is positive or negative.

In particular, if $a = b$, P^* divides AB in the ratio of $1/1$; hence the mean center of two points is midway between them.

Theorem. *If the set of points P_i associated with the numbers m_i is divided into two groups whose centroids are P', P'', then the centroid P^* of the entire set coincides with the centroid of P' and P'' when associated with the numbers m', m'' equal to sums of the m's for each group.*

Proof. Let $\mathbf{r}^*$, $\mathbf{r}'$, $\mathbf{r}''$ be the position vectors of P^*, P', P''. Then if Σ', Σ'' denote summations over the two separate groups, we have from (2)

$$\mathbf{r}' = \frac{\Sigma'\, m_i\mathbf{r}_i}{m'}, \qquad \mathbf{r}'' = \frac{\Sigma''\, m_i\mathbf{r}_i}{m''},$$

$$\mathbf{r}^* = \frac{\Sigma'\, m_i\mathbf{r}_i + \Sigma''\, m_i\mathbf{r}_i}{m' + m''} = \frac{m'\mathbf{r}' + m''\mathbf{r}''}{m' + m''}.$$

Example 3. To find the mean center G of three points A, B, C.

G is the centroid of A, B, C associated with the numbers 1, 1, 1. The centroid of A and B is the mid point N of the segment AB (Fig. 7*e*). Hence, by the theorem above, G is the centroid of C and N associated with the numbers 1 and 2; thus G divides CN in the ratio of $2/1$ (Ex. 2). As this statement applies to any median of the triangle ABC, we have proved that the medians of a triangle intersect in the mean center of its vertices and divide each other in the ratio of $2/1$.

PROBLEMS

1. Find the centroid of the points $(6, 1, 3)$, $(-3, 0, 4)$, $(3, -2, 0)$ associated with the numbers 2, 3, 1 respectively.

2. In Fig. 7*f*, P, Q, R, S are the middle points of the segments AB, CD, AD, BC. Show that PQ and RS intersect at the mean center of A, B, C, D.

3. In Fig. 7*f* let G denote the mean center of A, B, C. Prove that the mean center of A, B, C, D divides DG in the ratio of $3/1$.

4. If P^* and Q^* are the mean centers of the two sets of n points $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$, prove that

$$\sum \overrightarrow{P_iQ_i} = n\,\overrightarrow{P^*Q^*}.$$

5. Give a construction for finding the mean center of five points. Use the theorem above.

13. Products of Two Vectors. Hitherto we have only considered the products of vectors by numbers. We shall next define two operations between vectors, which are known as "products," because they have some properties in common with the products of numbers. These products of vectors, however, will also prove to have properties in striking disagreement with those of numbers.

Since one of these products gives rise to a scalar, the other to a vector, they are called the *scalar product* and *vector product* respectively. The definitions of these new products will seem rather arbitrary to one who is unfamiliar with the history of vector algebra. It must suffice at present to assure the student that both products are of great service in the development of mechanics, and moreover play an important rôle in other physical sciences, such as the theories of light, electricity and magnetism. Vector analysis, in fact, is growing in importance day by day, and will prove to be an invaluable aid to a genuine understanding of all those sciences that deal with directed quantities.

The scalar and vector products of two vectors $\mathbf{u}$ and $\mathbf{v}$ are written $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$ respectively, and are read $\mathbf{u}$ *dot* $\mathbf{v}$ and $\mathbf{u}$ *cross* $\mathbf{v}$. In view of this notation the products are sometimes called the *dot product* and *cross product.*

In vector algebra as here developed *the division of vectors by vectors is not defined.* This operation, therefore, is never used. Remember, however, that dividing a vector by a scalar $a \neq 0$ is the same as multiplying the vector by $1/a$ (§ 6).

14. Scalar Product of Two Vectors. *The scalar product of two vectors* $\mathbf{u}$ *and* $\mathbf{v}$, *written* $\mathbf{u}\cdot\mathbf{v}$, *is defined as the product of their lengths and the cosine of their included angle:*

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos(\mathbf{u}, \mathbf{v}). \tag{1}$$

The scalar product is therefore a *number*, which for *proper* vectors (i.e. vectors which are not zero) is positive, zero, or negative according as the angle $(\mathbf{u}, \mathbf{v})$ is acute, right or obtuse. For example let $|\mathbf{u}| = 2$, $|\mathbf{v}| = 3$; then if

$$\begin{aligned}
(\mathbf{u}, \mathbf{v}) &= 60^\circ, & \mathbf{u}\cdot\mathbf{v} &= 6 \cos 60^\circ = 3;\\
(\mathbf{u}, \mathbf{v}) &= 90^\circ, & \mathbf{u}\cdot\mathbf{v} &= 6 \cos 90^\circ = 0;\\
(\mathbf{u}, \mathbf{v}) &= 120^\circ, & \mathbf{u}\cdot\mathbf{v} &= 6 \cos 120^\circ = -3.
\end{aligned}$$

When a and b are numbers, $ab = 0$ only when $a = 0$ or $b = 0$. In contrast to this property, (1) shows that $\mathbf{u}\cdot\mathbf{v} = 0$ not only when $\mathbf{u} = 0$ or $\mathbf{v} = 0$, but also when $\cos(\mathbf{u}, \mathbf{v}) = 0$, that is, when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. Hence when $\mathbf{u}$ and $\mathbf{v}$ are *proper* vectors,

$$\mathbf{u}\cdot\mathbf{v} = 0 \quad \text{is equivalent to} \quad \mathbf{u} \perp \mathbf{v}. \tag{2}$$

When $\mathbf{u}$ and $\mathbf{v}$ are parallel

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \quad \text{or} \quad -|\mathbf{u}|\,|\mathbf{v}|$$

according as the vectors have the same or opposite directions. It is customary to write $\mathbf{u}^2$ instead of $\mathbf{u}\cdot\mathbf{u}$; thus

$$\mathbf{u}^2 = \mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2.$$

The square of a vector denotes the square of its length.

If $\mathbf{e}$ denotes a unit vector in the positive direction of s, the component of $\mathbf{u}$ on s is from § 10

$$u_s = |\mathbf{u}| \cos(s, \mathbf{u}) = |\mathbf{e}|\,|\mathbf{u}| \cos(\mathbf{e}, \mathbf{u}),$$

or, in view of (1),

$$u_s = \mathbf{e}\cdot\mathbf{u}. \tag{3}$$

The component of a vector on an axis is equal to its scalar product with a unit vector in the positive direction of the axis.

Since the right-hand side of (1) is not altered by interchanging the order of $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{v}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{v}. \tag{4}$$

The scalar product of two vectors is commutative.

We shall use the dot, as a symbol of operation, between two *vectors* to denote their scalar product or between scalars in the usual sense. Hence there is no question as to whether $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$ and $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$ are equal since both expressions are meaningless.

From (1) we see that

$$(-\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(-\mathbf{v}) = -\mathbf{u}\cdot\mathbf{v}, \qquad (-\mathbf{u})\cdot(-\mathbf{v}) = \mathbf{u}\cdot\mathbf{v},$$

$$(a\mathbf{u})\cdot(b\mathbf{v}) = ab\,\mathbf{u}\cdot\mathbf{v}. \tag{5}$$

The last result is obvious when a and b are positive numbers; the other cases then follow from the equations preceding.

15. Distributive Law for Scalar Products. If we denote the component of $\mathbf{u}$ in the direction of the vector $\mathbf{v}$ by u_v, we have from § 10

$$u_v = |\mathbf{u}| \cos(\mathbf{u}, \mathbf{v}), \quad v_u = |\mathbf{v}| \cos(\mathbf{u}, \mathbf{v}).$$

Hence the defining equation (§ 14, 1) for $\mathbf{u}\cdot\mathbf{v}$ may be written

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\, v_u = |\mathbf{v}|\, u_v; \tag{1}$$

that is, *the scalar product of two vectors is equal to the length of either one multiplied by the component of the other in its direction.*

We may now show that scalar products such as $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$ and $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}$ may be expanded just as in ordinary algebra; thus

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}, \qquad (\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}. \tag{2), (3}$$

To prove (2), for example, we note that the component of $\mathbf{v}+\mathbf{w}$ in the direction of $\mathbf{u}$ is $v_u + w_u$ (§ 10); hence from (1)

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = |\mathbf{u}|\,(v_u + w_u) = |\mathbf{u}|\,v_u + |\mathbf{u}|\,w_u = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}.$$

By repeated application of (2) and (3) we may expand the scalar product of any two vector sums; for example

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}+\mathbf{d}) &= \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) + \mathbf{a}\cdot\mathbf{d} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{d},\\
(\mathbf{a}+\mathbf{b})\cdot(\mathbf{c}+\mathbf{d}) &= (\mathbf{a}+\mathbf{b})\cdot\mathbf{c} + (\mathbf{a}+\mathbf{b})\cdot\mathbf{d}\\
&= \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{d},\\
(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= \mathbf{a}^2 + \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}^2 = \mathbf{a}^2 - \mathbf{b}^2,\\
(\mathbf{a}+\mathbf{b})^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}^2 + 2\,\mathbf{a}\cdot\mathbf{b} + \mathbf{b}^2.
\end{aligned}$$

As a further application of the distributive law let us expand

$$\mathbf{u}\cdot\mathbf{v} = (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\cdot(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}).$$

We note first that since **i**, **j**, **k** are mutually perpendicular unit vectors

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1, \qquad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$$

Therefore six of the nine terms in the expansion of $\mathbf{u}\cdot\mathbf{v}$ vanish, while the remaining three give

$$\mathbf{u}\cdot\mathbf{v} = u_x v_x + u_y v_y + u_z v_z. \tag{4}$$

This important result may be stated as follows:

The scalar product of two vectors is equal to the sum of the products of their corresponding components.

If $\mathbf{c} \neq 0$ in the equation

$$\mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c} \quad \text{or} \quad (\mathbf{a} - \mathbf{b})\cdot\mathbf{c} = 0,$$

we can conclude *either* that $\mathbf{a} - \mathbf{b} = 0$ *or* that $\mathbf{a} - \mathbf{b}$ and $\mathbf{c}$ are perpendicular vectors. Hence we cannot "cancel" $\mathbf{c}$ in the equation to obtain $\mathbf{a} = \mathbf{b}$ unless we know that $\mathbf{a} - \mathbf{b}$ and $\mathbf{c}$ are not perpendicular.

Example 1. If $\mathbf{u} = [2, -1, 3]$, $\mathbf{v} = [0, 2, 4]$,

$$\mathbf{u}\cdot\mathbf{v} = 0 - 2 + 12 = 10.$$

From (1) we see that the component of **u** in the direction of **v** is

$$u_v = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{10}{\sqrt{20}} = \sqrt{5} = 2.236,$$

while the component of **v** in the direction of **u** is

$$v_u = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|} = \frac{10}{\sqrt{14}} = \frac{5}{7}\sqrt{14} = 2.673.$$

The angle between **u** and **v** is given by

$$\cos(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{10}{\sqrt{280}} = 0.5976; \quad \text{hence} \quad (\mathbf{u}, \mathbf{v}) = 53^\circ\, 18'.$$

Example 2. In the circle of Fig. 15*a*, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OP} = \mathbf{r}$ denote fixed and variable vector radii. For any position of P on the circle

$$\mathbf{r}^2 = \mathbf{a}^2 \quad \text{or} \quad \mathbf{r}^2 - \mathbf{a}^2 = (\mathbf{r} - \mathbf{a})\cdot(\mathbf{r} + \mathbf{a}) = 0.$$

But since

$$\mathbf{r} - \mathbf{a} = \overrightarrow{OP} - \overrightarrow{OA} = \overrightarrow{AP}, \qquad \mathbf{r} + \mathbf{a} = \overrightarrow{OP} + \overrightarrow{BO} = \overrightarrow{BP},$$

we see that $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are perpendicular vectors. Thus we have proved that an angle inscribed in a semicircle is a right angle.

Example 3. In the triangle ABC of Fig. 15b we have

$$\mathbf{a} = \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \mathbf{b} - \mathbf{c}, \quad \mathbf{a}^2 = (\mathbf{b} - \mathbf{c})\cdot(\mathbf{b} - \mathbf{c}) = \mathbf{b}^2 - 2\,\mathbf{b}\cdot\mathbf{c} + \mathbf{c}^2.$$

Hence with the usual notation for the sides and angles of a triangle,

$$a^2 = b^2 + c^2 - 2\,bc \cos A.$$

This is the Cosine Law of plane trigonometry.

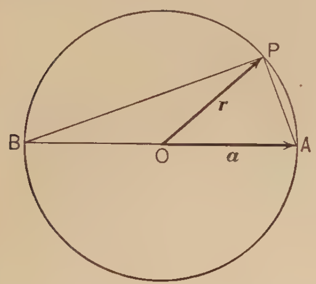

FIG. 15a.

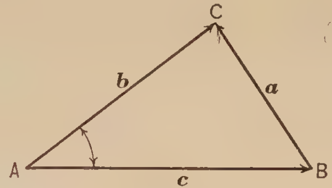

FIG. 15b.

Example 4. To find the shortest distance from a point A to the line through the points B and C.

Method. Find the component of the vector $\overrightarrow{AB}$ in the direction of $\overrightarrow{BC}$. The required distance d is then given by the equation

$$(\text{component of } \overrightarrow{AB} \text{ on } \overrightarrow{BC})^2 + d^2 = \overrightarrow{AB}^2.$$

Computation. If the points are

$$A\ (3, 1, -1), \qquad B\ (2, 3, 0), \qquad C\ (-1, 2, 4),$$

we have $$\overrightarrow{AB} = [-1, 2, 1], \qquad \overrightarrow{BC} = [-3, -1, 4].$$

From (1) the component of $\overrightarrow{AB}$ in the direction of $\overrightarrow{BC}$ is

$$\frac{\overrightarrow{AB}\cdot\overrightarrow{BC}}{|\overrightarrow{BC}|} = \frac{3 - 2 + 4}{\sqrt{26}} = \frac{5}{\sqrt{26}};$$

hence

$$d^2 = 6 - \frac{25}{26} = 5\frac{1}{26} = 5.038, \qquad d = 2.245.$$

Check. If we use the vector $\overrightarrow{AC}$ instead of $\overrightarrow{AB}$, we have $\overrightarrow{AC} = [-4, 1, 5]$,

$$\frac{\overrightarrow{AC}\cdot\overrightarrow{BC}}{|\overrightarrow{BC}|} = \frac{31}{\sqrt{26}}, \quad \text{and} \quad d^2 = 42 - \frac{961}{26} = 5\frac{1}{26}$$

as before.

Example 5. To find the Cartesian equation of a plane through the point P_1 and normal to the vector $\mathbf{n} = [A, B, C]$.

Let $\mathbf{r}_1$ and $\mathbf{r}$ denote the position vectors of P_1 and any point P of the plane. Then since $\mathbf{n}$ is perpendicular to

$$\overrightarrow{P_1P} = \mathbf{r} - \mathbf{r}_1 = [x - x_1, y - y_1, z - z_1],$$

the desired equation is $\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_1) = 0$, or

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$$

If the equation of the plane is given as

$$Ax + By + Cz + D = 0,$$

the vector $\mathbf{n} = [A, B, C]$ is normal to the plane. For if P and P_1 are points on the plane,

$$\mathbf{n}\cdot\mathbf{r} + D = 0, \qquad \mathbf{n}\cdot\mathbf{r}_1 + D = 0, \quad \text{and} \quad \mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_1) = 0.$$

Thus $\mathbf{n}$ is perpendicular to any vector $\overrightarrow{P_1P}$ in the plane and hence to the plane itself.

Example 6. To find the shortest distance from a point $P_1\,(x_1, y_1, z_1)$ to the plane whose Cartesian equation is

$$Ax + By + Cz + D = 0.$$

Method. If $P\,(x, y, z)$ is any point of the plane, the required distance d is numerically equal to the component of $\overrightarrow{PP_1}$ on the vector $\mathbf{n} = [A, B, C]$ normal to the plane.

Calculation. Since $\overrightarrow{PP_1} = [x_1 - x,\ y_1 - y,\ z_1 - z]$,

the component of $\overrightarrow{PP_1}$ on $\mathbf{n}$ is, from (1),

$$\frac{\mathbf{n}\cdot\overrightarrow{PP_1}}{|\mathbf{n}|} = \frac{A(x_1 - x) + B(y_1 - y) + C(z_1 - z)}{\sqrt{A^2 + B^2 + C^2}}.$$

Hence, on replacing $-Ax - By - Cz$ in the numerator by D, we have

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Remark. The problem of finding the shortest distance of the point $P_1\,(x_1, y_1)$ to the *line* $Ax + By + D = 0$ in the xy-plane is the same as finding the distance of $P_1\,(x_1, y_1, 0)$ to the *plane* $Ax + By + D = 0$ (which passes through the given line and is perpendicular to the xy-plane). The required distance is therefore

$$d = \frac{|Ax_1 + By_1 + D|}{\sqrt{A^2 + B^2}}.$$

PROBLEMS

1. Find the component of $[2, 3, -1]$ upon an axis in the direction of $[-1, -2, 2]$.

2. Find the component of the vector $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ upon a line directed from O into the first octant and making equal angles with the three coordinate axes.

3. Find the component of the vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ upon a line whose direction is the same as that of the vector $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

4. Find the angle between the vectors

(*a*) $[1, 1, 0]$, $[1, 0, 1]$; (*b*) $[1, 1, 1]$, $[1, 0, 0]$.

5. Find the shortest distance from the point $A\ (2, -3, -4)$ to the line through $B\ (1, 2, -3)$ and $C\ (3, 3, -5)$.

6. Find the shortest distance from the point $A\ (1, -2, 1)$ to the plane $4x - 3y + 12z - 8 = 0$.

7. Find the equation of a plane perpendicular to the line through $A\ (3, 4, -1)$ and $B\ (5, 2, 3)$ at its middle point.

8. Find the angle between the planes

$$x - y + z + 2 = 0, \qquad 2x + y - z + 1 = 0.$$

9. Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides.

10. Show that the equation of a sphere having AB as a diameter is $(\mathbf{r} - \mathbf{a})\cdot(\mathbf{r} - \mathbf{b}) = 0$, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{r}$ denote the position vectors of A, B, and any point P on the surface.

16. Vector Product of Two Vectors. When the vectors $\mathbf{u}$ and $\mathbf{v}$ are not parallel, $(\mathbf{u}, \mathbf{v})$ shall denote the smaller angle, regarded as positive, between $\mathbf{u}$ and $\mathbf{v}$; that is, $(\mathbf{u}, \mathbf{v}) > 0$ and $< 180°$. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, $(\mathbf{u}, \mathbf{v}) = 0$ or $180°$. Thus $\sin\,(\mathbf{u}, \mathbf{v})$ is either positive or zero.

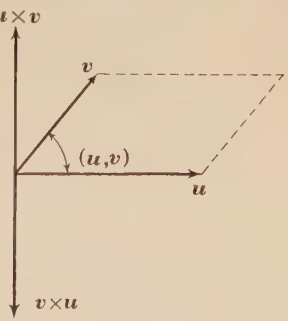

FIG. 16.

The vector product of two vectors $\mathbf{u}$ *and* $\mathbf{v}$, *written* $\mathbf{u}\times\mathbf{v}$, *is defined as the vector of length* $|\mathbf{u}|\,|\mathbf{v}|\sin\,(\mathbf{u}, \mathbf{v})$ *which is perpendicular to both* $\mathbf{u}$ *and* $\mathbf{v}$ *in the direction in which a right-handed screw advances when turned from* $\mathbf{u}$ *towards* $\mathbf{v}$ *through the angle* $(\mathbf{u}, \mathbf{v})$. In symbols,

$$(1) \qquad \mathbf{u}\times\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\sin\,(\mathbf{u}, \mathbf{v})\,\mathbf{e},$$

where $\mathbf{e}$ is a unit vector in the direction specified.

This definition fails to determine the direction of $\mathbf{u}\times\mathbf{v}$ when $\mathbf{u}$ and $\mathbf{v}$ are parallel; in this case, however, $\sin\,(\mathbf{u}, \mathbf{v}) = 0$

and $\mathbf{u} \times \mathbf{v} = 0$. Moreover $\mathbf{u} \times \mathbf{v} = 0$ only when $\mathbf{u} = 0$, $\mathbf{v} = 0$, or $\sin(\mathbf{u}, \mathbf{v}) = 0$. The last condition means that $\mathbf{u}$ and $\mathbf{v}$ are parallel. Hence when $\mathbf{u}$ and $\mathbf{v}$ are *proper* vectors,

(2) $$\mathbf{u} \times \mathbf{v} = 0 \quad \text{is equivalent to} \quad \mathbf{u} \parallel \mathbf{v}.$$ ∼ coll.

Note, in particular, that $\mathbf{u} \times \mathbf{u} = 0$.

If we construct a parallelogram on $\mathbf{u}$ and $\mathbf{v}$ as sides, the distance between the sides parallel to $\mathbf{u}$ is $|\mathbf{v}| \sin(\mathbf{u}, \mathbf{v})$. The area of the parallelogram is therefore

(3) $$A = |\mathbf{u}|\,|\mathbf{v}| \sin(\mathbf{u}, \mathbf{v}) \quad \text{and} \quad \mathbf{u} \times \mathbf{v} = A\mathbf{e}.$$

Thus *the length of the vector* $\mathbf{u} \times \mathbf{v}$ *is numerically the same as the area of the parallelogram determined by* $\mathbf{u}$ *and* $\mathbf{v}$. This area is zero when, and only when, $\mathbf{u}$ and $\mathbf{v}$ are parallel — in agreement with (2) above.

If $\mathbf{u}$ and $\mathbf{v}$ are interchanged in (1) the scalar factor $|\mathbf{u}|\,|\mathbf{v}| \sin(\mathbf{u}, \mathbf{v})$ is not altered; but as a screw turned from $\mathbf{v}$ toward $\mathbf{u}$ will advance in the direction opposite to that when turned from $\mathbf{u}$ toward $\mathbf{v}$, we see that $\mathbf{v} \times \mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$ have the same length but opposite directions:

(4) $$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}.$$

Therefore *the vector product of two vectors is not commutative.*

From (1) we see that

(5) $$(-\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (-\mathbf{v}) = -\mathbf{u} \times \mathbf{v}, \qquad (-\mathbf{u}) \times (-\mathbf{v}) = \mathbf{u} \times \mathbf{v},$$
$$(a\mathbf{u}) \times (b\mathbf{v}) = ab\, \mathbf{u} \times \mathbf{v}.$$

The last result is obvious when a and b are positive numbers; the other cases then follow from the equations preceding.

17. Distributive Law for Vector Products. The vector $\mathbf{u} \times \mathbf{v}$ may be formed as follows. Draw $\mathbf{u}$ and $\mathbf{v}$ from the same point A (Fig. 17a). Let p be a plane perpendicular to $\mathbf{u}$ at A, and regard rotations in this plane as positive when they advance a r-h screw in the direction of $\mathbf{u}$. Then $\mathbf{u} \times \mathbf{v}$ may be obtained by the sequence of three operations, namely

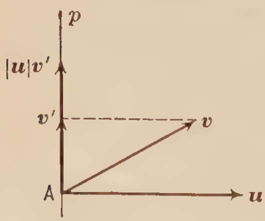

Fig. 17a.

(P) Project $\mathbf{v}$ on p, obtaining $\mathbf{v}'$;
(M) Multiply $\mathbf{v}'$ by $|\mathbf{u}|$, obtaining $|\mathbf{u}|\,\mathbf{v}'$;
(R) Revolve $|\mathbf{u}|\,\mathbf{v}'$ through $+90°$ about $\mathbf{u}$, obtaining $\mathbf{u} \times \mathbf{v}$.

In fact the resulting vector agrees with $\mathbf{u}\times\mathbf{v}$ in direction (upward in Fig. 17a) and also in magnitude since $|\mathbf{u}|\,|\mathbf{v}'| = |\mathbf{u}|\,|\mathbf{v}| \sin(\mathbf{u}, \mathbf{v})$. We may indicate this method of forming $\mathbf{u}\times\mathbf{v}$ by the notation

$$\mathbf{u}\times\mathbf{v} = RMP\mathbf{v}; \tag{1}$$

this simply means that $\mathbf{v}$ is first projected, then multiplied, and finally revolved as described above.

Now each of these operations is distributive; that is, the result of applying any one, as R, to the sum of two vectors is the same as if R were applied to the vectors separately and the resulting vectors added. Keeping this in mind, we have from (1)

$$\begin{aligned} \mathbf{u}\times(\mathbf{v} + \mathbf{w}) &= RMP\,(\mathbf{v} + \mathbf{w}) \\ &= RM\,(P\mathbf{v} + P\mathbf{w}) \\ &= R\,(MP\mathbf{v} + MP\mathbf{w}) \\ &= RMP\mathbf{v} + RMP\mathbf{w}, \quad \text{or} \end{aligned}$$

$$\mathbf{u}\times(\mathbf{v} + \mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}. \tag{2}$$

If the factors are interchanged in every term of (2) we again obtain a true equation:

$$(\mathbf{v} + \mathbf{w})\times\mathbf{u} = \mathbf{v}\times\mathbf{u} + \mathbf{w}\times\mathbf{u}; \tag{3}$$

for this simply amounts to multiplying (2) by -1.

Repeated application of (2) and (3) enables us to expand the vector product of two vector sums just as in ordinary algebra, *provided that the order of the factors is not altered;* for example

$$(\mathbf{a} + \mathbf{b})\times(\mathbf{c} + \mathbf{d}) = \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c} + \mathbf{a}\times\mathbf{d} + \mathbf{b}\times\mathbf{d}.$$

If $\mathbf{c} \neq 0$ in the equation

$$\mathbf{a}\times\mathbf{c} = \mathbf{b}\times\mathbf{c} \quad \text{or} \quad (\mathbf{a} - \mathbf{b})\times\mathbf{c} = 0,$$

we can conclude *either* that $\mathbf{a} - \mathbf{b} = 0$ *or* that $\mathbf{a} - \mathbf{b}$ and $\mathbf{c}$ are parallel vectors. Hence we cannot cancel $\mathbf{c}$ in the equation to obtain $\mathbf{a} = \mathbf{b}$ unless we know that $\mathbf{a} - \mathbf{b}$ and $\mathbf{c}$ are not parallel.

Since the axes of coördinates in § 11 were chosen so as to form a r-h system, a rotation of a r-h screw from $\mathbf{i}$ to $\mathbf{j}$, $\mathbf{j}$ to $\mathbf{k}$, or $\mathbf{k}$ to $\mathbf{i}$ will make it advance in the direction of $\mathbf{k}$, $\mathbf{i}$ or $\mathbf{j}$ respectively (see Fig. 11). Hence from the definition of a vector product,

$$\begin{aligned} &\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad &\mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad &\mathbf{k}\times\mathbf{i} = \mathbf{j}; \\ &\mathbf{j}\times\mathbf{i} = -\mathbf{k}, \quad &\mathbf{k}\times\mathbf{j} = -\mathbf{i}, \quad &\mathbf{i}\times\mathbf{k} = -\mathbf{j}. \end{aligned} \tag{4}$$

The relations (4) are easily remembered if we note that **i**, **j**, **k** occur in each equation in the same cyclical order (Fig. 17*b*).

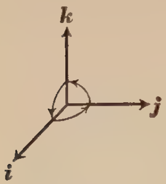

Fig. 17*b*.

We may now expand the product

$$\mathbf{u}\times\mathbf{v} = (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\times(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}).$$

Of the nine terms in the expansion, three vanish since

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0,$$

while the remaining six may be grouped as follows:

$$\mathbf{u}\times\mathbf{v} = (u_yv_z - u_zv_y)\,\mathbf{i} + (u_zv_x - u_xv_z)\,\mathbf{j} + (u_xv_y - u_yv_x)\,\mathbf{k}.$$

From this result we see that the components of $\mathbf{u}\times\mathbf{v}$ are the determinants formed by columns *2* and *3*, *3* and *1* (not *1* and *3*), *1* and *2* of the array

$$\begin{matrix} u_x & u_y & u_z \\ v_x & v_y & v_z. \end{matrix}$$

This rule is equivalent to writing $\mathbf{u}\times\mathbf{v}$ as the determinant

$$(5) \qquad \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}. \quad \begin{matrix} & \\ u_x & u_y \\ v_x & v_y \end{matrix}$$

Example 1. If $\mathbf{u} = [2, -3, 5]$, $\mathbf{v} = [-1, 4, 2]$, we may compute the components of $\mathbf{u}\times\mathbf{v}$ from the array

$$\begin{matrix} 2 & -3 & 5 \\ -1 & 4 & 2. \end{matrix}$$

Since

$$\begin{vmatrix} -3 & 5 \\ 4 & 2 \end{vmatrix} = -26, \qquad \begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix} = -9, \qquad \begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix} = 5,$$

$$\mathbf{u}\times\mathbf{v} = [-26,\ -9,\ 5].$$

As a check we may verify that $\mathbf{u}\times\mathbf{v}$ is perpendicular to both **u** and **v**:

$$(2)(-26) + (-3)(-9) + (5)(5) = -52 + 27 + 25 = 0,$$
$$(-1)(-26) + (4)(-9) + (2)(5) = 26 - 36 + 10 = 0.$$

Example 2. To find the equation of a plane through three points A, B, C not in a straight line.

Method. Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ lie in the plane, $\mathbf{n} = \overrightarrow{AB}\times\overrightarrow{AC}$ is normal to the plane. If P is any point of the plane, $\overrightarrow{AP}$ lies in the plane and is perpendicular to **n**. Hence $\mathbf{n}\cdot\overrightarrow{AP} = 0$ is the required equation.

Computation. If the points are

$$A\,(1, 0, 2), \quad B\,(2, 3, 0), \quad C\,(-1, -2, 5), \quad P\,(x, y, z),$$

we have

$$\overrightarrow{AB} = [1, 3, -2], \quad \overrightarrow{AC} = [-2, -2, 3], \quad \overrightarrow{AP} = [x - 1, y, z - 2],$$

and by the method of Ex. 1 we find

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = [5, 1, 4].$$

Hence the required equation is

$$5(x - 1) + y + 4(z - 2) = 0 \quad \text{or} \quad 5x + y + 4z = 13.$$

Check. This equation is satisfied by the points A, B, C.

Example 3. To find the shortest distance between two straight lines in space.

Method. If the straight lines are AB and CD, the vector $\mathbf{p} = \overrightarrow{AB} \times \overrightarrow{CD}$ is normal to both lines. Then the numerical value of the component of $\overrightarrow{AC}$ along $\mathbf{p}$ will give the required distance. Instead of $\overrightarrow{AC}$ we may use any vector, as $\overrightarrow{BD}$, which begins in one line and ends in the other.

Computation. If the lines AB and CD are given by the points

$$A\,(2, -3, 1), \quad B\,(1, 0, -2), \quad C\,(4, 2, 1), \quad D\,(-1, -2, 1),$$

we have

$$\overrightarrow{AB} = [-1, 3, -3], \quad \overrightarrow{CD} = [-5, -4, 0], \quad \mathbf{p} = \overrightarrow{AB} \times \overrightarrow{CD} = [-12, 15, 19].$$

The component of $\overrightarrow{AC} = [2, 5, 0]$ on $\mathbf{p}$ is, from (§ 15, 1),

$$\frac{\mathbf{p}\cdot\overrightarrow{AC}}{|\mathbf{p}|} = \frac{[-12, 15, 19]\cdot[2, 5, 0]}{\sqrt{730}} = \frac{-24 + 75}{\sqrt{730}} = \frac{51}{\sqrt{730}} = 1.889.$$

This is the distance required.

Check. The component of $\overrightarrow{BD} = [-2, -2, 3]$ on $\mathbf{p}$ is

$$\frac{\mathbf{p}\cdot\overrightarrow{BD}}{|\mathbf{p}|} = \frac{[-12, 15, 19]\cdot[-2, -2, 3]}{\sqrt{730}} = \frac{24 - 30 + 57}{\sqrt{730}} = \frac{51}{\sqrt{730}}.$$

PROBLEMS

1. Find the equation of a plane through the points $A\,(1, 0, 2)$, $B\,(2, 3, 0)$, $C\,(-1, -2, 5)$.

2. Find the components of a vector of length 39 which is perpendicular to the vectors $[4, -3, 0]$, $[-4, 6, 1]$.

3. Find the equation of a plane through the points $A\,(1, 2, 3)$, $B\,(2, -3, 4)$ and perpendicular to the plane $3x + 2y - z + 7 = 0$. [The vectors $\overrightarrow{AB}$ and $[3, 2, -1]$ are both parallel to the required plane.]

4. Find the equation of a plane through the point $A\ (-1, 2, 1)$ and perpendicular to the planes $4x + 3y - 2z + 7 = 0$, $3x - 2y - 5z + 6 = 0$.

5. Find the shortest distance from the point $P\ (1, 1, -3)$ to the plane through the points $A\ (1, -1, 2)$, $B\ (3, 2, 4)$, $C\ (6, 2, -2)$.

6. Find the shortest distance between the lines AB and CD:

(a) $A\ (-2, 4, 3)$, $B\ (2, -8, 0)$, $C\ (1, -3, 5)$, $D\ (4, 1, -7)$.

(b) $A\ (2, 3, 1)$, $B\ (0, -1, 2)$, $C\ (1, 2, 5)$, $D\ (-3, 1, 0)$.

18. Scalar Triple Product. Three vectors **u**, **v**, **w**, taken in this order, are said to form a right-handed (r-h) or left-handed (l-h) set according as the angle θ between $\mathbf{u}\times\mathbf{v}$ and **w** is acute or obtuse. This definition is easily remembered by noting that the thumb, index, and middle finger (bent upward) of the right hand form a r-h set.

The scalar product of $\mathbf{u}\times\mathbf{v}$ and **w** is written $\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}$* and often called the *scalar triple product* of these vectors. The geometric meaning of this product is stated in the

THEOREM. *The product* $\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}$ *is numerically equal to the volume* V *of a parallelopiped having* **u**, **v**, **w** *as concurrent edges; more precisely,*

$$(1)\qquad \mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = \begin{Bmatrix} V \\ -V \end{Bmatrix} \text{ when } \mathbf{u}, \mathbf{v}, \mathbf{w} \text{ form a } \begin{Bmatrix} \text{r-h} \\ \text{l-h} \end{Bmatrix} \text{ set.}$$

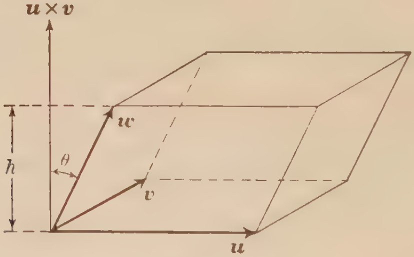

FIG. 18.

If A denotes the area of the faces of this parallelopiped parallel to **u** and **v**, and h the perpendicular distance between these faces (Fig. 18), we have

$$\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = |\mathbf{u}\times\mathbf{v}|\ |\mathbf{w}| \cos\theta = \pm Ah = \pm V$$

since $|\mathbf{u}\times\mathbf{v}| = A$ (§ 16), and $|\mathbf{w}| \cos\theta = h$ or $-h$ according as **u**, **v**, **w** form a r-h or a l-h set.

* No ambiguity can arise by omitting the parentheses in $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ since the grouping $\mathbf{u}\times(\mathbf{v}\cdot\mathbf{w})$ is meaningless.

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are proper vectors, $V = 0$ only when the vectors are coplanar. Therefore *three proper vectors are coplanar (parallel to the same plane) when and only when their scalar triple product is zero.* In particular, a scalar triple product containing two parallel vectors is zero; for the three vectors are then necessarily coplanar. For example, $\mathbf{u}\times\mathbf{v}\cdot\mathbf{u} = 0$.

The r-h or l-h character of a set $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is not changed by a *cyclical* change in their order, such as $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ or $\mathbf{w}$, $\mathbf{u}$, $\mathbf{v}$. The student should assure himself of this fact from a figure. Hence ~~from~~ (1)

~~(2)~~ $$\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = \mathbf{v}\times\mathbf{w}\cdot\mathbf{u} = \mathbf{w}\times\mathbf{u}\cdot\mathbf{v}.$$

However a r-h set becomes l-h, and vice versa, when their cyclical order is changed; hence

$$\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = -\mathbf{u}\times\mathbf{w}\cdot\mathbf{v}, \text{ etc.}$$

Thus if the set $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is r-h, the products in (2) all equal V while

$$\mathbf{u}\times\mathbf{w}\cdot\mathbf{v} = \mathbf{w}\times\mathbf{v}\cdot\mathbf{u} = \mathbf{v}\times\mathbf{u}\cdot\mathbf{w}$$

all equal $-V$.

The value of a scalar triple product is not altered by an interchange of the dot and cross. For from (2), $\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = \mathbf{v}\times\mathbf{w}\cdot\mathbf{u}$, and the last product may also be written $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$ (§14, 4); hence

~~(3)~~ $$\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{v}\times\mathbf{w}.$$

~~From (§ 14, 5) and (§ 16, 5)~~ we have

(4) $$(a\mathbf{u})\times(b\mathbf{v})\cdot(c\mathbf{w}) = abc\,\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}.$$

Finally, the distributive law for scalar and vector products shows that a scalar triple product of vector sums may be expanded just as in ordinary algebra, provided that the order of the factors is not altered. Thus if we expand

$$\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\times(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})\cdot(w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k})$$

we obtain 27 terms. Of these 21 vanish, as they contain triple products in which two or more equal vectors occur. The remaining 6 terms are those containing the products

$$\mathbf{i}\times\mathbf{j}\cdot\mathbf{k} = \mathbf{j}\times\mathbf{k}\cdot\mathbf{i} = \mathbf{k}\times\mathbf{i}\cdot\mathbf{j} = 1, \qquad \mathbf{i}\times\mathbf{k}\cdot\mathbf{j} = \mathbf{k}\times\mathbf{j}\cdot\mathbf{i} = \mathbf{j}\times\mathbf{i}\cdot\mathbf{k} = -1,$$

and the calculation shows that they are precisely the 6 terms in the expansion of the determinant below:

(5) $$\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}.$$

This result also follows at once from the determinant form of $\mathbf{u}\times\mathbf{v}$ (§ 17, 5).

Example 1. To compute the volume of the parallelopiped whose concurrent edges are AB, AC, AD when the rectangular coördinates of A, B, C, D are given.

Let the points be

$$A\,(-3, 1, 2), \quad B\,(-1, 0, -2), \quad C\,(2, 1, 4), \quad D\,(2, -3, 1).$$

Then $\quad \overrightarrow{AB} = [2, -1, -4], \quad \overrightarrow{AC} = [5, 0, 2], \quad \overrightarrow{AD} = [5, -4, -1],$

$$\overrightarrow{AB}\times\overrightarrow{AC} = [-2, -24, 5], \quad \overrightarrow{AB}\times\overrightarrow{AC}\cdot\overrightarrow{AD} = -10 + 96 - 5 = 81.$$

Thus, from (1), 81 is the required volume, and $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ form a r-h set. We might also compute this volume from (5):

$$\overrightarrow{AB}\times\overrightarrow{AC}\cdot\overrightarrow{AD} = \begin{vmatrix} 2 & -1 & -4 \\ 5 & 0 & 2 \\ 5 & -4 & -1 \end{vmatrix} = 81.$$

The volume of the tetrahedron $ABCD$ is $1/6$ of this, as we know from solid geometry.

Example 2. To find the point P where the line AB pierces the plane CDE.

Method. Since the vector $\overrightarrow{CP}$ is coplanar with $\overrightarrow{CD}$ and $\overrightarrow{CE}$,

$$\overrightarrow{CD}\times\overrightarrow{CE}\cdot\overrightarrow{CP} = 0.$$

But
$$\overrightarrow{CP} = \overrightarrow{AP} - \overrightarrow{AC} = \lambda\,\overrightarrow{AB} - \overrightarrow{AC}$$

where λ is a scalar. To find its value we have

$$\overrightarrow{CD}\times\overrightarrow{CE}\cdot(\lambda\,\overrightarrow{AB} - \overrightarrow{AC}) = 0, \qquad \lambda = \frac{\overrightarrow{CD}\times\overrightarrow{CE}\cdot\overrightarrow{AC}}{\overrightarrow{CD}\times\overrightarrow{CE}\cdot\overrightarrow{AB}}.$$

The point P is then given by

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + \lambda\,\overrightarrow{AB}.$$

Computation. Let the line and plane be given by the points

$$A\,(1, 2, 1), \; B\,(2, 1, 2) \quad \text{and} \quad C\,(0, -4, 4), \; D\,(2, -2, 2), \; E\,(4, 1, 2).$$

Then $\quad \overrightarrow{CD} = [2, 2, -2], \quad \overrightarrow{CE} = [4, 5, -2], \quad \overrightarrow{CD}\times\overrightarrow{CE} = [6, -4, 2];$

$$\overrightarrow{AC} = [-1, -6, 3], \qquad \overrightarrow{AB} = [1, -1, 1];$$

$$\lambda = \frac{[6, -4, 2]\cdot[-1, -6, 3]}{[6, -4, 2]\cdot[1, -1, 1]} = \frac{24}{12} = 2.$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \lambda\,\overrightarrow{AB} = [1, 2, 1] + [2, -2, 2] = [3, 0, 3].$$

The required point P is therefore (3, 0, 3).

Check. Since $\overrightarrow{DP}$ is coplanar with $\overrightarrow{DC}$ and $\overrightarrow{DE}$ we may use the relation $\overrightarrow{DC}\times\overrightarrow{DE}\cdot\overrightarrow{DP} = 0$, where

$$\overrightarrow{DP} = \overrightarrow{AP} - \overrightarrow{AD} = \lambda\,\overrightarrow{AB} - \overrightarrow{AD},$$

to determine λ.

19. Vector Triple Product. In order to expand the product $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ we first note that this vector is perpendicular to $\mathbf{u}\times\mathbf{v}$ and hence is coplanar with $\mathbf{u}$ and $\mathbf{v}$. Therefore, from § 8,

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = l\mathbf{u} + m\mathbf{v},$$

where l and m are numbers. To find l and m choose a r-h system of rectangular axes having the x-axis parallel to $\mathbf{u}$ and the y-axis in the plane of $\mathbf{u}$, $\mathbf{v}$. Then

$$\mathbf{u} = u_x\mathbf{i},\qquad \mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j},\qquad \mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k};$$

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= u_xv_y\mathbf{i}\times\mathbf{j} = u_xv_y\mathbf{k},\\
(\mathbf{u}\times\mathbf{v})\times\mathbf{w} &= u_xv_y\,(w_x\mathbf{k}\times\mathbf{i} + w_y\mathbf{k}\times\mathbf{j})\\
&= u_xv_y\,(w_x\mathbf{j} - w_y\mathbf{i})\\
&= u_xw_xv_y\mathbf{j} - v_yw_y\mathbf{u}\\
&= u_xw_x\,(\mathbf{v} - v_x\mathbf{i}) - v_yw_y\mathbf{u}\\
&= u_xw_x\mathbf{v} - (v_xw_x + v_yw_y)\,\mathbf{u};
\end{aligned}$$

or, since

$$\mathbf{u}\cdot\mathbf{w} = u_xw_x,\qquad \mathbf{v}\cdot\mathbf{w} = v_xw_x + v_yw_y,$$

(1) $$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}.$$

Since $\mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = -(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$, we have also

(2) $$\mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = (\mathbf{w}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{w}\cdot\mathbf{u})\,\mathbf{v}.$$

In the left-hand members of (1) and (2), one of the vectors in parenthesis is *adjacent* to the vector outside, the other *remote* from it. The right-hand members of both equations may be remembered from the scheme:

(Outer *dot* Remote) Adjacent *minus* (Outer *dot* Adjacent) Remote.

In general the vectors $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ and $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ are not equal; the former is coplanar with $\mathbf{u}$ and $\mathbf{v}$, the latter with $\mathbf{v}$ and $\mathbf{w}$. Hence *cross multiplication of vectors is not associative.*

Example 1. The vector projection of $\mathbf{u}$ in the direction of the unit vector $\mathbf{e}$ is $(\mathbf{e}\cdot\mathbf{u})\,\mathbf{e}$ from (§ 10, 1); hence the vector projection of $\mathbf{u}$ on a plane perpendicular to $\mathbf{e}$ is

$$\mathbf{u} - (\mathbf{e}\cdot\mathbf{u})\,\mathbf{e} = \mathbf{e}\times(\mathbf{u}\times\mathbf{e}).$$

Example 2. Find the vector $\mathbf{u}$ which satisfies the equations:

$$\mathbf{a}\cdot\mathbf{u} = \alpha, \qquad \mathbf{b}\times\mathbf{u} = \mathbf{c}.$$

The scalar α and the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are regarded as known. From the second equation we have

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{u}) = \mathbf{a}\times\mathbf{c} \quad \text{or} \quad (\mathbf{a}\cdot\mathbf{u})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{u} = \mathbf{a}\times\mathbf{c};$$

hence, in view of the first,

$$\mathbf{u} = \frac{\alpha\mathbf{b} - \mathbf{a}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}} \quad \text{provided} \quad \mathbf{a}\cdot\mathbf{b} \neq 0.$$

PROBLEMS

1. Prove the identities:

(*a*) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 0.$
(*b*) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}) = (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})^2.$
(*c*) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})\,(\mathbf{b}\cdot\mathbf{c}).$

2. Show that the lines AB, CD are coplanar and find the point P in which they meet:

$$A\,(-2, -3, 4), \quad B\,(2, 3, 0), \quad C\,(-2, 3, 2), \quad D\,(2, 0, 1).$$

[$\overrightarrow{CD}$ is parallel to $\overrightarrow{CP} = \overrightarrow{AP} - \overrightarrow{AC} = \lambda\,\overrightarrow{AB} - \overrightarrow{AC}$. Find λ by setting up a proportion between the components of these vectors; then $\overrightarrow{OP} = \overrightarrow{OA} + \lambda\,\overrightarrow{AB}$.]

3. Given the vectors $\mathbf{u}$ and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in § 9, compute l, m, n in (§ 9, 1). [To find l multiply the equation by $\cdot(\mathbf{b}\times\mathbf{c})$.]

4. Prove that the equation of a plane through the points P_1, P_2 and parallel to the vector $\mathbf{a}$ is

$$\mathbf{r}\cdot(\mathbf{r}_2 - \mathbf{r}_1)\times\mathbf{a} = \mathbf{r}_1\cdot\mathbf{r}_2\times\mathbf{a}.$$

5. Find the point P where a line through P_1 and parallel to $\mathbf{l}$ pierces a plane through P_2 and normal to $\mathbf{n}$. [The equations of the line and plane are $\mathbf{r} - \mathbf{r}_1 = \lambda\mathbf{l}$, $(\mathbf{r} - \mathbf{r}_2)\cdot\mathbf{n} = 0$. At the point P show that $\lambda = (\mathbf{r}_2 - \mathbf{r}_1)\cdot\mathbf{n}/\mathbf{l}\cdot\mathbf{n}$.]

20. Summary, Chapter I. A vector is a segment of a straight line having a definite length and direction. *Equal vectors* have the same length and direction. The *negative of a vector* is a vector of the same length but opposite direction. Note that

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}.$$

Vectors are added according to the *parallelogram law.* Vector addition is commutative and associative:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}, \qquad (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

Subtracting a vector is the same as adding its negative.

The product of a vector and a positive number p is a vector p times as long and having the same direction. The product of a vector and a negative number $-p$ is a vector p times as long and having the opposite direction. The multiplication of vectors by numbers is commutative, associative and distributive:

$$a\mathbf{u} = \mathbf{u}a, \qquad (ab)\,\mathbf{u} = a\,(b\mathbf{u}),$$
$$(a + b)\,\mathbf{u} = a\mathbf{u} + b\mathbf{u}, \qquad a\,(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, any vector coplanar with them may be expressed as $a\mathbf{u} + b\mathbf{v}$.

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are not coplanar, any vector may be expressed as $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

The *component* of $\mathbf{u}$ on an axis s is the number

$$u_s = |\mathbf{u}| \cos (s, \mathbf{u}).$$

The component of $\mathbf{u} + \mathbf{v}$ on s is $u_s + v_s$.

The point P is said to divide the segment AB in the ratio a/b when

$$\overrightarrow{AP} = \frac{a}{b}\overrightarrow{PB}; \quad \text{and} \quad \overrightarrow{OP} = \frac{b\,\overrightarrow{OA} + a\,\overrightarrow{OB}}{a + b}.$$

The *centroid* of a set of points P_i associated with the numbers m_i is the point P^* defined by

$$\sum m_i\,\overrightarrow{P^*P_i} = 0; \quad \text{and} \quad \overrightarrow{OP^*} = \frac{\sum m_i \overrightarrow{OP_i}}{\sum m_i}.$$

The *scalar product* $\mathbf{u}\cdot\mathbf{v}$ is defined as

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos (\mathbf{u}, \mathbf{v}); \quad \text{and} \quad \mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\mathbf{v}_u = |\mathbf{v}|\mathbf{u}_v.$$

Dot multiplication is commutative and distributive:

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}, \qquad \mathbf{u}\cdot(\mathbf{v} + \mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}.$$

If $\mathbf{u}$ and $\mathbf{v}$ are not zero, $\mathbf{u}\cdot\mathbf{v} = 0$ implies that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, and conversely.

If $\mathbf{e}$ is a unit vector in the positive direction of an axis s, the component of $\mathbf{u}$ on s is $\mathbf{e}\cdot\mathbf{u}$.

The *vector product* $\mathbf{u}\times\mathbf{v}$ is defined as

$$\mathbf{u}\times\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \sin (\mathbf{u}, \mathbf{v})\,\mathbf{e},$$

where **e** is a unit vector normal to the plane of **u** and **v** whose direction is given by the rule of the r-h screw. If A is the area of the parallelogram with **u** and **v** as sides, $\mathbf{u}\times\mathbf{v} = A\mathbf{e}$. Cross multiplication is distributive, but not commutative:

$$\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}, \qquad \mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}.$$

If **u** and **v** are not zero, $\mathbf{u}\times\mathbf{v} = 0$ implies that **u** and **v** are parallel, and conversely.

The *scalar triple product* $\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}$ is numerically equal to the volume of a parallelopiped having **u**, **v**, **w** as edges. If **u**, **v**, **w** are not zero, $\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = 0$ implies that **u**, **v**, **w** are coplanar, and conversely. The value of $\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}$ is not affected by a change in cyclical order of the vectors or by interchanging the dot and cross.

The *vector triple product*

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}.$$

If **u** is referred to rectangular axis x, y, z,

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k};$$
$$u_x = |\mathbf{u}| \cos(x, \mathbf{u}), \text{ etc.}, \qquad |\mathbf{u}|^2 = u_x^2 + u_y^2 + u_z^2.$$

If **u** lies in the xy-plane, $u_z = 0$ and $\cos(y, \mathbf{u}) = \sin(x, \mathbf{u})$;

$$u_x = |\mathbf{u}| \cos(x, \mathbf{u}), \qquad u_y = |\mathbf{u}| \sin(x, \mathbf{u}); \qquad \frac{u_y}{u_x} = \tan(x, \mathbf{u}).$$

In terms of rectangular components we have

$$\mathbf{u} + \mathbf{v} = [u_x + v_x,\ u_y + v_y,\ u_z + v_z],$$
$$a\mathbf{u} = [au_x,\ au_y,\ au_z];$$
$$\mathbf{u}\cdot\mathbf{v} = u_xv_x + u_yv_y + u_zv_z;$$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}; \qquad \mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}.$$

CHAPTER II

STATICS. FUNDAMENTAL PRINCIPLES

21. Force. Everybody is familiar with the notion of a force from the muscular sensations due to pushing, pulling, or supporting heavy bodies. This rough notion may be made quantitative by measuring such efforts on a spring balance. Some definite force must first be chosen as a unit — for example, the tension produced in a spring, held vertically, when supporting some standard body in a particular locality, and the elongation of the spring marked. Other bodies may now be made which produce this same elongation, and the position of the pointer corresponding to two, three, etc. of these bodies marked 2, 3, . . . , thus forming a scale of force magnitude. Either of two bodies, which singly produce equal deflections, and which together produce the unit deflection, will locate the graduation 1/2. Other sub-multiple graduations may be located in a similar manner.* Thus, theoretically at least, an instrument may be constructed to measure forces. If such an instrument seems crude, it should be remembered that all instruments are to some extent inaccurate. Moreover it is not difficult to imagine a standard spring balance, constructed with great care, by which forces over a limited range could be measured with extreme precision.

It should be noted that the conception of a force as a push or pull involves the idea of direction as well as magnitude. Thus a force, when being measured on a spring balance, must be applied along the axis of the spring. In brief, *force is a vector quantity.* A force, however, cannot be represented by a *free* vector; for the effect produced by a force on a body depends upon the point at which it is applied. Forces are therefore represented by vectors acting at specific points, their *points of application.*

* This method of graduating a spring balance does not presuppose a knowledge of the law connecting the elongation of the spring with the force applied. It does assume, however, that within the range of forces measured, the same force will always produce the same deflection.

We may now define force as a vector quantity localized at a point which represents the action of one body on another.* It must be admitted that this definition is of slight service unless the nature of force is already known from experience. This is due to the fact that force, as one of the primitive concepts of mechanics, is hardly capable of being defined.

As forces are vector quantities they will be represented by letters in heavy type, as **F**, **W**, **P**, etc. The corresponding letters in italics will usually be used to denote their magnitude; thus $F = |\mathbf{F}|$. The notation will always show whether a vector or its magnitude is being considered.

22. Local and Standard Weight. The most familiar of all forces is the pull of the earth (force of gravity) on all bodies on or near its surface. The *weight* of material bodies is due to this earth pull.

Definition 1. *The weight of a body in a given locality — its local weight — is the force the body exerts on its supports when at rest relative to the earth in that locality.*

If a body is resting on a table, the downward force it exerts on the table is its weight; if it is hung from a cord, its weight is the downward force it exerts on the cord. Note that weight is a force and consequently a vector quantity. Since the direction of the weight is always known, being along the plumb-line at the locality in question, the weight of a body is usually given as a scalar magnitude. Thus we shall speak of a body of weight W, meaning that the magnitude of the weight is W.

The local weight of a body may be determined by suspending it from a spring-balance graduated in terms of an assumed unit force. Experiment has shown that the weight of a given body varies with the locality, increasing with the latitude and decreasing slightly with the height above sea-level. The greatest variation of local weight in the United States proper is about 0.2 per cent; this, of course, is negligible in most engineering calculations.

If, however, we wish to define the unit of force as the weight of some standard body, it is necessary for the sake of precision to specify the locality at which the weight is measured. For the present we may regard any place at sea-level in latitude 45° N. as a *standard locality.*

* Force is often defined as the agency which changes, or tends to change, the motion of material bodies.

Definition 2. *The standard weight of a body is its weight at a standard locality.*

The British gravitational unit of force is the *pound force;* this is defined as the standard weight of the standard pound body (a platinum body kept at the Standards Office of the Board of Trade in London).

The metric gravitational unit of force is the *kilogram force;* this is defined as the standard weight of the international prototype kilogram (a platinum cylinder kept at the International Bureau of Weights and Measures, Sèvres, France).

23. Particles and Rigid Bodies. If in a certain problem in mechanics the matter composing a certain body may be regarded as concentrated in a single point, the body is called a *particle.* As our subject develops we shall see when it is allowable to regard a body as a particle and when this is inadmissible. Thus the motion of a billiard ball thrown in the air may be determined quite accurately by treating it as a particle; but this is clearly out of the question if we are dealing with its motion on a billiard table, which involves, perhaps, rolling, sliding and spinning.

We shall consider any material body as composed of an aggregate of very small particles. The size and shape of any actual body may be altered by suitably applied forces. But bodies vary considerably in their ability to resist deformation, some yielding readily to small forces while others are not visibly affected by very large ones. In the latter the relative positions of the component particles are only very slightly altered. We may therefore imagine bodies so rigid that they suffer no deformation whatever no matter what forces they are subjected to; these are the ideal *rigid bodies* of mechanics. Of course no actual bodies are *rigid* in this sense; the ideal *rigid body* is a pure abstraction. The justification for regarding actual bodies as *rigid* in certain problems is that their deformation is negligible or unimportant in the problems in question.

In certain branches of mechanics, as the *Theory* of *Elasticity* and *Hydrodynamics,* the deformations of bodies are of the first importance and are carefully taken into account. These branches constitute what is known as the *Mechanics of Deformable Bodies.* In this book, however, we shall be principally concerned with the *Mechanics of a Particle and of Rigid Bodies.* It is important, therefore, to keep in mind the following definition:

A rigid body is an ideal body whose particles always maintain the same relative position, irrespective of the forces applied; its size and shape are therefore invariable.

24. The Fundamental Principles of Statics. We shall base our development of Statics on four fundamental principles. These are collected below for the sake of convenient reference. Each principle will be discussed in detail in a subsequent article.

PRINCIPLE A (VECTOR ADDITION OF FORCES). *A system of forces acting simultaneously on the same particle may be replaced by a single force, acting on this particle, equal to their vector sum.*

PRINCIPLE B (TRANSMISSIBILITY OF A FORCE). *A force acting on a rigid body may be shifted along its line of action so as to act on any particle in this line.*

PRINCIPLE C (STATIC EQUILIBRIUM). *If the forces acting on a particle or rigid body, initially at rest, can be reduced to zero by means of Principles A and B, the particle or body will remain at rest.*

PRINCIPLE D (ACTION AND REACTION). *The interaction between two particles, whether in direct contact or at a distance from each other, may be represented by two forces of equal magnitude and opposite direction acting along their joining line.**

These principles may be justified in part by direct experiment. The most conclusive evidence of their correctness, however, is that the calculations based on them are in agreement with observation, the agreement becoming more and more exact with increased accuracy in the measurements.

We proceed now to a more detailed examination of these principles and the results that may be deduced from them.

25. Principle A: Vector Addition of Forces. *A system of forces acting simultaneously on the same particle may be replaced by a single force, acting on this particle, equal to their vector sum.*†

We have seen that a force may be represented as a vector localized at a point. In view of Principle A, we may combine any

* It is understood that the forces representing the interaction of two particles P_1 and P_2 *act on these particles;* obviously the action of P_1 on P_2 is a force acting on P_2.

† It would have been sufficient to state this principle for *two* forces; in this form the principle is known as *The Parallelogram of Forces* (see § 3). The case of any number of forces follows readily from this. From a purely logical point of view, it is only necessary to state this principle for two special cases: (1) two collinear forces, (2) two perpendicular forces. The general case of any two forces may be built up from these.

set of forces acting on a particle into a single force, their *resultant*, by the polygon construction of § 3. Thus in Fig. 25a, the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$ acting at P may be replaced by the force $\mathbf{R}$ at P. In particular, any set of forces acting on the same particle may be canceled if their vector sum is zero.

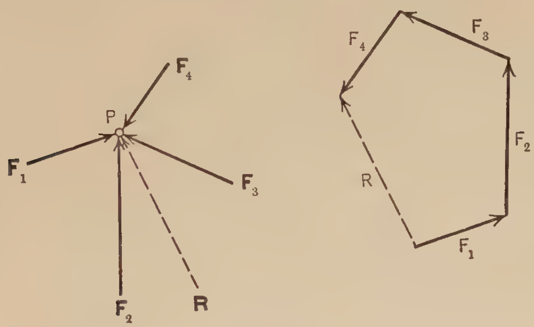

FIG. 25a.

From Principle A we may prove the following useful

THEOREM A. *A system of forces, acting simultaneously on a particle, may be replaced by any other system of forces, acting on this particle, having the same vector sum.*

For both systems may be replaced by one and the same single force, their vector sum; they must therefore have the same dynamic effect on the particle.

If the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ act on the particle, their resultant

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \sum \mathbf{F}_i$$

may be found graphically by the polygon construction, as in Fig 25a, provided that the forces lie in a plane. But when the forces are not coplanar, the polygon formed by the vectors $\mathbf{F}_i$ and $\mathbf{R}$ is not plane and the methods of descriptive geometry must be used to find $\mathbf{R}$ graphically. If we project this skew polygon on any plane a closed plane polygon is obtained. Thus in Fig. 25b the horizontal and vertical projections of three forces, concurrent at a point P, are shown to scale on the left. On the right the usual polygon construction gives the projections $\mathbf{R}_h$, $\mathbf{R}_v$ of the resultant. From these we may find the magnitude R of the resultant; one method, shown in the figure, consists in revolving $\mathbf{R}$ about a vertical line until it is parallel to the vertical plane.

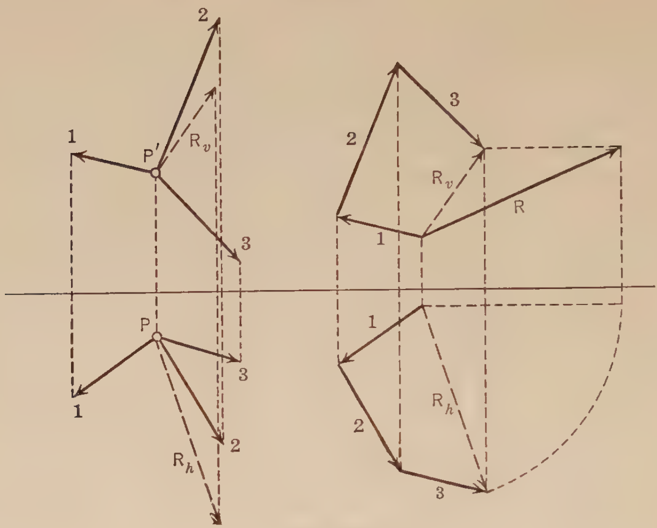

FIG. 25b.

26. Computation of Resultant. If a system of forces $\mathbf{F}_i$ act on a particle, the magnitude and direction of their resultant $\mathbf{R}$ may be computed as follows. Choose a system of rectangular axes and express each force $\mathbf{F}_i$ in terms of its rectangular components. Using the bracket notation of § 11 we write

$$\mathbf{R} = [X, Y, Z], \quad \mathbf{F}_i = [X_i, Y_i, Z_i]$$

Since the component of $\mathbf{R}$ on any axis is equal to the sum of the components of the forces $\mathbf{F}_i$ on this axis (§ 10),

$$X = \sum X_i, \quad Y = \sum Y_i, \quad Z = \sum Z_i.$$

Then from § 11 we have

$$R = \sqrt{X^2 + Y^2 + Z^2}; \quad \cos(x, \mathbf{R}) = \frac{X}{R}, \text{ etc.}$$

Thus both magnitude and direction of $\mathbf{R}$ are determined.

If the forces $\mathbf{F}_i$ are coplanar, their plane may be chosen as the xy-plane; then $Z_i = 0$ and $Z = 0$. If we denote the angle $(x, \mathbf{F}_i)$ by θ_i, we have from § 11,

$$X = \sum X_i = \sum F_i \cos\theta_i, \quad Y = \sum Y_i = \sum F_i \sin\theta_i;$$
$$R = \sqrt{X^2 + Y^2}; \quad \tan(x, \mathbf{R}) = \frac{Y}{X}.$$

The precise quadrant in which the angle $(x, \mathbf{R})$ lies is determined by the signs of X and Y.

The resultant of two concurrent forces may be found from the Law of Cosines. Thus in Fig. 26a,

$$AC^2 = AB^2 + BC^2 - 2\,(AB)\,(BC) \cos \theta, \quad \text{or}$$
$$R^2 = F_1^2 + F_2^2 + 2\,F_1F_2 \cos\,(\mathbf{F}_1, \mathbf{F}_2)$$

since $\theta = 180° - (\mathbf{F}_1, \mathbf{F}_2)$. The angle $\phi = (\mathbf{F}_1, \mathbf{R})$ is now found from the Law of Sines:

$$\frac{\sin \phi}{F_2} = \frac{\sin \theta}{R}.$$

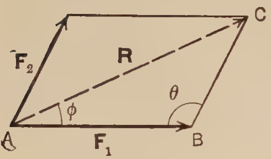

FIG. 26a.

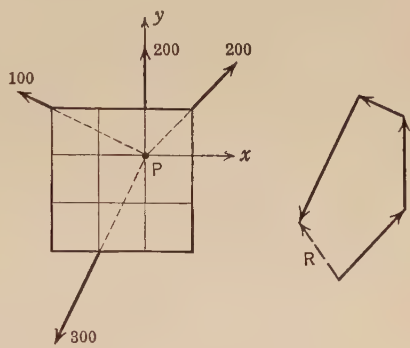

FIG. 26b.

Example. The four coplanar forces acting on the body shown in Fig. 26b are concurrent at P and therefore have a resultant $\mathbf{R} = [X, Y]$. The sines and cosines of the angles $(x, \mathbf{F}_i)$ may be found at once from the figure. Thus we have

$$\begin{aligned} X &= 200\frac{1}{\sqrt{2}} + 0 - 100\frac{2}{\sqrt{5}} - 300\frac{1}{\sqrt{5}} \\ &= 100\sqrt{2} - 100\sqrt{5} = -82.2. \\ Y &= 200\frac{1}{\sqrt{2}} + 200 + 100\frac{1}{\sqrt{5}} - 300\frac{2}{\sqrt{5}} \\ &= 100\sqrt{2} + 200 - 100\sqrt{5} = 117.8. \end{aligned}$$

Hence
$$R = \sqrt{(82.2)^2 + (117.8)^2} = 143.6 \text{ lb.},$$
$$(x, \mathbf{R}) = \tan^{-1}\frac{117.8}{-82.2} = 180° - 55°\,06' = 124°\,54';$$

the angle is in the second quadrant since $X < 0$, $Y > 0$. As a check we compute

$$R = \frac{X}{\cos\,(x, \mathbf{R})} = \frac{82.2}{\cos 55°\,06'} = 143.6 \text{ lb.}$$

The figure also shows the graphical construction for $\mathbf{R}$.

PROBLEMS

1. Two concurrent forces of 500 lb. and 300 lb. make an angle of 60°; find the magnitude and direction of their resultant.

2. The resultant of three concurrent forces is zero; if $F_1 = 80$, $F_2 = 70$, $F_3 = 50$ lb., find the angles between the forces.

3. If A, B, C, D, E, F are the vertices of a regular hexagon, show that the five forces $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$, $\overrightarrow{AF}$ have a resultant equal to $3\,\overrightarrow{AD}$.

4. Find the resultant of three forces equal to $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in Fig. 18.

5. Find graphically and analytically the magnitude and direction of the resultant of the following sets of forces in the xy-plane directed outward from the origin. The first number gives the magnitude of $\mathbf{F}_i$ in pounds; then the direction of $\mathbf{F}_i$ is specified by giving the angle $(x, \mathbf{F}_i)$ measured counterclockwise or by the coördinates of a point on its line of action:

	$\mathbf{F}_1$	$\mathbf{F}_2$	$\mathbf{F}_3$	$\mathbf{F}_4$
(*a*)	11, 180°	20, 90°	15, (−3, −4)	25, (3, −4)
(*b*)	60, 180°	50, (−4, 3)	130, (12, 5)	59, 270°
(*c*)	68, (8, 15)	50, (3, −4)	48, 270°	52, (−12, −5)
(*d*)	50, 30°	70, 135°	100, 240°	

27. Principle B: Transmissibility of a Force. *A force acting on a rigid body may be shifted along its line of action so as to act on any particle in this line.*

It should be noted that this principle applies only to bodies which may be regarded as rigid (§ 23).

From Principles A and B we now deduce a theorem which is very useful in the reduction of forces acting on a rigid body:

THEOREM B. *Two forces of equal magnitude and opposite directions acting along the same line in a rigid body, may be introduced or canceled at pleasure.*

Proof. We may introduce two forces $\mathbf{F}$ and $-\mathbf{F}$ acting on the same particle P of the body since their vector sum is zero (Prin. A): then we may shift $-\mathbf{F}$ so that it acts on any other particle Q of their common line of action (Prin. B). Again, if $\mathbf{F}$ acts at P and $-\mathbf{F}$ at Q, both forces having the same line of action PQ, we may shift $-\mathbf{F}$ to P (Prin. B) and then cancel both forces at P (Prin. A).

28. Equivalent Systems of Forces. Resultants. Two systems of forces S and S' are said to be *equivalent* when either system may be replaced by the other by applying Principles A and B (or the derived Theorems A and B). Note that if we can pass from S to

S' by a certain chain of operations allowed by these principles, we can also pass back from S' to S by exactly retracing our steps.

If the systems S and S' are equivalent, the vector sum of the forces — the *force-sum* — is the same in each system. For the very statement of Principles A and B shows that the force-sum is not altered when they are applied.

If the forces applied to a body are equivalent to a single force, this force is called their *resultant*. If the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ have a resultant $\mathbf{R}$,

$$(1) \qquad \mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \sum \mathbf{F}_i,$$

and the magnitude and direction of $\mathbf{R}$ may be found graphically or analytically as explained in the preceding articles. To completely determine $\mathbf{R}$, however, we must also find its line of action. Both graphical and analytical methods to accomplish this will be given later on. In particular the forces $\mathbf{F}_i$ acting on a *particle* P always have the resultant $\mathbf{R}$ given by (1); in this case $\mathbf{R}$ is completely determined since it necessarily acts upon P.

A system of forces applied to a rigid body may not have a resultant — in fact, it is only by exception that the system can be reduced to a single force. To give a simple example, consider a pair of forces $\mathbf{F}$ and $-\mathbf{F}$, of equal magnitude, opposite directions, and with different (but parallel) lines of action. Such a pair of forces is called a *couple*. If a couple had a resultant $\mathbf{R}$, then from (1), $\mathbf{R} = \mathbf{F} - \mathbf{F} = 0$; that is, a couple acting as a rigid body would be equivalent to no force acting on it. But this can not be true since a couple obviously exerts a turning effect on the body to which it is applied.

29. Resultant of Parallel Forces. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two parallel forces acting on a rigid body at the points P_1 and P_2. If the forces do not form a couple they have a resultant that may be found as follows. At P_1 and P_2 introduce two opposite forces, $\mathbf{Q}$ and $-\mathbf{Q}$, acting along the line P_1P_2 (Theorem B). Combine $\mathbf{F}_1$, $\mathbf{Q}$ and $\mathbf{F}_2$, $-\mathbf{Q}$ to the forces $\mathbf{R}_1$, $\mathbf{R}_2$ respectively (Prin. A), shift $\mathbf{R}_1$ and $\mathbf{R}_2$ to the point O where their lines of action meet (Prin. B) and combine them at O (Prin. A) to the force

$$\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2 = \mathbf{F}_1 + \mathbf{Q} + \mathbf{F}_2 - \mathbf{Q} = \mathbf{F}_1 + \mathbf{F}_2.$$

$\mathbf{R}$ is the resultant of the given forces $\mathbf{F}_1$, $\mathbf{F}_2$. Figures 29*a* and 29*b* show the construction when the forces have the same direction

and when opposite directions. If $\mathbf{F}_1$ and $\mathbf{F}_2$ form a *couple* the above construction fails to yield a resultant; the original forces are then merely replaced by an equivalent couple formed by $\mathbf{R}_1$ and $\mathbf{R}_2$.

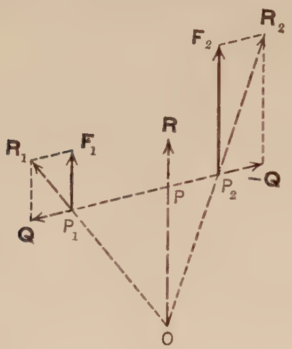

FIG. 29*a*.

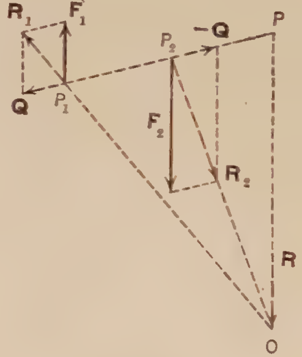

FIG. 29*b*.

Let the line of action of $\mathbf{R}$ cut P_1P_2 in the point P; then from similar triangles (in both figures)

$$\frac{P_1P}{PO} = \frac{Q}{F_1}, \qquad \frac{PP_2}{PO} = \frac{Q}{F_2}$$

where F_1, F_2, Q denote magnitudes of the forces. On dividing the first equation by the second we obtain

$$\frac{P_1P}{PP_2} = \frac{F_2}{F_1};$$

that is, P divides the segment P_1P_2 in the ratio F_2/F_1, the division being internal or external according as $\mathbf{F}_1$ and $\mathbf{F}_2$ have the same or opposite directions. In the latter case $\mathbf{R}$ lies beyond the larger force.

Examples. In Fig. 29*c* the forces of 5 lb. and 3 lb. applied to the body have a resultant of 8 lb. whose line of action divides P_1P_2 internally in the ratio of 3:5.

In Fig. 29*d* the forces of 2 lb. and 5 lb. have a resultant of 3 lb. whose line of action divides P_1P_2 externally in the ratio of 5:2.

A system of three parallel forces can always be reduced to two by the above construction; for two of these forces necessarily have

the same direction and may be replaced by a single force. If the forces of this new system do not form a couple they may in turn be replaced by a single force — the resultant of the given system.

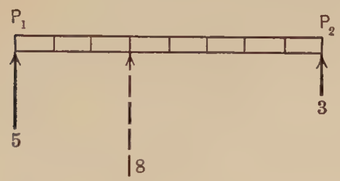

FIG. 29c.

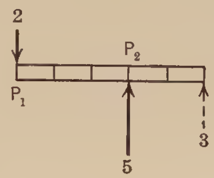

FIG. 29d.

By successive applications of this process we may reduce any system of parallel forces $\mathbf{F}_i$ to two forces, say $\mathbf{R}_1$ and $\mathbf{R}_2$, such that

$$\mathbf{R}_1 + \mathbf{R}_2 = \sum \mathbf{F}_i.$$

If $\sum \mathbf{F}_i \neq 0$, $\mathbf{R}_1$ and $\mathbf{R}_2$ cannot form a couple and may therefore be replaced by a single force $\mathbf{R}$, the resultant of the system.

If, however, $\sum \mathbf{F}_i = 0$, then $\mathbf{R}_2 = -\mathbf{R}_1$ and the forces form a couple provided they have different lines of action. But if $\mathbf{R}_1$ and $\mathbf{R}_2$ act along the same line they may be canceled; the original system of forces is then reduced to zero.

30. Principle C: Static Equilibrium. The next fundamental principle deals with the central problem of statics: If a body, acted on by certain forces, is at rest, under what conditions will it remain at rest?

PRINCIPLE C (STATIC EQUILIBRIUM). *If the forces acting on a particle or a rigid body, initially at rest, can be reduced to zero by means of Principles A and B, the particle or body will remain at rest.*

A set of forces that can be reduced to zero by means of Principles A and B is said to be *in equilibrium.* From § 28 we see that the vector sum of any set of forces in equilibrium must be zero. It does not follow, however that we have equilibrium when this condition is fulfilled; consider, for example, a body acted on by a couple.

Example 1. In Fig. 30*a* the upward forces may be replaced by a force of 8 lb. which is exactly opposed to the downward force (see § 29, Example)

and therefore cancels it. The forces are thus reduced to zero and the body is in equilibrium.

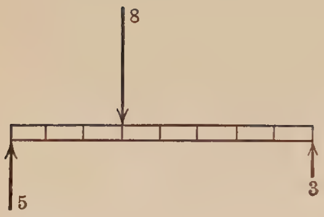

FIG. 30a.

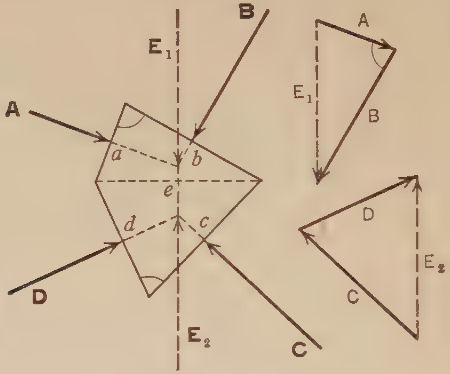

FIG. 30b.

Example 2. In Fig. 30b the four coplanar forces **A**, **B**, **C**, **D** are normal to the sides of the quadrilateral at the middle points and proportional to the lengths of these sides:

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{D}{d}.$$

Replace **A** and **B** by their resultant $\mathbf{E}_1$. If in the force triangle **A**, **B**, $\mathbf{E}_1$ we choose the scale so that $A = a$, $B = b$, it will be the same as the triangle *abe* turned through 90°, and $E_1 = e$. Moreover $\mathbf{E}_1$ is normal to the diagonal e at its middle point; for the perpendicular bisectors of a triangle meet in a point. Similarly replace **C** and **D** by their resultant $\mathbf{E}_2$, drawing the force triangle **C**, **D**, $\mathbf{E}_2$ to the same scale so that $C = c$, $D = d$. We find that $\mathbf{E}_2 = -\mathbf{E}_1$ and that these forces have the same line of action. If we cancel $\mathbf{E}_1$ and $\mathbf{E}_2$ the force system is reduced to zero. The body represented by the quadrilateral is therefore in equilibrium.

31. Principle D: Action and Reaction. Two particles P_1 and P_2 may act on each other by direct contact pressures or at a distance through the agency of gravitation, magnetism, or electrical attraction or repulsion. The force $\mathbf{F}_{12}$ exerted by P_1 on P_2 acts at P_2; the force $\mathbf{F}_{21}$ exerted by P_2 on P_1 acts at P_1. Principle D now states that:

$$\text{(1)} \qquad \mathbf{F}_{12} = -\mathbf{F}_{21} \quad \text{or} \quad \mathbf{F}_{12} + \mathbf{F}_{21} = 0,$$

and that both forces act along the line P_1P_2.

Principle D (Action and Reaction). *The interaction between two particles, whether in direct contact or at a distance from each other, may be represented by two forces of equal magnitude and opposite direction acting along their joining line.*

The forces acting on a given body may be divided into two classes: (1) the *external forces* exerted by bodies whose particles do not form a part of the given body; and (2) the *internal forces* consisting of the interactions between its component particles. According to Principle D the internal forces may all be grouped in pairs of equal magnitude and opposite direction, each pair having a common line of action. In view of Theorem B of § 27 all such pairs of internal forces may be canceled (i.e. disregarded) *provided that the body is rigid.* We therefore have the important

Theorem D. *In considering the equilibrium of a rigid body, only the external forces need be taken into account.*

According to the Principle of Action and Reaction all forces occur in pairs of equal magnitude and opposite direction. Thus if the force exerted by a bat on a ball is **F**, the force exerted by the ball on the bat is $-$**F**. The gravitational pull of the earth on the moon is a force **G** directed from the moon to the earth along the line joining their centers; the pull of the moon on the earth is a force $-$**G** acting along the same line. If a heavy ball resting on a table exerts a downward force of 10 lb. at the point of contact, the table exerts an upward force of 10 lb. on the ball at this point. In brief, *to every action there corresponds an opposite reaction of equal magnitude.*

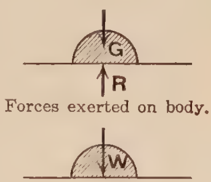

Fig. 32*a*.

32. Contact Forces. Friction. Consider a body at rest on a horizontal plane (Fig. 32*a*). Every particle of the body is attracted to the earth by a certain force and the resultant of this system of (virtually) parallel forces is a single force **G** directed vertically downward. This resultant earth-pull may be called the *gravity* of the body.

As the body is in equilibrium, **G** must be balanced by an upward force **R**, the resultant pressure of the plane on the body; that is, **R** = $-$**G** and acts along the same vertical line. Now by the Principle of Action and Reaction, the resultant pressure of the body on the plane is a force **W** = $-$**R**, exactly opposed to **R**. From § 22, Def. 1, **W** is the *local weight* of the body. As **R** may

be regarded as the "reaction" corresponding to the "action" **W**, **R** is called the *reaction of the plane on the body.*

From the above we have **W** = **G**; the same vector thus serves to represent both the gravity and the weight of a body. In the future, therefore, we shall usually denote the gravity by the letter **W**. Nevertheless the gravity of a body and its weight are distinct forces; the former is exerted by the earth on the body, the latter by the body on its supports.

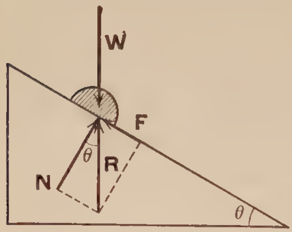

FIG. 32b.

We know from experience that a body may also remain at rest on an inclined plane (Fig. 32b), and that the rougher the plane the more steeply it may be pitched before the body begins to slide. As before, the gravity **W** of the body must be balanced by the reaction **R** = −**W** of the plane in order to maintain equilibrium. The reaction **R** now makes an angle θ with the normal to the plane equal to the inclination of the plane to the horizontal. If we replace **R** by forces **N** and **F**, respectively normal and parallel to the plane (Theorem A), we have

$$N = R\cos\theta, \qquad F = R\sin\theta, \qquad \tan\theta = \frac{F}{N}.$$

F and **N**, the tangential and normal projections of the reaction, are called respectively the *friction* and the *normal pressure* of the plane on the body.

If the inclination of the plane is increased beyond a certain limiting value, the body will begin to slip. The greatest angle ϕ that **R** can make with **N** before the body slips is called the *angle of friction*. The tangent of the angle of friction is called the *coefficient of friction* and denoted by μ; that is

(1) $$\mu = \tan\phi$$

where ϕ is the greatest value of the angle $\theta = (\mathbf{R}, \mathbf{N})$. The body will not slip as long as

(2) $$\theta \leqq \phi \qquad \text{or} \qquad \tan\theta \leqq \mu.$$

Since $\tan\theta = F/N$ this condition may also be written

(3) $$F \leqq \mu N.$$

We state this important result as follows:

The friction exerted by a rough plane on a body at rest upon it can not exceed μ times the normal pressure.

In the case above, the angle of friction ϕ is equal to the greatest inclination that the plane can have before the body slips. This fact furnishes a convenient method for finding μ experimentally and has given rise to the name *angle of repose* for ϕ.

Another simple method for obtaining the value of μ consists in finding the greatest horizontal force P that can be applied to a body of known weight W at rest on a horizontal plane without causing it to slip (Fig. 32*c*). The resultant **R′** of the gravity **W** of the body and the force **P** must be exactly balanced by the reaction **R** of the plane. On replacing **R** by the forces **N** and **F** we have, since $\mathbf{R} = -\mathbf{R}'$,

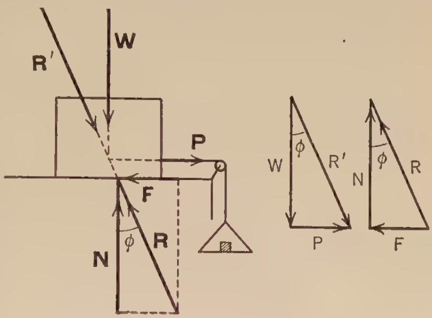

FIG. 32*c*.

$$\mathbf{N} = -\mathbf{W}, \quad \mathbf{F} = -\mathbf{P}, \quad \text{and} \quad \mu = \tan\phi = \frac{F}{N} = \frac{P}{W}.$$

Thus if a body weighing 10 lb. will slip when the horizontal force exceeds 1 lb., $\mu = 0.1$. In performing this experiment the force **P** is applied through the tension of a horizontal string attached to the body. The string passes over a smooth pulley and carries a scale pan to which weights may be added. Then if W_1 is the weight of the pan and W_2 the greatest weight that may be added to it before slipping occurs, we may take $P = W_1 + W_2$ in computing μ as above. The string is assumed to be so light that its weight may be neglected.

Equations (2) and (3) hold for all cases of bodies in contact along *plane* surfaces. If the surfaces in contact are curved the equations apply to elements of surface that may be regarded as practically plane.

Numerous experiments have shown that the coefficient of friction μ depends essentially on the nature of surfaces in contact, but not on their shape or area. Moreover μ is independent of the normal pressure. These "laws of friction," however, are only approximate.

Values of μ are tabulated in various handbooks for engineers. For the contact of dry metals μ ranges from 0.1 to 0.3, for wood on wood from 0.2 to 0.5, depending largely on the grain. For lubricated surfaces the values are much smaller.

If we wish to consider a problem under the assumption that $\mu = 0$ we shall call the surfaces "smooth." This, of course, is an ideal case since $\mu > 0$ for all actual surfaces. As $\phi = 0$ when $\mu = 0$, the reaction of a "smooth" surface is always along its normal.

33. Axial Stress. Consider a thin bar AB in equilibrium under the action of opposed forces applied at its ends A, B (Fig. 33*a*). If one of these forces is **F**, equilibrium requires that the other shall be $-$**F** and that both shall act along the line AB.

If we imagine the bar to consist of two parts, AC and CB, formed by the particles to the left and to the right of any plane section C, these parts are individually in equilibrium. In order

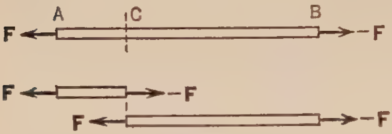

Fig. 33*a*.

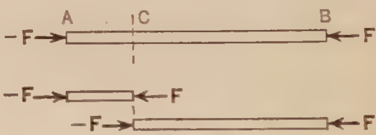

Fig. 33*b*.

that part AC shall be in equilibrium the resultant of the cohesive forces exerted by the particles just to the right of C on those just to the left must equal $-$**F**. Similarly in order that part CB shall be in equilibrium the resultant of the cohesive forces exerted by the particles just to the left of C on those just to the right must equal **F**. This pair of opposite forces of equal magnitude at C is called the *stress* across the section. When, as in Fig. 33*a*, the forces at A and B tend to stretch the bar, the stress is called a *tension*; but when these forces tend to compress the bar, is an Fig. 33*b*, the stress is called a *compression*.

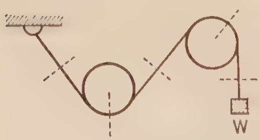

Fig. 33*c*.

A cord or a flexible cable can sustain a tension, but obviously not a compression. We shall prove in § 94 that if a flexible cord is stretched across a *smooth* surface, a smooth peg or pulley for example, the tension is the same across any section. For the present we shall assume this fact. Thus if in Fig. 33*c* the cord supports a weight of W lb., the tension in

the cord across any of the sections shown is W lb., provided that the surfaces are smooth ($\mu = 0$).

A stress along a bar or cable is specified by giving the magnitude of one of its forces. Compression is sometimes distinguished from tension by prefixing a minus sign to its value.

The twin forces of a stress are internal forces and may therefore be neglected in considering the equilibrium of the bar as a whole (Theorem D). But for the equilibrium of a part, as AC, that force of the stress which represents the action of CB on AC must be taken into account; the other force does not then concern us as it acts on CB.

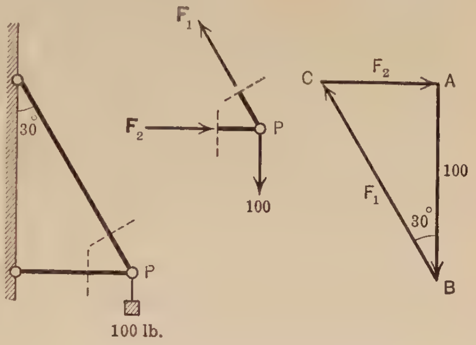

FIG. 33*d*.

Example. In Fig. 33*d* a weight of 100 lb. is hung from a wall bracket. To find the stresses in its bars consider the equilibrium of the section shown. If we shift the forces so that they act at P (Prin. B) they may be replaced by a single force, their vector sum (Prin. A). For equilibrium this sum must be zero. Hence draw a vector $\overrightarrow{AB}$ to represent the 100-lb. weight and from its ends draw lines parallel to the bars of the bracket (the lines of action of their stresses) thus forming a triangle ABC. If we place arrowheads on BC and CA so that

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0 \quad \text{and take} \quad \mathbf{F}_1 = \overrightarrow{BC}, \quad \mathbf{F}_2 = \overrightarrow{CA},$$

equilibrium is assured; hence

$$F_1 = \frac{100}{\cos 30^\circ} = 115.4 \text{ lb.}, \quad F_2 = 100 \tan 30^\circ = 57.7 \text{ lb.}$$

The directions of $\mathbf{F}_1$ and $\mathbf{F}_2$ show that the upper bar is in tension, the lower in compression.

34. Summary, Chapter II. A force may be represented by a vector localized at a point.

The (local) *weight* of a body is the force that it exerts on its supports when at rest. The (local) *gravity* of a body is the resultant pull of the earth upon it; the gravity of a body and its weight are

equal forces. The *standard weight* of a body is its weight at a standard locality — approximately any place at sea-level in latitude 45° N.

A *particle* is a body which, in the problems considered, may be regarded as concentrated in a single point.

A *rigid body* is an ideal body whose shape and size are not altered by the forces applied to it.

The four fundamental principles of statics deal with

A. The Vector Addition of Forces,
B. The Transmissibility of Force,
C. Static Equilibrium,
D. Action and Reaction;

they are collected for reference in § 24. According to Principle B, a force acting on a *rigid* body may be represented by a vector localized in a line, its *line of action.*

A body at rest on a plane (or on a portion of a surface so small that it may be regarded as plane) is acted on by impressed forces, including gravity, that have a resultant $\mathbf{R}'$; then the *reaction* of the plane on the body is the force $\mathbf{R} = -\mathbf{R}'$ that balances $\mathbf{R}'$. When $\mathbf{R}$ is replaced by two forces $\mathbf{N}$ and $\mathbf{F}$, normal and tangent to the plane, $\mathbf{N}$ is called the *normal pressure*, $\mathbf{F}$ the *friction*. If the body begins to slip when the angle $(\mathbf{R}, \mathbf{N})$ exceeds a certain value ϕ, the condition for equilibrium is

$$F \leqq \mu N \quad \text{where} \quad \mu = \tan \phi;$$

ϕ is called the *angle of friction*, μ the *coefficient of friction.* The coefficient μ is practically independent of N and of the size and shape of the area of contact.

In a bar or cable subject to axial forces at its ends, the pair of opposed forces of equal magnitude that represent the mutual actions of the matter on opposite sides of any section is called a *stress* (*tension* or *compression*).

The tension in a flexible cable is not altered in passing over a "smooth" surface.

CHAPTER III

STATICS OF A PARTICLE

35. Equilibrium of a Particle. A body, either rigid or deformable, of sufficiently small dimensions may usually be regarded as a particle situated at a certain point P. In accordance with this view, the lines of action of all external forces acting on the body pass through P; or, to use a briefer phrase, *the forces are concurrent* at P. However, a *rigid* body of arbitrary dimensions may be regarded as a particle whenever the external forces applied to it are concurrent at a point P. For the forces may be shifted along their lines of action until they act on the particle of the body at P (Prin. B) and then replaced by a single force, their vector sum (Prin. A). The body will be in equilibrium only when this resultant vanishes (Prin. C), a condition entirely independent of the size or shape of the body. In view of Theorem D (§ 30) we may therefore state the

THEOREM. *A particle at rest will remain at rest when, and only when, the vector sum of the external forces acting on it is zero.*

The entire theory of the equilibrium of a particle is comprised in this theorem.

If the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ act on the particle, the above condition for equilibrium is expressed by the vector equation

$$(1) \qquad \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = 0 \quad \text{or} \quad \sum \mathbf{F}_i = 0.$$

From the definition of vector addition (§ 3) this condition may be put in the geometric form:

The forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ *acting on a particle will be in equilibrium when, and only when, the vectors* $\mathbf{F}_i$, *drawn successively (beginning to end) in any order whatever, form a closed polygon.*

This polygon — the *force-polygon* — will be a plane figure only when the forces $\mathbf{F}_i$ are coplanar; otherwise it will be skew and the methods of descriptive geometry are required to represent it on paper.

If a skew force polygon is projected on any plane a closed vector polygon is again obtained. Hence if a system of concurrent

forces is in equilibrium, their projections on any plane also represent a system of concurrent forces in equilibrium.

Example 1. What horizontal force P is required to support a body of weight W on a smooth plane of inclination α?

The body will be in equilibrium under three forces: its gravity **W**, the force **P** and the reaction **R** of the smooth plane, normal to its surface. Draw a vector to represent **W** and complete the force triangle by drawing the sides parallel to **P** and **R** (Fig. 35*a*). From this triangle we have

$$P = W \tan \alpha.$$

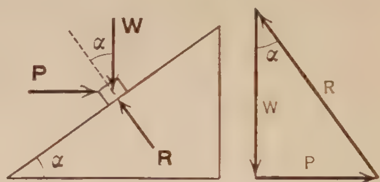

FIG. 35*a*.

Example 2. A body of weight W is hung from the apex of a tripod. Find graphically the stresses in the legs of the tripod.

The tripod $OABC$ is shown in plan and elevation in Fig. 35*b*. If we take a section of the tripod as shown, the upper part will be in equilibrium under the weight **W** and the three upward thrusts along its legs. The weight of the tripod itself is neglected. As **W** projects into a point on the horizontal plane, the horizontal projection of the force-quadrilateral

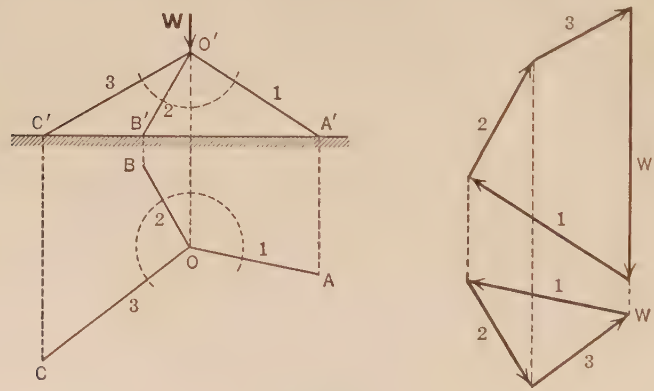

FIG. 35*b*.

is a triangle. This triangle is drawn, to an arbitrary scale, with its sides parallel to the legs OA, OB, OC. We next draw the vertical projections of these forces as shown; the closing side of the force-quadrilateral must represent **W** in magnitude and direction. The sides 1, 2, 3 of the triangle and quadrilateral represent the horizontal and vertical projections of the thrusts along the legs of the tripod to the same scale that **W** is repre-

sented in the vertical projection. The actual magnitudes of these forces may now be found from their projections by one of the methods of descriptive geometry for obtaining the true length of a line.

Example 3. Find graphically the stresses in the legs of a tripod $OABC$ when a given force $\mathbf{P}$ is applied to its apex O (Fig. 35*c*).

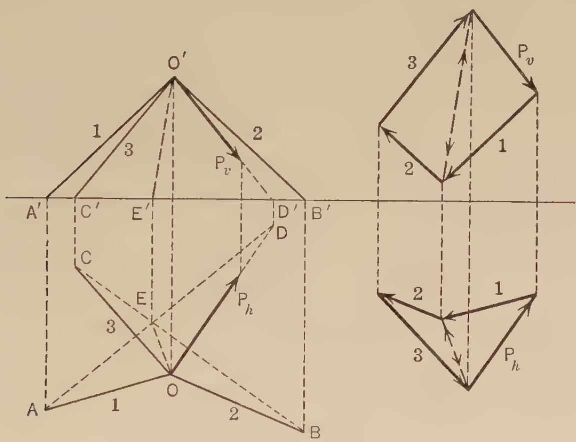

FIG. 35*c*.

Let $\mathbf{R}$ denote the resultant of $\mathbf{F}_2$ and $\mathbf{F}_3$, the stresses in legs *2* and *3*; then

$$\mathbf{F}_2 + \mathbf{F}_3 - \mathbf{R} = 0, \quad \mathbf{P} + \mathbf{F}_1 + \mathbf{R} = 0.$$

Thus $\mathbf{R}$ lies in the planes of $\mathbf{F}_2$ and $\mathbf{F}_3$ and also in the plane of $\mathbf{P}$ and $\mathbf{F}_1$; hence $\mathbf{R}$ acts along the line of intersection of these planes. One point of this line is O; another is the point E where the horizontal traces of the planes meet. Therefore $\mathbf{R}$ is parallel to OE. Both projections of the force-triangle $\mathbf{P}$, $\mathbf{F}_1$, $\mathbf{R}$ can now be drawn and then the projections of the triangle $-\mathbf{R}$, $\mathbf{F}_2$, $\mathbf{F}_3$ joined on. The values of the stresses are found from their projections. Note that leg *1* is in tension.

36. Free-Body Diagram. Before solving a problem in Statics it is necessary to know just what forces are acting on each body involved. The best way to make this perfectly clear is to draw each body *free* from its supports and then represent the action of the latter on the body, as well as the other forces, known and unknown, by vectors drawn in the proper directions. This constitutes the *free-body diagram* of the body.

When a surface supporting a body is removed, its "reaction" $\mathbf{R}$ on the body must be shown by a vector. If the surface is smooth,

R will be normal to the surface. But if friction is considered and the body is on the verge of slipping, **R** will make an angle $\phi = \tan^{-1} \mu$ with the normal in the direction to oppose the motion.

If a cord supporting a body is cut, its action on the body must be shown by a vector along the cord directed away from the body.

A portion of a structure is often "cut off" and regarded as a *free body*. A bar of a framed structure may be cut in this process *provided it is only subject to axial stress.* This is always the case when all the external forces acting on the bar are applied at two points, A and B. For let the forces at these points have the resultants $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively; then since the bar is in equilibrium, $\mathbf{R}_1 + \mathbf{R}_2 = 0$. These forces must also act along the same line AB, the axis of the rod; otherwise they would form a couple. The stress across any section of the rod is therefore axial (§ 33). When such a rod is cut, the forces across the section pull on its parts when the stress is a tension, push on its parts when the stress is a compression (Fig. 33a, b).

In most numerical problems it is simpler to first solve for the unknown quantities in general terms, and then substitute the given numerical values in the results. We shall usually follow this plan.

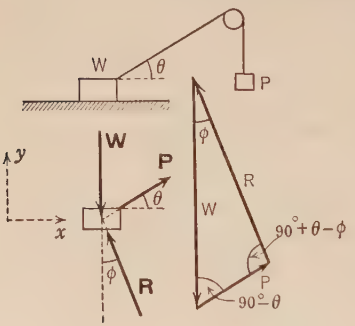

FIG. 36.

Example. In Fig. 36 what is the greatest force P that can be applied to the string before the body of weight $W = 100$ lb. slips along the plane? The coefficient of friction is $\mu = 0.25$ and $\theta = 30°$.

The figure shows the free-body diagram and force-triangle. Two of the angles of this triangle are clearly ϕ, the angle of friction, and $90° - \theta$; the third is therefore $90° + \theta - \phi$. From the Law of Sines

$$\frac{P}{W} = \frac{\sin \phi}{\sin (90° + \theta - \phi)} = \frac{\sin \phi}{\cos (\theta - \phi)}.$$

Since $\tan \phi = \mu = 0.25, \quad \phi = 14° 2', \quad$ and

$$P = 100 \frac{\sin 14° 2'}{\cos 15° 58'} = 25.2 \text{ lb.}$$

37. Scalar Conditions of Equilibrium. From the vector condition of equilibrium, $\sum \mathbf{F}_i = 0$, for concurrent forces we may

obtain three equivalent scalar equations by taking components along any three axes not parallel to the same plane. Thus if we adopt a system of rectangular axes and write $\mathbf{F}_i = [X_i, Y_i, Z_i]$ we obtain

$$\sum X_i = 0, \qquad \sum Y_i = 0, \qquad \sum Z_i = 0, \tag{1}$$

as necessary and sufficient conditions for equilibrium.

When the forces $\mathbf{F}_i$ are coplanar, their plane may be chosen as the xy-plane; the third equation of (1) then reduces to $0 = 0$ and may be omitted.

Example. If in the Example of § 36 we take components along the x- and y-axes shown, we get

$$\begin{aligned} P\cos\theta - R\sin\phi \qquad\quad &= 0, \\ P\sin\theta + R\cos\phi - W &= 0. \end{aligned}$$

On solving these equations for P and R we find

$$P = \frac{W\sin\phi}{\cos(\theta - \phi)}, \qquad R = \frac{W\cos\theta}{\cos(\theta - \phi)}.$$

38. Equilibrium of Concurrent, Coplanar Forces. If the forces $\mathbf{F}_i$ are coplanar and we denote the angle $(x, \mathbf{F}_i)$ by θ_i, the two scalar conditions of equilibrium may be written

$$\sum X_i = \sum F_i \cos\theta_i = 0, \quad \sum Y_i = \sum F_i \sin\theta_i = 0.$$

From these two equations we may determine *two* unknown quantities; these may be

1. the magnitude and direction of one force: F_1, θ_1;
2. the magnitudes of two forces: F_1, F_2;
3. the directions of two forces: θ_1, θ_2;
4. the magnitude of one force and the direction of another: F_1, θ_2.

We shall give in turn the graphical solution in these four cases.

Case 1: To find F_1, θ_1. Construct the resultant $\mathbf{R} = \mathbf{F}_2 + \mathbf{F}_3 + \cdots$ of the known forces. Then $\mathbf{F}_1 = -\mathbf{R}$ and is completely determined.

In the remaining cases we first construct the resultant $\mathbf{R} = \mathbf{F}_3 + \mathbf{F}_4 + \cdots$ of the known forces. The unknown elements of $\mathbf{F}_1$ and $\mathbf{F}_2$ are then found as follows.

Case 2: To find F_1, F_2. From the ends of **R** draw lines parallel to $\mathbf{F}_1$ and $\mathbf{F}_2$; their intersection determines the force triangle **R**, $\mathbf{F}_1$, $\mathbf{F}_2$. This construction is applied in the Example of § 36.

Case 3: To find θ_1, θ_2. From the ends of **R** as centers describe circles of radii F_1, F_2. If they intersect they determine *two* force-triangles (Fig. 38*a*); one of these, however, or possibly both, may prove to be inadmissible. If the circles are tangent there is but one solution; the force-triangle then degenerates into a segment of a straight line described twice. If the circles fail to intersect equilibrium is impossible.

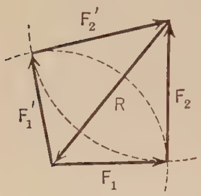

FIG. 38*a*.

Case 4: To find F_1, θ_2. From one end of **R** as center draw a circle of radius F_2, from the other end draw a line parallel to $\mathbf{F}_1$. If the line cuts the circle there are two solutions (Fig. 38*b*); one of these, or both, may prove to be inadmissible. If the line is tangent to the circle there is one solution. If the line fails to meet the circle equilibrium is impossible.

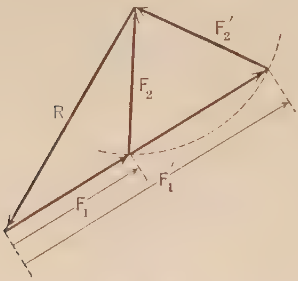

FIG. 38*b*.

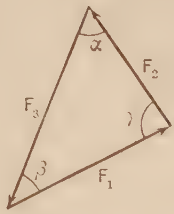

FIG. 38*c*.

Three concurrent forces in equilibrium are necessarily coplanar; for the force-polygon is a triangle and therefore plane. As to their magnitudes, we have on applying the Law of Sines to the force-triangle (Fig. 38*c*),

$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma}.$$

Since $\sin \alpha = \sin (\mathbf{F}_2, \mathbf{F}_3)$, etc., we have proved

LAMY'S THEOREM. *If three concurrent forces are in equilibrium, the magnitude of each force is proportional to the sine of the angle between the other two.*

Example 1. A sphere weighing W lb. rests in the angle between two smooth planes inclined α and β to the horizontal (Fig. 38*d*). Find the reactions of the plane on the sphere.

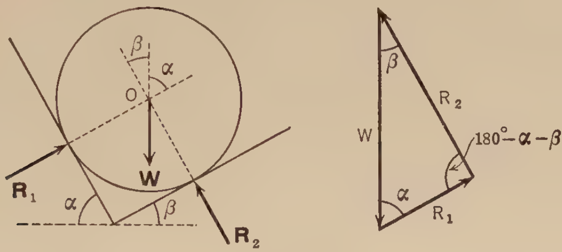

FIG. 38*d*.

The sphere is in equilibrium under three forces: its gravity **W**, which by symmetry acts through the center O, and the reactions $\mathbf{R}_1$, $\mathbf{R}_2$ of the *smooth* planes, whose lines of action also pass through O. As R_1, R_2 are unknown, the problem falls under Case 2. From the Law of Sines applied to force-triangle (or from Lamy's Theorem), we have

$$R_1 = \frac{W \sin \beta}{\sin (\alpha + \beta)}, \quad R_2 = \frac{W \sin \alpha}{\sin (\alpha + \beta)}.$$

If, for example, $W = 100$ lb., $\alpha = 60°$, $\beta = 30°$, we find $R_1 = 50$ lb., $R_2 = 86.6$ lb.

Example 2. A cord which passes over two smooth pegs A, B has the weights $W_1 = 3$ lb., $W_2 = 5$ lb., $W = 7$ lb. attached as shown in Fig. 38*e*. Find the position of equilibrium and the pressures on the pegs.

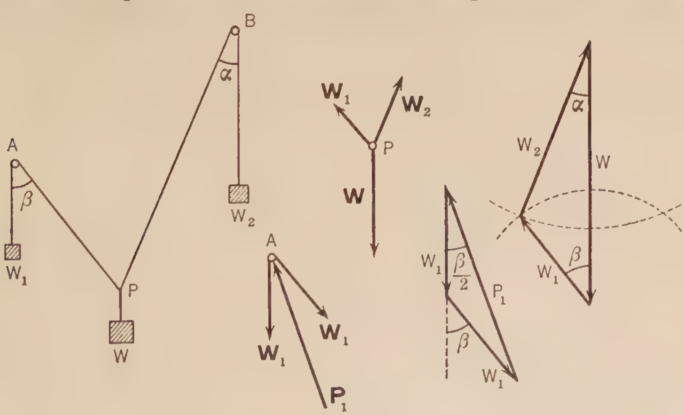

FIG. 38*e*.

The particle P of the cord where W is attached is in equilibrium under three forces of known magnitude. As the directions of two forces are

unknown, the problem falls under Case 3. Although the circle is cut in two points there is only one admissible solution. To find the inclinations of the two parts of the string we must solve the force-triangle for the angles α and β. As its three sides are known, this is a standard problem in trigonometry. The student may verify that $\alpha = 21° 47'$, $\beta = 38° 13'$.

The pressure on the peg A is a force exactly balanced by the pressure $\mathbf{P}_1$ exerted by the peg on the cord. As the peg is smooth the tension of the cord on both sides of A is W_1; and as the directions of those forces are known, the determination of $\mathbf{P}_1$ falls under Case 1. Since the force-triangle is isosceles we have

$$P_1 = 2\,W_1 \cos \tfrac{1}{2}\,\beta = 5.67 \text{ lb.};$$

and the inclination of $\mathbf{P}$ to the vertical is $\frac{1}{2}\,\beta$. Similarly

$$P_2 = 2\,W_2 \cos \tfrac{1}{2}\,\alpha = 9.82 \text{ lb.}$$

Example 3. A weight $W = 15$ lb. is supported on a smooth plane, inclined $\alpha = 30°$ to the horizontal, by a string which passes over a smooth pulley above the plane and carries a weight of 10 lb. hanging vertically (Fig. 38*f*). What angle does the string make with the plane?

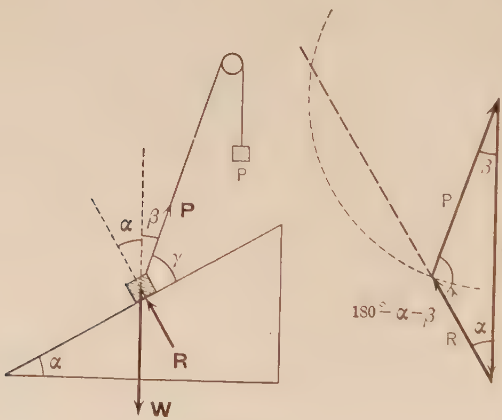

Fig. 38*f*.

As the magnitude of the reaction and the direction of the string are unknown, the problem falls under Case 4. Only one of the force-triangles gives an admissible solution. From it we find by the Law of Sines

$$\sin(\alpha + \beta) = \frac{W \sin \alpha}{P} = \frac{15 \times 0.5}{10} = 0.75;$$

hence
$$\alpha + \beta = 48° 35', \quad \gamma = 90° - (\alpha + \beta) = 41° 25'.$$

PROBLEMS

1. A weight of 80 lb. is supported by two ropes 5 ft. long attached at two points on the same level and 6 ft. apart. Find the tensions of the ropes.

2. A weight W is supported by two ropes making angles of α and β with the horizontal. Find the tensions of the ropes.

Compute the tensions when $W = 100$ lb., $\alpha = 30°$, $\beta = 45°$.

3. A weight of 2000 lb. hangs from the end of a boom inclined at an angle of 30° to the vertical. The boom is supported by a cable at its upper end making an angle β with the horizontal. For what value of β will the cable tension be a minimum? Compute this tension and corresponding stress in the boom.

4. In Problem 1 compute the tensions in the ropes when a horizontal force of 24 lb. is applied to the weight.

5. What is the least horizontal force P that will support a body weighing 100 lb. on a rough plane ($\mu = \frac{1}{4}$) inclined at an angle of 30° to the horizontal?

What is the greatest horizontal force P' that can be applied to the body without causing it to slip upward?

6. Solve Problem 5 for forces P and P' parallel to the plane.

7. A box weighing 100 lb. is supported on a rough plane ($\mu = 0.3$), inclined at an angle of $\alpha = 30°$ the horizontal, by a force making an angle of $\theta = 20°$ with the plane. Find the limits P, P' between which this force can vary.

8. If $\phi = \tan^{-1} \mu$ is the angle of friction in Problem 7 show that the limiting values of the supporting force are

$$P = \frac{\sin(\alpha - \phi)}{\cos(\theta + \phi)} W, \qquad P' = \frac{\sin(\alpha + \phi)}{\cos(\theta - \phi)} W.$$

Obtain general solutions of Problems 5 and 6 from these results.

39. Equilibrium of Concurrent Forces in Space. For a system of concurrent forces in space we have three scalar conditions of equilibrium (§ 37, 1). From these we may, in general, solve for three unknown quantities — magnitudes of forces or their direction angles. The most important cases arise when the unknown quantities are

1. the magnitude of one force $\mathbf{F}_1$ and its direction angles;
2. the magnitudes of three forces, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ whose (known) lines of action are not coplanar.

In these cases there is just one solution for the unknowns.

In Case 1 we have

$$X_1 = -X_2 - X_3 - \cdots, \quad Y_1 = -Y_2 - Y_3 - \cdots,$$
$$Z_1 = -Z_2 - Z_3 - \cdots;$$

then F_1 and the direction angles may be computed as in § 11.

In Case 2 let **R** denote the resultant of the known forces; then

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = -\mathbf{R}.$$

Hence if we express $-\mathbf{R}$ as the sum of three forces parallel to the known lines of action of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ (§ 8) the magnitudes of these forces will give the desired solution. The solution is therefore unique.*

As already noted in § 35, the projections of the forces in equilibrium on any line or plane may also be regarded as a set of concurrent forces in equilibrium. The use of this fact, as shown in the examples following, usually facilitates the solution of problems.

Example 1. Find the stresses in the three rods of the wall bracket shown in Fig. 39*a*. The suspended weight $W = 100$ lb. and $\alpha = \beta = 30°$.

The pin at the junction of the rods is taken as the free body. $\mathbf{F}_3$ must be a tension in order to furnish a vertical component to balance **W**; and $\mathbf{F}_1$ and $\mathbf{F}_2$, which by symmetry are numerically equal, must be compressions in order to balance the horizontal component of $\mathbf{F}_3$. If we project the forces on the plane of **W** and $\mathbf{F}_3$, the projections of $\mathbf{F}_1$ and $\mathbf{F}_2$ are both of magnitude $F_1 \sin \alpha$. From the force-triangle in this plane we find

$$F_3 \cos \beta = W, \quad 2\,F_1 \sin \alpha = W \tan \beta;$$

hence

$$F_1 = F_2 = \frac{W \tan \beta}{2 \sin \alpha}, \quad F_3 = \frac{W}{\cos \beta}.$$

Substituting the values given above we find

$$F_1 = F_2 = \frac{50 \tan 30°}{\sin 30°} = \frac{50}{\cos 30°} = 57.7 \text{ lb.},$$
$$F_3 = \frac{100}{\cos 30°} = 115.5 \text{ lb.}$$

* The vectorial solution is as follows. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be unit vectors along the lines of action of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$; then

$$F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 = -\mathbf{R}.$$

To find F_1 take the scalar product of both members by $\mathbf{e}_2\times\mathbf{e}_3$; thus

$$F_1\mathbf{e}_1\cdot\mathbf{e}_2\times\mathbf{e}_3 = -\mathbf{R}\cdot\mathbf{e}_2\times\mathbf{e}_3, \qquad F_1 = -\frac{\mathbf{R}\cdot\mathbf{e}_2\times\mathbf{e}_3}{\mathbf{e}_1\cdot\mathbf{e}_2\times\mathbf{e}_3}.$$

If F_1 turns out to be negative its direction is given by $-\mathbf{e}_1$. F_2 and F_3 are obtained in similar fashion.

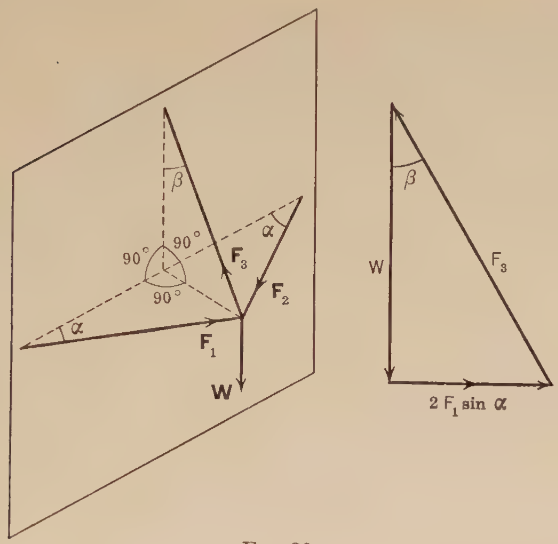

Fig. 39a.

Example 2. Find the stresses in the mast AB and supporting cables, BC, BD, when the force **P** acts in the vertical plane midway between the cables (Fig. 39*b*).

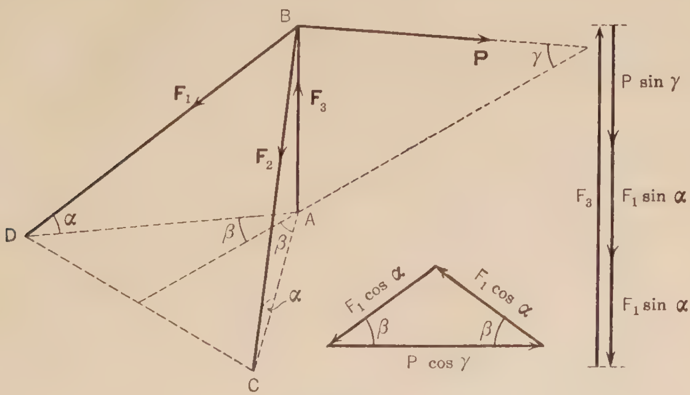

Fig. 39*b*.

Consider the point B as a free body. By symmetry $F_1 = F_2$; these stresses must be tensions in order to balance the horizontal component of **P**. Hence $\mathbf{F}_3$ must be directed upward in order to balance the downward components of $\mathbf{F}_1$, $\mathbf{F}_2$ and **P**. Project the forces on the horizontal

plane; the projections of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{P}$ have the magnitudes $F_1 \cos \alpha$, $F_1 \cos \alpha$, 0, $P \cos \gamma$. From the force-triangle in this plane we find

$$2 F_1 \cos \alpha \cos \beta = P \cos \gamma, \quad F_1 = F_2 = \frac{P \cos \gamma}{2 \cos \alpha \cos \beta}$$

Now, take components vertically; then

$$F_3 = P \sin \gamma + 2 F_1 \sin \alpha = P\left(\sin \gamma + \frac{\cos \gamma \tan \alpha}{\cos \beta}\right).$$

If $$P = 1000 \text{ lb.}, \quad \alpha = \beta = 45°, \quad \gamma = 30°,$$

$$F_1 = F_2 = \frac{1000 \cos 30°}{2 \cos^2 45°} = 500 \sqrt{3} = 866 \text{ lb.},$$

$$F_3 = 1000\left(\tfrac{1}{2} + \frac{\cos 30°}{\cos 45°}\right) = 500\,(1 + \sqrt{6}) = 1725 \text{ lb.}$$

Example 3. A particle of weight $W = 10$ lb. is at rest on a rough plane inclined at an angle $\alpha = 10°$ to the horizontal. If the coefficient of friction is $\mu = 0.3$, what is the greatest horizontal force P, parallel to the plane, that the particle can sustain without slipping?

Replace the reaction of the plane by the normal pressure $\mathbf{N}$ and the friction $\mathbf{F}$ (Fig. 39*c*). $\mathbf{F}$ will be directed up the plane and away from $\mathbf{P}$ so as to oppose both downward and horizontal slipping; denote the

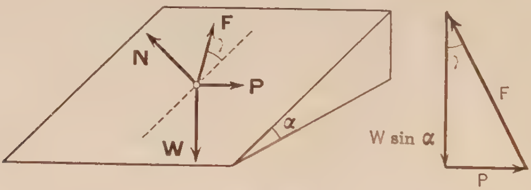

Fig. 39*c*.

angle between $\mathbf{F}$ and a line of greatest slope on the plane by γ. Project the forces $\mathbf{W}$, $\mathbf{P}$, $\mathbf{F}$, $\mathbf{N}$, on the plane; the magnitudes of their projections are $W \sin \alpha$, P, F, 0 respectively. From the force-triangle of these projections we find

$$F^2 = P^2 + W^2 \sin^2 \alpha, \quad \tan \gamma = \frac{P}{W \sin \alpha}$$

On taking components normal to the plane we have also

$$N = W \cos \alpha.$$

When F has its limiting value μN,

$$F^2 = \mu^2 N^2 \quad \text{or} \quad P^2 + W^2 \sin^2 \alpha = \mu^2 W^2 \cos^2 \alpha,$$

whence $$P = W \cos \alpha \sqrt{\mu^2 - \tan^2 \alpha}.$$

Since the particle was originally at rest on the plane, $\tan \alpha \leqq \mu$ (§ 32); the radical is therefore real.

From the data of our problem

$$P = 10 \cos 10° \sqrt{.09 - \tan^2 10°} = 2.39 \text{ lb.}$$

If desired we could also compute N, F and γ from the results above, thus completely determining the reaction of the plane.

PROBLEMS

1. A weight of 100 lb. is hung from three ropes 6 ft. long which are fastened above to three hooks on the same level, each being 4 ft. from the other two. Find the tension in the ropes.

2. Find the stresses in the legs of the tripod shown in Fig. 39*d* when it is loaded with

(*a*) the horizontal force $P = 1000$ lb.,

(*b*) the vertical force $W = 1000$ lb.

Solve both graphically and analytically.

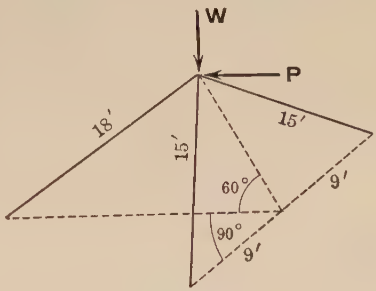

FIG. 39*d*.

3. The tripod $OABC$ rests on a horizontal plane. The apex O projects into the point Q on the plane and the angles

$$OAQ = A, \quad OBQ = B, \quad OCQ = C; \quad BQC = \alpha, \quad CQA = \beta, \quad AQB = \gamma.$$

When the tripod supports a weight W at its apex, prove that the stresses in the legs OA, OB, OC are

$$\frac{W \sin \alpha}{k \cos A}, \qquad \frac{W \sin \beta}{k \cos B}, \qquad \frac{W \sin \gamma}{k \cos C},$$

where
$$k = \sin \alpha \tan A + \sin \beta \tan B + \sin \gamma \tan C.$$

40. Systems of Particles. Problems in the equilibrium of a system of particles are solved by treating in turn each particle as a free body. The Principle of Action and Reaction now comes into play; this asserts that the forces exerted mutually by two particles on each other are equal in magnitude, opposite in direction and act along the same line. These mutual forces are usually the reactions of surfaces in contact or stresses in bars or cables that join the particles of a system.

Example 1. The framework ABC consists of two bars AC, BC connected by a pin at C while the ends A and B are connected by a rope.

If it rests on a smooth plane and sustains a weight W at C, find the stresses in the bars and rope (Fig. 40*a*).

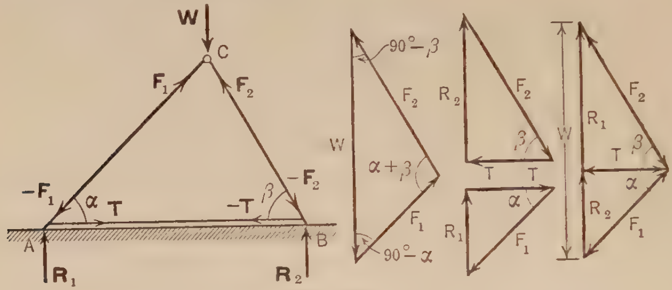

FIG. 40*a*.

Draw first the force-triangle **W**, $\mathbf{F}_1$, $\mathbf{F}_2$ for the forces in equilibrium at C and apply the Law of Sines:

$$\frac{F_1}{\sin(90° - \beta)} = \frac{F_2}{\sin(90° - \alpha)} = \frac{W}{\sin(\alpha + \beta)};$$

hence $$F_1 = \frac{W\cos\beta}{\sin(\alpha+\beta)}, \qquad F_2 = \frac{W\cos\alpha}{\sin(\alpha+\beta)}.$$

Knowing $\mathbf{F}_1$ we may draw the force-triangle $-\mathbf{F}_1$, $\mathbf{R}_1$, **T** for the forces in equilibrium at A; as this is right-angled

$$T = F_1\cos\alpha = \frac{W\cos\alpha\cos\beta}{\sin(\alpha+\beta)}.$$

We might equally well have drawn the force-triangle $-\mathbf{F}_2$, $-\mathbf{T}$, $\mathbf{R}_2$ for the forces in equilibrium at B; from this we obtain the same expression for T.

These three force-triangles may be combined in a single figure in which each pair of mutual forces $\pm\mathbf{F}_1$, $\pm\mathbf{F}_2$, $\pm\mathbf{T}$ is represented by a single line. From this diagram we have

$$W = R_1 + R_2 = T\tan\alpha + T\tan\beta, \quad T = \frac{W}{\tan\alpha + \tan\beta}.$$

This result is easily seen to be in agreement with that above.

If we take $P = 1000$ lb., $\alpha = 45°$, $\beta = 60°$, we find

$$F_1 = 518 \text{ lb.}, \quad F_2 = 732 \text{ lb.}, \quad T = 366 \text{ lb.}$$

Example 2. Two smooth spheres of weight W_1, W_2 rest in contact with each other in the angle between two planes inclined at angles α and β to the horizontal. Find the reactions R_1, R_2 of the planes on the spheres and the mutual pressures $\pm\mathbf{R}$ of the spheres on each other (Fig. 40*b*).

Each sphere is in equilibrium under the action of three concurrent forces:

$$\mathbf{W}_1 + \mathbf{R}_1 + \mathbf{R} = 0, \qquad \mathbf{W}_2 + \mathbf{R}_2 - \mathbf{R} = 0.$$

On adding these equations we have

$$\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{R}_1 + \mathbf{R}_2 = 0;$$

hence the vectors $\mathbf{W}_1 + \mathbf{W}_2$, $\mathbf{R}_1$, $\mathbf{R}_2$ form a closed triangle. From the Law of Sines:

$$R_1 = \frac{(W_1 + W_2) \sin \beta}{\sin (\alpha + \beta)}, \qquad R_2 = \frac{(W_1 + W_2) \sin \alpha}{\sin (\alpha + \beta)}.$$

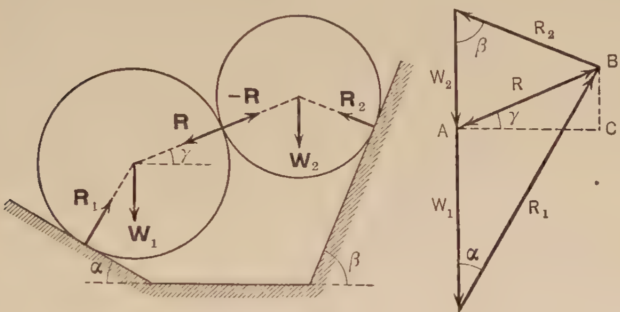

FIG. 40*b*.

We must also have $\mathbf{R} = \overrightarrow{BA}$ in order to satisfy the two original equations of equilibrium. The angle γ, giving the direction of $\mathbf{R}$, is found from

$$\tan \gamma = \frac{BC}{AC} = \frac{W_2 - R_2 \cos \beta}{R_2 \sin \beta} = \frac{W_2 \cot \alpha - W_1 \cot \beta}{W_1 + W_2}.$$

Finally R may be computed from the two expressions for AC:

$$R \cos \gamma = R_2 \sin \beta.$$

PROBLEMS

1. Particles weighing 2 and 3 lb., connected by a cord, rest on the surface of a smooth circular cylinder. If the cord subtends a central angle of 90°, find the equilibrium position of the particles and the tension of the cord.

2. Weights of 10 and 20 lb. hang from the points B, C of a cord $ABCD$ whose ends A, D are fixed. If AB and CD are inclined at angles of 30° and 45° to the horizontal, find the tensions in AB, BC, CD and the inclination of BC to the horizontal.

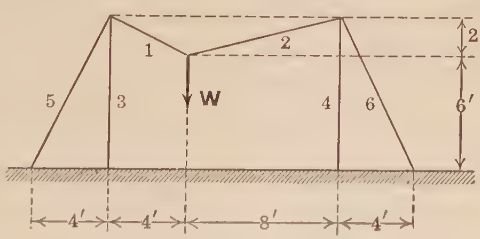

FIG. 40*c*.

3. A weight $W = 100$ lb. hangs from a wire supported by two posts as shown in Fig. 40c. Find both graphically and analytically the stresses in the wires 1, 2, the posts 3, 4 and the guys 5, 6.

4. Find both graphically and analytically the stresses F_1, F_2, F_3, F_4 in the truss shown in Fig. 40d when $P = 5000$ lb.

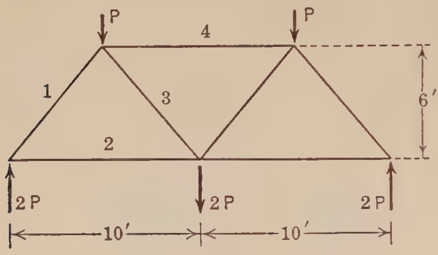

FIG. 40d.

5. Find the stresses F_1, F_2, . . ., F_6 in the framework shown in Fig. 40e when $\alpha = 45°$ and $P = 2000$ lb. [First compute β.]

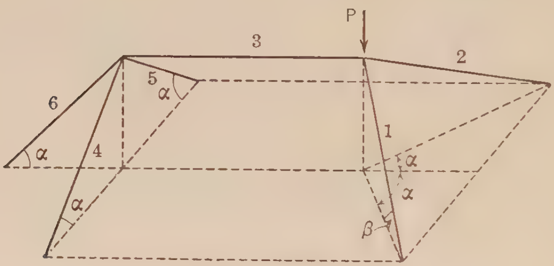

FIG. 40e.

41. Summary, Chapter III. In order that a particle shall be in equilibrium it is necessary and sufficient that the vector sum of the forces acting on it equal zero; in brief, the force-polygon must close.

If a set of concurrent forces in equilibrium is projected on any line or plane, their projections also represent a set of forces in equilibrium.

The vector equation

$$\sum \mathbf{F}_i = 0$$

for the equilibrium of the concurrent forces $\mathbf{F}_i$ is equivalent to the three scalar equations

$$\sum X_i = 0, \qquad \sum Y_i = 0, \qquad \sum Z_i = 0.$$

If the forces all lie in the xy-plane, the third of these equations is omitted.

Before solving a problem in Statics a *free-body diagram* should be drawn for each body involved. In this diagram the body is shown *free* from its supports, and all the forces acting on it, known and unknown, are represented by vectors. In a system of particles the mutual forces exerted by any two on each other are equal in magnitude, opposite in direction and act along their joining line.

CHAPTER IV

PLANE STATICS

42. The Law of the Lever. If AB is a light* uniform bar free to turn about a horizontal axis O, it may be shown by direct experiment that a weight W_1 supported at a distance d_1 from O may be exactly balanced by a weight W_2 at a distance d_2 from O on the other side (Fig. 42a), provided that

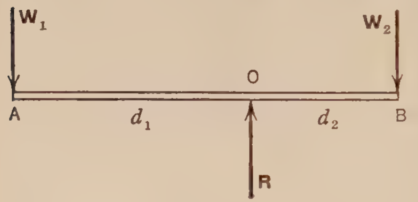

Fig. 42a.

(1) $$W_1 d_1 = W_2 d_2.$$

The equilibrium may be readily explained. From (1) we have

$$\frac{AO}{OB} = \frac{d_1}{d_2} = \frac{W_2}{W_1};$$

that is, O divides AB internally in the ratio $W_2 : W_1$. Hence from § 29 the resultant of the parallel forces $\mathbf{W}_1$ and $\mathbf{W}_2$ is a force $\mathbf{W}_1 + \mathbf{W}_2$ acting downward at O. If then the support O exerts an upward reaction $\mathbf{R} = -(\mathbf{W}_1 + \mathbf{W}_2)$ on the bar, it will be in equilibrium under the forces $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{R}$.

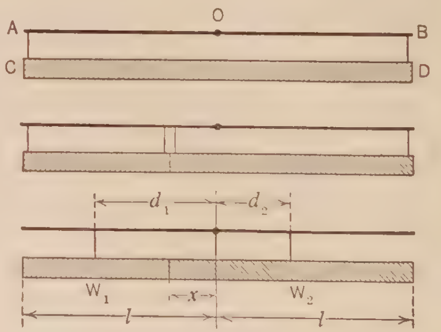

Fig. 42b.

Archimedes (287–212 B.C.) deduced the *law of the lever* expressed in (1) by a very ingenious line of reasoning. The argument, as modified by Galileo, is as follows. Let AB be a light rod free to turn about a central axis O and supporting a uniform beam CD by strings at its ends (Fig. 42b). If CD is cut in two parts of weight W_1, W_2 and each part is hung from AB by

* We speak of a body as "light" when its weight is neglected in the problem under consideration.

strings at its ends, the equilibrium will not be disturbed. Let us now replace the strings on each part by a single string at its center; it is plausible that this change also will not disturb the equilibrium of AB. From the figure we see that

$$d_1 = \tfrac{1}{2}(l + x), \quad d_2 = \tfrac{1}{2}(l - x), \quad \frac{W_1}{W_2} = \frac{l - x}{l + x} = \frac{d_2}{d_1};$$

this gives equation (1) above.

We may regard the products $W_1 d_1$ and $W_2 d_2$ as measures of the turning effects exerted by the weights $\mathbf{W}_1$ and $\mathbf{W}_2$ about the axis; for equilibrium they must be equal in magnitude but opposite in sense. These products are called the *moments* of the forces $\mathbf{W}_1$ and $\mathbf{W}_2$ about the axis. In the present case the forces are both perpendicular to the axis.

In any case a force $\mathbf{F}$ which acts on a body capable of turning about an axis s may always be replaced by two forces $\mathbf{F}'$ and $\mathbf{F}''$, respectively perpendicular and parallel to the axis. It is obvious that only $\mathbf{F}'$ contributes to the turning effect or *moment* of $\mathbf{F}$ about s, while the tendency of $\mathbf{F}''$ is to slide the body along s. The moment of $\mathbf{F}$ about s is then measured by the product of the magnitude of $\mathbf{F}'$ into its perpendicular distance from the axis.

43. Moment of a Force about an Axis. In order to specify the sense of a rotation about a line s, we choose a positive direction on the line, that is, we regard s as an *axis* (§ 10). Then the positive sense of rotation about the axis s is such that a right-handed screw (in a fixed nut) revolved in this sense will advance in the positive direction along s (Fig. 43a). A positive rotation, observed from the positive end of the axis, will appear as counterclockwise.

FIG. 43a.

We are now in position to give a general definition of the moment of a force about an axis that takes account of the sense as well as the magnitude of the turning effect.

DEFINITION. *If* $\mathbf{F}$ *is a force and* $\mathbf{F}'$ *its projection on a plane normal to the axis s, the moment of* $\mathbf{F}$ *about s is the product $F'p$ of the magnitude of* $\mathbf{F}'$ *and its perpendicular distance p from s, taken with a positive or negative sign in agreement with the sense of rotation about s indicated by* $\mathbf{F}'$.*

* This is also the definition of the moment of *any vector* $\mathbf{F}$, localized in a line, about an axis s.

If in Fig. 43b, $F' = 3$ lb., $p = 2$ ft., the moment of **F** about s is 6 lb.-ft.; the moment is positive because the rotation indicated by **F**′ is counterclockwise when viewed from the $+s$ side of the plane. If the direction of the axis s or of the force **F** is reversed the moment changes sign.

The moment of **F** about s is not changed by shifting the force along its line of action; for such a shift does not change F', p, or the sense of rotation indicated by **F**′.

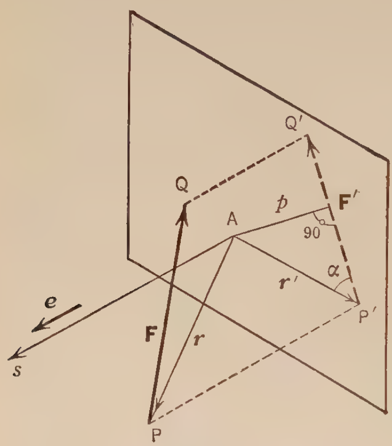

Fig. 43b.

If $\mathbf{F} \neq 0$, the moment of **F** about s will vanish only when $F' = 0$ or $p = 0$. In the former case **F** is parallel to s, in the latter its line of action passes through s. In both cases **F** and s lie in the same plane; therefore:

The moment of a force about an axis vanishes when, and only when, the force and axis are coplanar.

44. Computation of Moments. In Fig. 43b **F**′ and $\mathbf{r}' = \overrightarrow{AP'}$ are the projections of **F** and $\mathbf{r} = \overrightarrow{AP}$ on a plane normal to the axis at A. Denote the moment of **F** about s by M_s. In the figure **F**′ tends to produce a positive rotation about s; hence, from the definition of § 43,

$$M_s = F'p = F'r' \sin \alpha;$$

and if **e** is a unit vector in the direction of s

$$M_s\mathbf{e} = r'F' \sin(\mathbf{r}', \mathbf{F}')\,\mathbf{e} = \mathbf{r}' \times \mathbf{F}' \qquad (\S\ 16, 1).$$

If we reverse the direction of s, *both* M_s and **e** change sign, so that $M_s\mathbf{e}$ still equals $\mathbf{r}'\times\mathbf{F}'$. Hence in both cases

$$M_s = \mathbf{e}\cdot\mathbf{r}'\times\mathbf{F}'.$$

If we put

$$\mathbf{r}' = \mathbf{r} + \overrightarrow{PP'}, \qquad \mathbf{F}' = \overrightarrow{P'P} + \mathbf{F} + \overrightarrow{QQ'}$$

in this result and expand, five terms are triple products containing two of the parallel vectors **e**, $\overrightarrow{PP'}$, $\overrightarrow{QQ'}$ and therefore vanish (§ 18). The remaining term gives

$$(1) \qquad M_s = \mathbf{e}\cdot\mathbf{r}\times\mathbf{F}.$$

Clearly A may be chosen at pleasure on the axis; and since **F** may be shifted along its line of action without changing M_s, P may be chosen at pleasure on this line. In brief, *any vector* AP *from a point on the axis to the line of action may be taken as* **r** *in* (1).*

In view of (§ 14, 3) the important result (1) may be stated as follows:

THEOREM. *The moment of a force* **F** *about an axis is equal to the component of the vector* $\mathbf{r}\times\mathbf{F}$ *on this axis, where* **r** *is any vector from a point on the axis to a point on the force's line of action.*

Example. The line of action of the force $\mathbf{F} = [1, -1, 2]$ passes through the point $P(2, 4, -1)$. Find the moment of **F** about an axis s through the point $A(3, -1, 2)$ and having the direction of the vector $[2, -1, 2]$.

Computation.

$$\mathbf{r} = \overrightarrow{AP} = [2 - 3, 4 + 1, -1 -2] = [-1, 5, -3],$$

$$\mathbf{r}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 5 & -3 \\ 1 & -1 & 2 \end{vmatrix} = [7, -1, -4]; \quad \mathbf{e} = \frac{[2, -1, 2]}{\sqrt{4 + 1 + 4}} = \tfrac{1}{3}[2, -1, 2];$$

$$M_s = \mathbf{e}\cdot\mathbf{r}\times\mathbf{F} = \tfrac{1}{3}(14 + 1 - 8) = 2\tfrac{1}{3}.$$

Check. The points $A_1(5, -2, 4)$ and $P_1(3, 3, 1)$ lie on s and the line of action respectively. If we take

$$\mathbf{r} = \overrightarrow{A_1P_1} = [-2, 5, -3], \quad \text{then} \quad \mathbf{r}\,\mathbf{F} = [7, 1, -3]$$

and $\mathbf{e}\cdot\mathbf{r}\times\mathbf{F}$ has the value above.

45. Moments about the Coördinate Axes. Let the force $\mathbf{F} = [X, Y, Z]$ act through the point $P(x, y, z)$. Then the moments of **F** about the coördinate axes are the components of the vector

$$\overrightarrow{OP}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ X & Y & Z \end{vmatrix}$$

* This may be shown directly. For if $\mathbf{r}_1 = \overrightarrow{A_1P_1}$ is another such vector and we put

$$\mathbf{r} = \overrightarrow{AP} = \overrightarrow{AA_1} + \overrightarrow{A_1P_1} + \overrightarrow{P_1P}$$

in (1) and expand, we find $M_s = \mathbf{e}\cdot\mathbf{r}_1\times\mathbf{F}$ since $AA_1 \| \mathbf{e}$ and $P_1P \| \mathbf{F}$.

on these axes. Hence

$$M_x = \begin{vmatrix} y & z \\ Y & Z \end{vmatrix}, \quad M_y = \begin{vmatrix} z & x \\ Z & X \end{vmatrix}, \quad M_z = \begin{vmatrix} x & y \\ X & Y \end{vmatrix}.$$

Example. If $\mathbf{F} = [2, 2, -3]$ acts through the point $P(1, -2, 1)$, the moments of $\mathbf{F}$ about the coördinate axes are the components of the vector

$$[1, \ -2, \ 1]\times[2, \ 2, \ -3] = [4, \ 5, \ 6].$$

Thus $M_x = 4$, $M_y = 5$, $M_z = 6$.

PROBLEMS

1. In § 17, Problem 6*a*, compute

(*a*) the moment of $\overrightarrow{AB}$ about the axis CD,

(*b*) the moment of $\overrightarrow{CD}$ about the axis AB,

(*c*) the moments of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ about the coördinate axes.

2. In § 19, Problem 2, compute the moment of $\overrightarrow{AB}$ about the axis CD. Interpret your result.

3. Compute the moment of $[3, -4, -2]$, localized at $P(1, -1, 3)$, about the axis through $A(2, 1, 0)$ in the direction of $+z$.

46. Theorem of Moments. *The sum of the moments about any axis of a set of forces concurrent at the point P is equal to the moment of their vector sum, acting through P, about this axis.*

Proof. Let A be a point on the axis s and $\mathbf{r} = \overrightarrow{AP}$. Then if the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots$, concurrent at P, have the vector sum $\mathbf{R}$,

(1) $$\mathbf{r}\times\mathbf{F}_1 + \mathbf{r}\times\mathbf{F}_2 + \cdots = \mathbf{r}\times(\mathbf{F}_1 + \mathbf{F}_2 + \cdots) = \mathbf{r}\times\mathbf{R}$$

by (§ 17, 2). Let $\mathbf{e}$ be a unit vector in the direction of s. Then since

$$\mathbf{e}\cdot\mathbf{r}\times\mathbf{F}_1 + \mathbf{e}\cdot\mathbf{r}\times\mathbf{F}_2 + \cdots = \mathbf{e}\cdot\mathbf{r}\times\mathbf{R},$$

our theorem is proved.

In any system of forces the sum of the moments about an axis s is called the *moment-sum about s*. Consider now two *equivalent* systems of forces, S and S'. We have seen in § 28 that the *force-sum* is the same in each system. The moment-sum about any given axis is also the same in each system. For in reducing S to S', the application of Principle A does not affect the moment-

sum by virtue of the Theorem of Moments; and the application of Principle B, the shifting of forces along their lines of action, leaves their moments unaltered.

Two equivalent systems of forces have the same force-sum and the same moment-sum about any given axis.

47. Couples. A pair of forces **F** and $-\mathbf{F}$, equal in magnitude, opposite in direction, and having different (parallel) lines of action is called a *couple* (§ 28).

Let P and Q be any two points on the lines of action of **F** and $-\mathbf{F}$ respectively (Fig. 47). Then the sum of the moments of **F** and $-\mathbf{F}$ about an axis s in the direction of the unit vector **e** is

$$\mathbf{e}\cdot\overrightarrow{AP}\times\mathbf{F} + \mathbf{e}\cdot\overrightarrow{AQ}\times(-\mathbf{F}) = \mathbf{e}\cdot(\overrightarrow{AP} - \overrightarrow{AQ})\times\mathbf{F} = \mathbf{e}\cdot\overrightarrow{QP}\times\mathbf{F}.$$

As this moment-sum depends only on the direction of the axis and not on its position we may state the

THEOREM. *A couple has the same moment (i.e. moment-sum) about all parallel axes in the same direction.*

This theorem shows that a couple is essentially irreducible. For if a couple could be reduced to a single force **R**, we would have $\mathbf{R} = 0$ (§ 28) and hence the moment of the couple would be zero about any axis whatever. As this is obviously not the case, the proposed reduction is impossible.

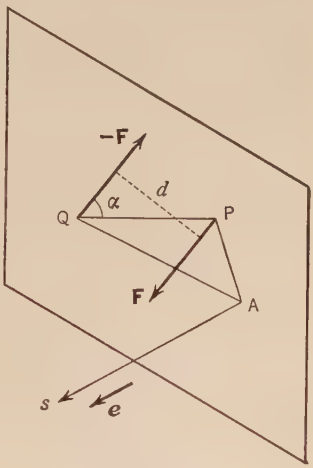

FIG. 47.

The vector $\overrightarrow{QP}\times\mathbf{F}$ is perpendicular to the plane of the couple. Hence the moment of a couple has its greatest numerical value about axes perpendicular to its plane, namely $F\cdot QP \sin\alpha = Fd$. The perpendicular distance d between the forces is called the *arm* of the couple. Hence, about a normal axis, the

$$\text{Moment of a Couple} = \pm\ \text{Force} \times \text{Arm}.$$

48. Reduction of Coplanar Forces. In dealing with a system of coplanar forces we shall always take moments about axes in a fixed direction normal to their plane. If such an axis cuts the

plane in the point A, a moment about the axis is called a moment about A. If the plane of the forces is chosen as the xy-plane, moments will be taken about axes in the direction of $+z$.

Consider, now, a system of coplanar forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . . acting on a rigid body. Denote their force-sum and their moment-sum about A by $\mathbf{F}$ and M_A. If there are three or more forces, two at least, say $\mathbf{F}_1$ and $\mathbf{F}_2$, do not form a couple. If the lines of action of $\mathbf{F}_1$ and $\mathbf{F}_2$ meet at P, shift the forces so that they act at P (Prin. B) and replace them by their vector sum (Prin. A); but if the forces are parallel, combine them by the construction of § 29. Similarly combine the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$ with a third force $\mathbf{F}_3$ and continue this process until the system is reduced to two forces, $\mathbf{F}'$ and $\mathbf{F}''$. The force-sum and moment-sum of this equivalent system are $\mathbf{F}$ and M_A (§ 46).

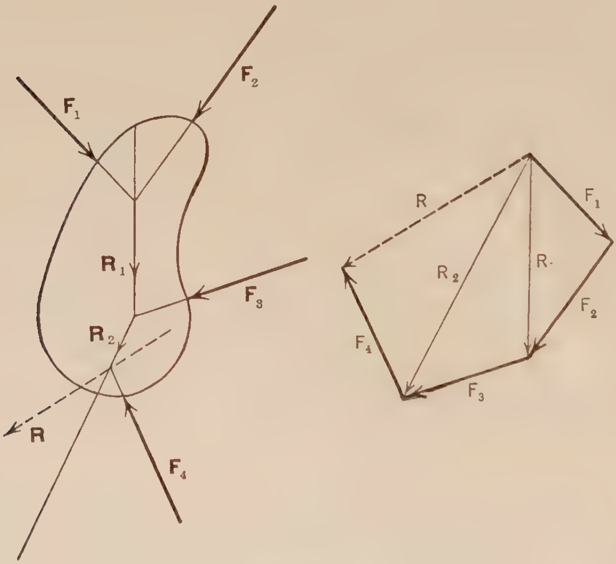

FIG. 48.

Case 1: $\mathbf{F} \neq 0$. Then $\mathbf{F}'$ and $\mathbf{F}''$ do not form a couple and may be replaced by their resultant, $\mathbf{R} = \mathbf{F}' + \mathbf{F}''$.

Case 2: $\mathbf{F} = 0$, $M_A \neq 0$. Now $\mathbf{F}' + \mathbf{F}'' = 0$, but the forces can not have a common line of action and reduce to zero; for in that case M_A would vanish. Hence $\mathbf{F}'$ and $\mathbf{F}''$ form a couple of moment M_A.

Case 3: $\mathbf{F} = 0$, $M_A = 0$. Again $\mathbf{F}' + \mathbf{F}'' = 0$, but now the forces have a common line of action and reduce to zero; for if they formed a couple, M_A would not vanish. The forces of the system are therefore in equilibrium (§ 30).

Example. Fig. 48 represents four coplanar forces acting on a rigid body. With the aid of a force-polygon, which gives the direction of the successive resultants,

$$\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{R}_1, \quad \mathbf{R}_1 + \mathbf{F}_3 = \mathbf{R}_2, \quad \mathbf{R}_2 + \mathbf{F}_4 = \mathbf{R},$$

the resultant $\mathbf{R}$ of the system is determined as shown. Only lines of action need be drawn in the space-diagram to the left, in which $\mathbf{R}$'s line of action is shown; its magnitude and direction are given by the force-polygon.

49. Funicular Polygon. The graphical reduction of a system of coplanar forces by the method of the previous example is inconvenient in practice whenever the forces to be combined are parallel

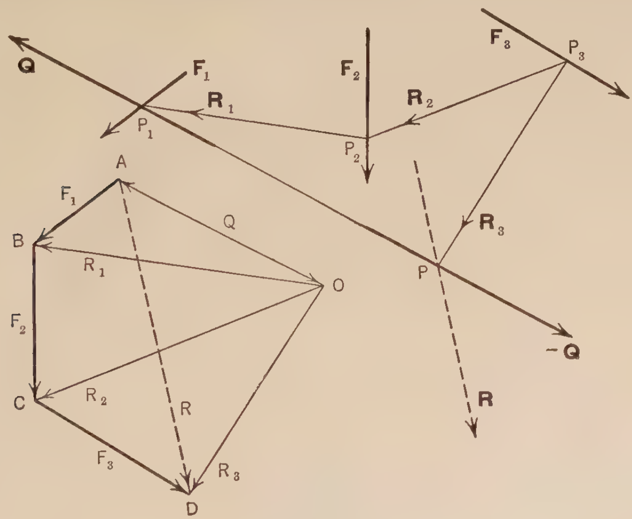

FIG. 49*a*.

or nearly parallel. We therefore give a more serviceable method which is based on the simple device, already employed in § 29, of introducing two forces $\mathbf{Q}$ and $-\mathbf{Q}$ acting along the same line (Theorem B).

Consider, for example, three forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ acting along the lines shown in Fig. 49*a*. To find their resultant $\mathbf{R}$, we first con-

struct the force-polygon $ABCD$ shown at the right. Then from any convenient point O as *pole* we draw lines to the vertices A, B, C, D. We now introduce the forces

$$\mathbf{Q} = \overrightarrow{OA} \quad \text{and} \quad -\mathbf{Q} = \overrightarrow{AO}$$

into the system, and construct the successive resultants

$$\mathbf{Q} + \mathbf{F}_1 = \mathbf{R}_1, \quad \mathbf{R}_1 + \mathbf{F}_2 = \mathbf{R}_2, \quad \mathbf{R}_2 + \mathbf{F}_3 = \mathbf{R}_3,$$

and finally

$$-\mathbf{Q} + \mathbf{R}_3 = \overrightarrow{AO} + \overrightarrow{OD} = \overrightarrow{AD} = \mathbf{R}.$$

The directions of $\mathbf{Q}$, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ and $\mathbf{R}$ are given by $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ and $\overrightarrow{AD}$.

Turning to the space-diagram, we draw any line parallel to $\overrightarrow{OA}$ to serve as the line of action of $\mathbf{Q}$ and $-\mathbf{Q}$. Let this line cut the line of $\mathbf{F}_1$ at P_1. Now draw successively

$$P_1P_2 \parallel OB, \qquad P_2P_3 \parallel OC, \qquad P_3P \parallel OD;$$

these are clearly the lines of action of $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$. The point P, where the lines of $-\mathbf{Q}$ and $\mathbf{R}_3$ meet, lies on the line of action of $\mathbf{R}$. Thus $\mathbf{R}$ is completely determined in magnitude, direction and position. The construction is suggested by the following scheme in which the sign $\equiv$ denotes *equivalence* in the sense of § 28:

$$\begin{aligned}
\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 &\equiv \mathbf{Q} + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + (-\mathbf{Q}) \\
&\equiv \mathbf{R}_1 + \mathbf{F}_2 + \mathbf{F}_3 + (-\mathbf{Q}) \\
&\equiv \mathbf{R}_2 + \mathbf{F}_3 + (-\mathbf{Q}) \\
&\equiv \mathbf{R}_3 + (-\mathbf{Q}) \\
&\equiv \mathbf{R}.
\end{aligned}$$

The figure $P_1P_2 \ldots$, whose sides are parallel to the *rays* $OA, OB, \ldots$ drawn from the pole O to the vertices of the force-polygon, is called a *funicular* or *link polygon*. Note that the two sides of the funicular which meet on the line of action of a force $\mathbf{F}_i$ are parallel to the rays that subtend $\mathbf{F}_i$ in the force-polygon.

The construction above corresponds to Case 1 of § 48, in which $\sum \mathbf{F}_i \neq 0$. If, however, $\sum \mathbf{F}_i = 0$, the force-polygon closes and the system will reduce to a couple or to zero. The construction for four forces is shown in Fig. 49*b*. The successive resultants

$$\mathbf{Q} + \mathbf{F}_1 = \mathbf{R}_1, \quad \mathbf{R}_1 + \mathbf{F}_2 = \mathbf{R}_2, \quad \mathbf{R}_2 + \mathbf{F}_3 = \mathbf{R}_3, \quad \mathbf{R}_3 + \mathbf{F}_4 = \mathbf{R}_4$$

are constructed as before. Since the force-polygon is closed,

$\mathbf{R}_4 = \mathbf{Q}$; the system is thus reduced to the couple $\mathbf{R}_4$, $-\mathbf{Q}$ (Case 2, § 48).

If the line of action of $\mathbf{R}_4$ had coincided with that of $\mathbf{Q}$, the forces $\mathbf{R}_4$ and $-\mathbf{Q}$ would have canceled each other. The original forces would then have been in equilibrium (Case 3, § 48).

In carrying out the above construction we may vary the force-polygon by changing the *order* of its sides. Moreover the pole may be chosen at pleasure; and lastly, for a given pole the line of action of the forces $\mathbf{Q}$, $-\mathbf{Q}$ may be any line parallel to the first ray from O.

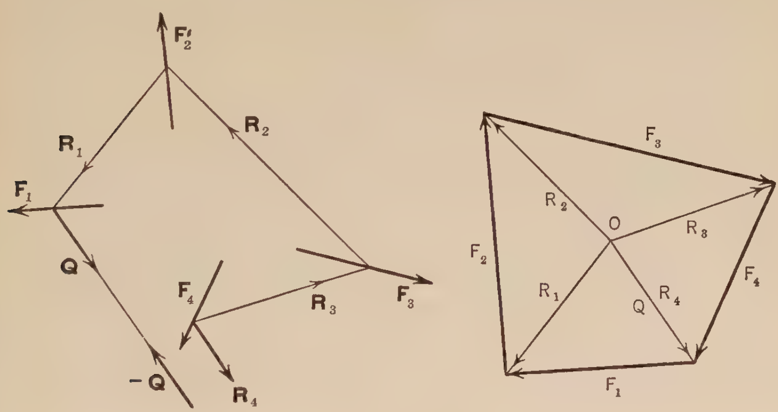

FIG. 49*b*.

The funiculars drawn for different poles have the following property:

The corresponding sides of two funicular polygons, drawn for the same forces but with different poles O, O', meet on a line parallel to OO'.

Consider for example the sides of the polygons meeting on the line of action of the force $\mathbf{F}_2$ (Fig. 49*c*). In the space diagram we have

$$\mathbf{R}_1 + \mathbf{F}_2 \equiv \mathbf{R}_2, \quad \mathbf{R}_1' + \mathbf{F}_2 \equiv \mathbf{R}_2';$$

hence on reversing the forces in the second equivalence and adding them to the first,

$$\mathbf{R}_1 + (-\mathbf{R}_1') \equiv \mathbf{R}_2 + (-\mathbf{R}_2').$$

As the forces in the two members are equivalent, their resultants must be equal vectors acting along the same line. The force

diagram shows that both resultants equal $\overrightarrow{OO'}$; as the first acts through L, the second through M, a line through L parallel to OO' must pass through M. Similarly $\mathbf{R}_3$ and $\mathbf{R}_3'$ meet on a line through M parallel to OO', i.e. the line LM, and so on.

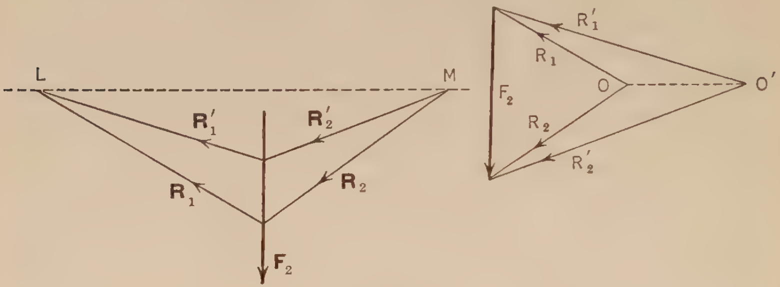

Fig. 49c.

50. Resultant of Coplanar Forces. Consider a system of coplanar forces $\mathbf{F}_i$ acting on a rigid body. If the force-sum is not zero, these forces have a resultant $\mathbf{R} = \sum \mathbf{F}_i$ whose line of action may be found from the equation expressing that the moment of $\mathbf{R}$ about any axis is equal to the moment-sum of the system about the axis.

We use rectangular axes and write $\mathbf{R} = [X, Y]$; then

$$X = \sum X_i, \quad Y = \sum Y_i. \tag{1}$$

The moment of $\mathbf{R}$ about the origin (i.e. the z-axis) is, from § 45, $xY - yX$.* If the forces $[X_i, Y_i]$ act at the points (x_i, y_i), their moment-sum about the origin is

$$M_O = \sum (x_i Y_i - y_i X_i). \tag{2}$$

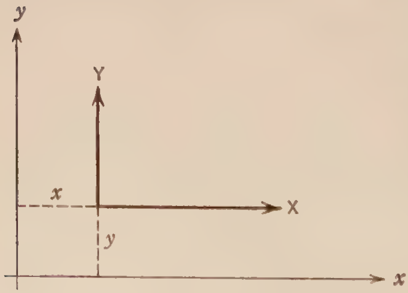

Fig. 50a.

Hence if (x, y) is any point on $\mathbf{R}$'s line of action

$$xY - yX = M_O. \tag{3}$$

This equation of the line of action is known as soon as X, Y and M_O are computed from (1) and (2)

*This result may be read from Fig. 50a in which x, y, X, Y are all positive.

An important case arises when the forces $\mathbf{F}_i$ are all parallel. If we take the x-axis perpendicular to the forces, $X_i = 0$ and $X = 0$ The line of action of the resultant then has the equation

$$xY = M_O = \sum x_i Y_i \quad \text{or} \quad x = \frac{\sum x_i Y_i}{\sum Y_i}.$$

Example 1. The three forces shown acting on the block in Fig. 50*b* have a resultant $\mathbf{R}$ whose components are

$$X = -20 + 50 \cos 30° = -20 + 43.3 = 23.3 \text{ lb.},$$
$$Y = -100 + 50 \sin 30° = -100 + 25 = -75.0 \text{ lb.};$$

hence

$$R = \sqrt{23.3^2 + 75^2} = 78.5 \text{ lb.}$$

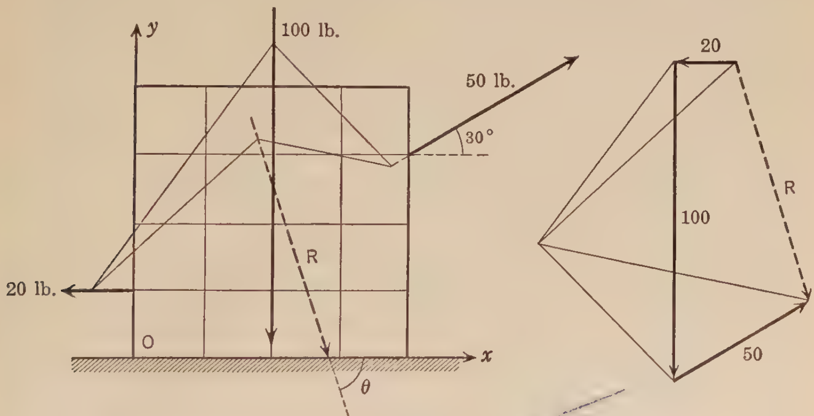

FIG. 50*b*.

Since the 50-lb. force has the components [43.3, 25] and acts through the point (4, 3), the moment-sum about O is

$$M_O = 20 \times 1 - 100 \times 2 + 25 \times 4 - 43.3 \times 3 = -209.9 \text{ lb.-ft.}$$

The equation of $\mathbf{R}$'s line of action is therefore

$$xY - yX = -209.9 \quad \text{or} \quad 75\,x + 23.3\,y = 209.9.$$

The intercept of this line on the x-axis is 2.8; its direction is given by

$$\tan\theta = \frac{-75}{23.3} = -3.22, \quad \theta = -72° \, 45'.$$

The figure also shows how $\mathbf{R}$ may be found graphically by means of a force-polygon and funicular polygon.

Example 2. Fig. 50*c* represents a beam 10 ft. long acted upon by three forces. Find their resultant.

Taking the y-axis as shown, we have

$$Y = -100 + 100 - 300 = -300 \text{ lb.}$$

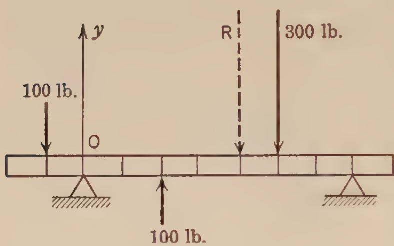

FIG. 50*c*.

for the resultant. The moment-sum of the given forces about O is

$$M_O = 100 \times 1 + 100 \times 2 - 300 \times 5 = -1200 \text{ lb.-ft.}$$

Hence

$$Yx = -300x = -1200, \quad x = 4 \text{ ft.}$$

The resultant is therefore a downward force of 300 lb. acting 4 ft. to the right of the left support.

PROBLEMS

Solve both graphically and analytically.

1. $ABCD$ is a square. Show that the forces represented by $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{BD}$, $\overrightarrow{CA}$ have the resultant $2\,\overrightarrow{CD}$.

2. Forces of 1, 2 and 3 lb., acting at the vertices A, B, C of an equilateral triangle having sides of 3 feet, are perpendicular to the sides AB, BC, CA. Each force tends to turn the triangle in the sense ABC about its center. Show that their resultant is a force of $\sqrt{3}$ lb. in the direction BC and cutting AB produced 8 feet from A.

3. The forces $\mathbf{F}_i$ in the xy-plane act through the points P_i and make an angle θ_i with the $+x$-axis. Find their resultant in the following cases:

(a)	$\mathbf{F}_i$	20	10	30	15 lb.
	P_i	(1, 0)	(3, 0)	(3, 3)	(0, 4) ft.
	θ_i	135°	45°	−90°	0°
(b)	$\mathbf{F}_i$	30	15	10	15 lb.
	P_i	(0, 0)	(2, 0)	(3, 0)	(4, 0) ft.
	θ_i	90°	−90°	−90°	90°

4. $ABCD$ are the vertices of a parallelogram taken in order. Prove that the forces represented by $\overrightarrow{AB}$, $\overrightarrow{BD}$, $\overrightarrow{DC}$, $\overrightarrow{CA}$, are in equilibrium.

5. Prove that the forces represented by the sides of a plane polygon, traversed counterclockwise, reduce to a couple whose moment equals twice the area of the polygon. (Forces and lengths are drawn to the same scale.)

51. Center of Gravity. The *gravity* of a body was defined in § 32 as the resultant force of attraction exerted on the body by the earth. We shall show later that no matter how a body is placed with respect to the earth, the line of action of its gravity always passes through a certain point, the *center of gravity* of the body, whose position relative to the body is fixed.

At present we shall only consider bodies that are symmetrical with respect to a point O. Every point P of such a body corresponds to another P' lying on the line PO prolonged and at the same distance from O. If a *homogeneous* body has central symmetry, its center of gravity coincides with its center of symmetry O. For if we regard the body as composed of symmetrical pairs of particles of equal gravity, the resultant gravity of each pair will pass through O; hence the gravity of the entire body acts through O.

In solving problems, the gravity of uniform bars, spheres, cylinders, etc. will be represented by forces acting through their centers of symmetry. Since the gravity of a body is *equivalent* to the distributed attractions on its particles, we may use the gravity instead of the latter in forming the sums of forces or their moments (§ 46).

52. Equilibrium of a Rigid Body: Coplanar Forces. In considering the equilibrium of a rigid body only the external forces need be taken into account (§ 31). If the external forces acting on the body are equivalent to zero, the force-sum and moment-sum about any axis vanish (§ 46). Conversely, if the force-sum and moment-sum about any axis normal to the plane vanish, the system of forces reduce to zero (§ 48, Case 3). Hence in view of Principle C (§ 30), we have the following

THEOREM. *In order that a rigid body shall be in equilibrium under the action of a coplanar system of external forces, it is necessary and sufficient that the force-sum and the moment-sum about any one axis normal to the plane both vanish.*

Thus if $\mathbf{F}$ denotes the force-sum and M_A the moment-sum about the point A, the body will be in equilibrium when and only when

$$\mathbf{F} = 0, \qquad M_A = 0.$$

The vector equation $\mathbf{F} = 0$ may be replaced by two scalar equations by taking components along any two intersecting axes

of the plane. If we choose the axes of x and y for this purpose we obtain the three scalar equations of equilibrium:

$$(1) \qquad F_x = 0, \quad F_y = 0, \quad M_A = 0.$$

When the forces $[X_i, Y_i]$ of the system act through the points (x_i, y_i) and we take moments about the origin, these conditions, when written out in full, become

$$\sum X_i = 0, \quad \sum Y_i = 0, \quad \sum (x_i Y_i - y_i X_i) = 0.$$

Either of the following sets of conditions

$$(2) \qquad F_x = 0, \quad M_A = 0, \quad M_B = 0 \quad (x \text{ not } \perp AB),$$
$$(3) \qquad M_A = 0, \quad M_B = 0, \quad M_C = 0 \quad (A, B, C \text{ not in a line})$$

are necessary and sufficient for equilibrium. If the system is equivalent to zero it is obvious that these conditions must be fulfilled. Conversely if either conditions (2) or (3) are fulfilled, the system is equivalent to zero.

Proof. In either case the system can not reduce to a couple. Suppose, then, that the forces have a resultant **R**.

When conditions (2) hold, **R** must act along the line AB, as its moment about A and B is zero. Also $F_x = R \cos(x, \mathbf{R}) = 0$, and since x is not perpendicular to AB or **R**, $R = 0$.

When conditions (3) hold, the moment of **R** about A, B and C is zero, that is, **R**'s line of action passes through these points. This is impossible as the points do not lie on a line; hence we must again conclude that $R = 0$.

When all the forces of the system are parallel the conditions for equilibrium reduce to two scalar equations. For if we take the x-axis perpendicular to the forces, the equation $F_x = 0$ is automatically satisfied, and the conditions (1) and (2) reduce to

$$(1)' \qquad F_y = 0, \quad M_A = 0,$$
$$(2)' \qquad M_A = 0, \quad M_B = 0 \quad (AB \text{ not } \parallel \text{ the forces}).$$

Example 1. In Fig. 52a the bars AB and BC are hinged to a wall at A and C, and joined by a pin at B. In order to find the reaction at A and the stress in BC we draw a free-body diagram of AB as shown. **S** denotes the thrust on the pin B and **R** the reaction at A, which must be directed to the left in order to balance the horizontal component of **S**.

Using conditions (1), we have

(i) $$F_x = -R\cos\theta + S\cos\alpha = 0,$$
(ii) $$F_y = R\sin\theta + S\sin\alpha - 600 - 900 = 0,$$
(iii) $$M_A = S \times 6\sin\alpha - 600 \times 2 - 900 \times 4 = 0.$$

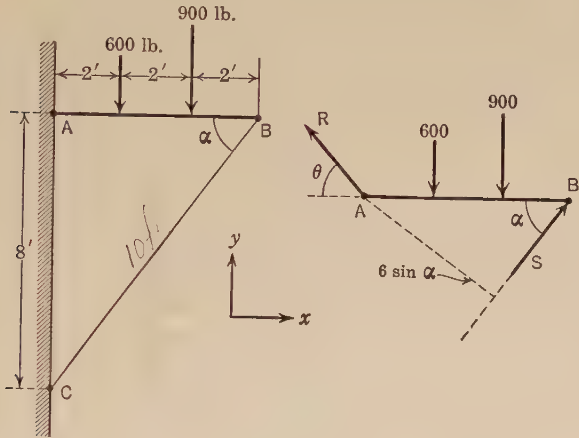

Fig. 52*a*.

Since $BC = 10$ ft.,

$$\sin\alpha = \frac{AC}{BC} = \frac{4}{5}, \qquad \cos\alpha = \frac{AB}{BC} = \frac{3}{5}.$$

From (iii)

$$S = \frac{4800 \times 5}{24} = 1000 \text{ lb.},$$

and hence from (i) and (ii),

$$R\cos\theta = 600, \qquad R\sin\theta = 700;$$

$$\tan\theta = \frac{7}{6} = 1.1667, \quad \theta = 49^\circ\, 24';$$

$$R = 100\sqrt{36 + 49} = 922 \text{ lb.}$$

We might have used the equation

$$M_B = R \times 6\sin\theta - 600 \times 4 - 900 \times 2 = 0$$

instead of (ii). This gives $R \sin\theta = 700$ as before.

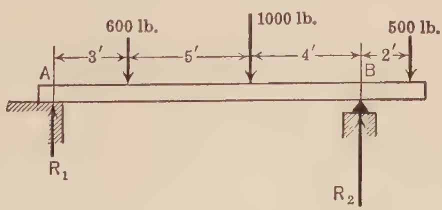

Fig. 52*b*.

Example 2. Fig. 52*b* represents a beam loaded with three concentrated loads. If the left support is smooth, its reaction $\mathbf{R}_1$ will be vertical, and as all the loads are vertical, the same is true of $\mathbf{R}_2$.

To find the reactions we use conditions (1)′:

(i) $F_y = R_1 + R_2 - 600 - 1000 - 500 = 0,$

(ii) $M_B = -12\,R_1 + 600 \times 9 + 1000 \times 4 - 500 \times 2 = 0.$

From (ii) $12\,R_1 = 8400, \quad R_1 = 700$ lb.;

from (i) $R_1 + R_2 = 2100, \quad R_2 = 1400$ lb.

Instead of (i) we might have used the equation

$$M_A = 12\,R_2 - 600 \times 3 - 1000 \times 8 - 500 \times 14 = 0;$$

this gives the above value for R_2

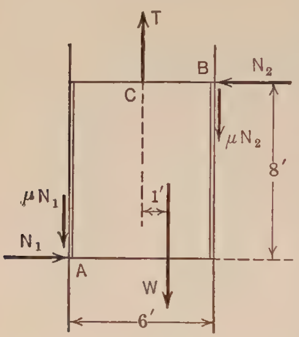

FIG. 52c.

Example 3. Fig. 52c represents an elevator car eccentrically loaded. If the weight of the car and contents is $W = 4000$ lb., find the tension T in the cable just before the car starts upward. The coefficient of friction between the car and guides is $\mu = \frac{1}{4}$.

Let us suppose that the eccentric load causes the car to have point contact with the guides at A and B. In the figure the reactions at these points are replaced by their normal projections N_1, N_2 and the forces of friction μN_1, μN_2 acting downward in order to oppose the impending motion. On applying conditions (1) we have

(i) $F_x = N_1 - N_2 = 0,$

(ii) $F_y = T - W - \mu N_1 - \mu N_2 = 0,$

(iii) $M_C = 8\,N_1 + 3\,\mu N_1 - 3\,\mu N_2 - W = 0.$

From (i) and (ii)

$$N_1 = N_2, \qquad T = W + 2\,\mu N_1;$$

and from (iii), $N_1 = W/8 = 500$ lb. Hence

$$T = 4000 + 2 \times \tfrac{1}{4} \times 500 = 4250 \text{ lb.}$$

PROBLEMS

1. A 10-ft. horizontal beam AB weighing 30 lb. per foot is supported at its ends, the left support being smooth. If it carries vertical loads of 100, 300, and 200 lb. at 2, 4, and 9 ft. from A respectively, find the reactions at A and B.

2. A uniform bar 8 ft. long weighing 100 lb. is supported horizontally between two smooth pins A, B, 1 ft. apart. The bar has one end under A and passes over B. Find the pin reactions.

3. A uniform rod 12 ft. long weighing 120 lb. rests on a smooth floor at an angle of 30° to the vertical. It is supported between two smooth

pins A, B, 2 ft. apart, the lower pin A being above the rod. Find the pin and floor reactions.

4. A uniform rod AB of weight W rests with the end A on a smooth floor, the end B on a smooth vertical wall. It is kept from slipping by a *horizontal* string tied at A and fastened to the wall. If the rod is inclined at an angle α to the floor, show that the tension in the string is $\frac{1}{2} W \cot \alpha$.

5. What horizontal force P is required to support the box in Fig. 52*d* if its weight of 900 lb. acts through its center? Find the reaction at A.

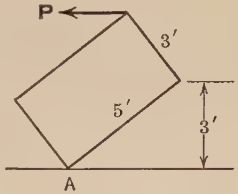

Fig. 52*d*.

6. A 10-ft. ladder weighing 32 lb. rests on a cement walk and leans against a smooth wall so that its top is just 8 ft. above the ground. If the coefficient of friction at the base is $\frac{1}{4}$, what horizontal force applied to the ladder at a point one-third of the way up will just keep it from slipping outward?

How great may this force be before the ladder slips inward?

7. The radii of a wheel and axle are respectively r_1 and r_2. A rope wrapped in opposite directions around the wheel and axle carries a pulley weighing W lb., the free parts of the rope being vertical. What force F applied at right angles to the end of a crank of length l will just support the pulley?

When $r_1 = 1$ ft., $r_2 = 3$ in., $l = 2$ ft., find the ratio F/W.

8. A boat davit rests in step-bearing at A, passes through a smooth hole B in a rail 4 ft. vertically above A and then curves outward so that the falls are 5 ft. from the center-line AB. If the ropes carry a load of 3000 lb., find the reactions at A and B.

9. A window-sash 5 ft. wide and $4\frac{1}{2}$ ft. high has an unbalanced weight of 10 lb. It has two grips 1 ft. from the sides to raise it. If the coefficient of friction at the sides is $\frac{1}{4}$, what pull applied at the right-hand grip will just raise it? (Assume contact at the lower right and upper left corners.)

10. When unloaded the beam of a balance has its three knife-edges on a horizontal line at a distance d from each other. If a weight w in one pan deflects the beam (of weight W) through an angle θ, prove that the center of gravity of the beam is at a distance $(w/W)\, d \cot \theta$ below the central knife-edge.

53. Three Forces in Equilibrium. Suppose that the three coplanar forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ are in equilibrium and that the lines of action of two, say $\mathbf{F}_1$ and $\mathbf{F}_2$, meet at a point A. For equilibrium it is necessary and sufficient that

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0, \quad \text{and} \quad M_A = 0.$$

Since the moments of $\mathbf{F}_1$ and $\mathbf{F}_2$ about A are zero, the moment of $\mathbf{F}_3$ about A must also be zero, that is, the line of action of $\mathbf{F}_3$ passes through A. The forces are therefore concurrent at A. Conversely, when this is true we must have $M_A = 0$. Therefore:

In order that three non-parallel coplanar forces shall be in equilibrium it is necessary and sufficient that they form a closed triangle and that their lines of action meet in a point.

Example 1. A light ladder AB of length l, with its lower end on a rough horizontal plane, leans against a smooth vertical wall (Fig. 53a). A man of weight W stands on the ladder at C. For a given inclination α of the ladder, how far may the man ascend before the ladder slips?

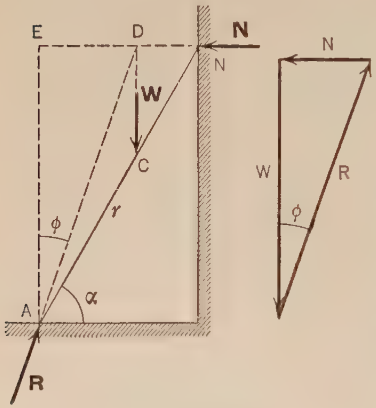

Fig. 53a.

Write $AC = r$; the problem, then, is to find the greatest value of r consistent with equilibrium. If we neglect its weight, the ladder is acted on by three forces: the weight $\mathbf{W}$, the normal reaction $\mathbf{N}$ of the wall, and the reaction $\mathbf{R}$ of the ground. As these must be concurrent, the line of $\mathbf{R}$ passes through the point D where the lines of $\mathbf{W}$ and $\mathbf{N}$ meet. At the point of slipping, the inclination of $\mathbf{R}$ to the vertical is equal to angle of friction ϕ. Hence

$$\tan\phi = \frac{ED}{AE} = \frac{r\cos\alpha}{l\sin\alpha} = \frac{r}{l}\cot\alpha,$$

or, since $\tan\phi = \mu$, the coefficient of friction,

$$r = \mu l \tan\alpha.$$

For example, when $\mu = 0.2$, $\alpha = 60°$, $l = 9$ ft.,

$$r = 0.2 \times 9 \times 1.73 = 3.1 \text{ ft.}$$

From the force-triangle we find

$$N = W\tan\phi, \qquad R = W\sec\phi.$$

Example 2. Fig. 53b represents a crane supporting a load $P = 5$ tons. If we neglect the weight of its members, the tension T in the rod BC and the reaction R on the pin A may be found as follows.

In the free-body diagram for AB the direction of **R** is determined since the forces **P**, **T**, **R** must be concurrent. From the figure

$$\tan\alpha = \frac{4}{12} = \frac{1}{3}, \qquad \tan\beta = \frac{4\tan\alpha}{8} = \frac{1}{6};$$
$$\alpha = 18°\,26', \qquad \beta = 9°\,28'.$$

Therefore in the force-triangle the angles have the values shown, and from the Law of Sines

$$\frac{T}{\sin 80°\,32'} = \frac{R}{\sin 71°\,34'} = \frac{P}{\sin 27°\,54'}.$$

Since $P = 5 \times 2000 = 10{,}000$ lb., we find

$$T = 21{,}080 \text{ lb.}, \qquad R = 20{,}270 \text{ lb.}$$

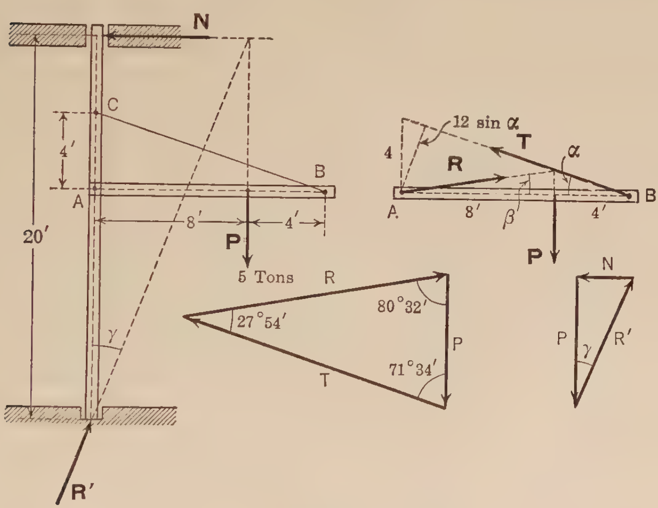

FIG. 53*b*.

These results also follow from the conditions of equilibrium:

(i) $$F_x = R\cos\beta - T\cos\alpha = 0,$$
(ii) $$M_A = 12\sin\alpha\, T - 8\,P = 0,$$
(iii) $$M_B = 4\,P - 12\sin\beta R = 0.$$

From (ii) we find

$$T = \frac{2\,P}{3\sin\alpha} = \frac{20{,}000}{3\sin 18°\,26'} = 21{,}080 \text{ lb.},$$

and from (i) and (iii)

$$\tan\beta = \frac{P}{3\,T\cos\alpha}, \qquad R = \frac{P}{3\sin\beta}.$$

The student may show that β and R computed from these equations have the values given above.

Suppose, now, that the upper end of the crane-column turns in a smooth bearing. Under the influence of the load **P** the column will be pressed against the right side of the bearing so that the bearing will exert a horizontal reaction **N** on the column. The crane as a whole is in equilibrium under three forces, the load **P**, the reaction **N** at the top, and the reaction **R′** of the step-bearing below. Hence **R′** must pass through the point where the lines of **P** and **N** meet. The direction of **R′** is thus given by

$$\tan\gamma = \tfrac{8}{20} = 0.4, \qquad \gamma = 21^\circ\,48';$$

and from the force-triangle **P**, **R′**, **N**, we have

$$N = P\tan\gamma = 4000 \text{ lb.}, \qquad R' = \frac{P}{\cos\gamma} = 10{,}770 \text{ lb.}$$

As a check, take moments about the base of the column; then $20\,N - 8\,P = 0$ and $N = 4000$ lb.

Example 3. *The Suspended Rod.* A uniform rod AB of length l and weight W is supported at its ends by two strings of lengths a, b which are tied to a hook O above (Fig. 53*c*). Find the tensions T_a, T_b of the strings.

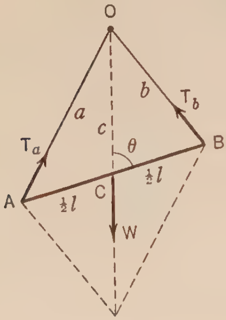

FIG. 53*c*.

The rod is in equilibrium under the forces **W**, $\mathbf{T}_a$, $\mathbf{T}_b$. Since the forces must be concurrent the rod will assume a position so that the line OC is vertical. If we draw the parallelogram $OAPB$, the triangle OPB has its sides parallel to the three forces and may be chosen as the force-triangle. Hence

$$\frac{T_a}{a} = \frac{T_b}{b} = \frac{W}{2\,c}$$

where c is the length of OC. Thus the tensions of the strings are proportional to their lengths.

To find c we note that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals; hence

$$4\,c^2 + l^2 = 2\,a^2 + 2\,b^2, \qquad 2\,c = \sqrt{2\,a^2 + 2\,b^2 - l^2}.$$

Knowing c we may readily compute the inclination θ of the rod.

For example if $a = 4$, $b = 3$, $l = 5$ ft., and $W = 100$ lb., $2\,c = 5$ ft.; and we find $T_a = 80$, $T_b = 60$ lb., $\theta = 73^\circ\,44'$.

Example 4. A block of weight W resting on a rough floor is subjected to a horizontal force P acting at a distance h above the floor (Fig. 53*d*). Under what conditions will it remain at rest?

If the block remains at rest, the three forces **P**, **W**, and the reaction **R**

of the table will be in equilibrium. Then the magnitude and direction of **R** are determined by the force-triangle **P**, **W**, **R**; thus

$$R = \sqrt{P^2 + W^2}, \qquad \tan\theta = \frac{P}{W}\cdot$$

If μ is the coefficient of friction between floor and block, the block will not slip as long as $\tan\theta \leqq \mu$ (§ 32, 2). The condition for no slipping is therefore

$$\frac{P}{W} \leqq \mu \quad \text{or} \quad P \leqq \mu W.$$

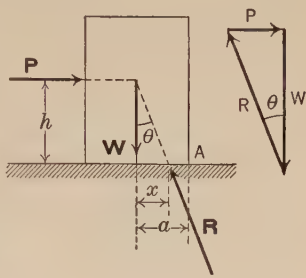

FIG. 53*d*.

For equilibrium it is also necessary that the lines of action of **P**, **W** and **R** shall meet in a point. From the figure we see that line of action of **R** will meet the base at a distance $x = h \tan\theta$ from the line of **W**. In order that **R** may act at a point in the base of the block we must have $x \leqq a$, that is

$$h\tan\theta \leqq a \quad \text{or} \quad hP \leqq aW.$$

If this condition is fulfilled the block will not overturn about the edge A.

In order that the block shall neither slip nor tip over, *both* of the conditions above must be fulfilled.

Example 5. *The Three-Hinged Arch.* Fig. 53*e* represents schematically an arch consisting of two rigid parts *1* and *2*, hinged to the abutments at A and B and to each other at C. Let $\mathbf{P}_1$ and $\mathbf{P}_2$ denote the resultants of all the loads applied to parts *1* and *2* respectively.

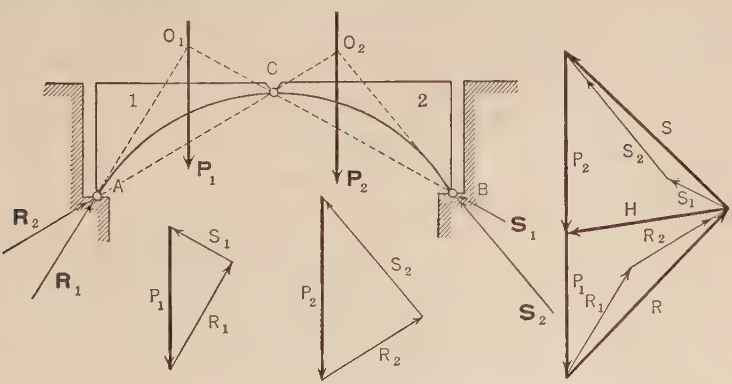

FIG. 53*e*.

We shall first find the reactions $\mathbf{R}_1$ at A and $\mathbf{S}_1$ at B when $\mathbf{P}_1$ alone is applied to the arch ($\mathbf{P}_2$ removed). Then part *2* is in equilibrium under

two forces, $\mathbf{S}_1$ at B and the hinge-pressure at C. The hinge-pressure of *1* on *2* is therefore equal to $-\mathbf{S}_1$ and acts along CB. Consequently the hinge-pressure of *2* on *1* equals $\mathbf{S}_1$ and acts along BC. Now part *1* is in equilibrium under the three forces, $\mathbf{P}_1$, $\mathbf{R}_1$ at A, and $\mathbf{S}_1$ at C; hence $\mathbf{R}_1$ must pass through the point O_1 where the lines of action of $\mathbf{P}_1$ and $\mathbf{S}_1$ meet. The force-triangle $\mathbf{P}_1$, $\mathbf{R}_1$, $\mathbf{S}_1$ may now be drawn and $\mathbf{R}_1$ and $\mathbf{S}_1$ determined.

We next find the reactions $\mathbf{R}_2$ and $\mathbf{S}_2$ when $\mathbf{P}_2$ alone is applied to the arch ($\mathbf{P}_1$ removed). Now part *2* is in equilibrium under the three forces $\mathbf{P}_2$, $\mathbf{S}_2$, and the hinge-pressure of *1* on *2* at C. The latter equals $\mathbf{R}_2$, acts along AC, and meets the line of action of $\mathbf{P}_2$ at O_2. Since $\mathbf{S}_2$ must pass through O_2, its direction is known. The force-triangle $\mathbf{P}_2$, $\mathbf{R}_2$, $\mathbf{S}_2$ may now be drawn and $\mathbf{R}_2$ and $\mathbf{S}_2$ determined.

Now since

$$\mathbf{P}_1 + \mathbf{R}_1 + \mathbf{S}_1 = 0 \quad \text{and} \quad \mathbf{P}_2 + \mathbf{R}_2 + \mathbf{S}_2 = 0$$

we must have

$$\mathbf{P}_1 + \mathbf{P}_2 + (\mathbf{R}_1 + \mathbf{R}_2) + (\mathbf{S}_1 + \mathbf{S}_2) = 0.$$

This shows that the entire arch is in equilibrium under the loads $\mathbf{P}_1$, $\mathbf{P}_2$ and the reactions

$$\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2 \text{ at } A, \qquad \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \text{ at } B.$$

$\mathbf{R}$ and $\mathbf{S}$ may now be constructed and the complete force-diagram drawn as shown. $\mathbf{H}$ gives the hinge-pressure at C exerted by *2* on *1*.

The entire construction admits of a simple graphical check. For if we draw a line (not shown) through the hinge C parallel to $\mathbf{H}$ and cutting $\mathbf{P}_1$ and $\mathbf{P}_2$ in the points Q_1 and Q_2, then $\overrightarrow{AQ_1}$ and $\overrightarrow{BQ_2}$ must have the directions of $\mathbf{R}$ and $\mathbf{S}$ respectively.

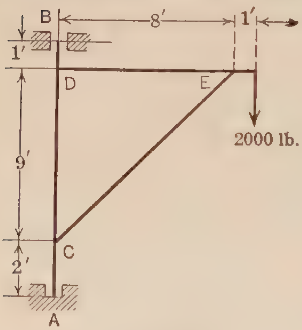

FIG. 53*f*.

PROBLEMS

1. A uniform rod AB 8 ft. long is hinged at A; its upper end B rests against a smooth vertical wall. If the rod weighs 100 lb. and makes an angle of 30° with the horizontal, find the reactions at the hinge and wall.

2. A uniform rod AB 10 ft. long is hinged at A. A cord BC 6 ft. long, tied to its upper end B, is fastened to a point C 10 ft. vertically above A. If the rod weighs 60 lb., find the tension of the string and the hinge reaction.

3. Fig. 53*f* represents a crane carrying a load of 2000 lb. at the end of the boom. Find the reactions at *A* and *B* and the pin pressures at *C*, *D*, and *E* due to the load.

4. A light rod *AB*, hinged at its *upper* end *A* is inclined at an angle of 30° to the horizontal. It rests at *C* on a smooth horizontal pin and carries a vertical load of 70 lb. at its lower end *B*. If $AC = 7$ ft., $CB = 3$ ft., find the reactions at *A* and *C*.

5. The roof truss of Fig. 58*c* is supported on horizontal rollers at *A* and is hinged at *H*. What are the reactions at *A* and *H* due to a wind load of 10,000 lb. perpendicular to *AD* and regarded as concentrated at *C*? Solve graphically and analytically.

6. In the arch of Fig. 53*e*, *AC* and *BC* have horizontal and vertical projections of 10 and 4 ft. (span = 20 ft.). If a single load $P_1 = 16$ tons cuts *AB* 5 ft. from *A*, find the reactions at *A* and *B*. What is the horizontal thrust on the arch?

7. The rods *AB* and *CD* in Fig. 53*g* are pinned to a wall at *A*, *C* and to each other at *D*. Find the reactions at *A* and *C* due to a downward load of 300 lb.

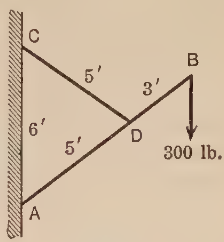

FIG. 53*g*.

8. A uniform rod *AB* weighing 60 lb. is hinged at *A*. A cord attached at *B* passes over a smooth pulley *C* 6 ft. vertically above *A* and carries a weight of 50 lb. hanging freely. Show that in equilibrium the portion *BC* of the cord is 10 ft. long. What is the length of *BC* when the hanging weight is 30 lb.?

9. A uniform rod *AB* 3 ft. long is supported with its upper end *A* against a smooth vertical wall by means of a string 5 ft. long joining *B* to a point *C* of the wall above *A*. If the rod weighs 10 lb., find the tension of the string and the reaction of the wall on the rod.

54. Problem of Three Forces. Let **R** denote the resultant of the known forces acting on a rigid body. Then if *1*, *2*, *3* are three lines, not all parallel, lying in a plane with **R**, we can always find three forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ acting along these lines and forming with **R** a system in equilibrium.

Suppose, first, that no two of the lines are parallel. Then if *2* and *3*, *3* and *1*, *1* and *2* meet in the points *A*, *B*, *C* (Fig. 54*a*) the conditions of equilibrium (§ 52, 3)

$$M_A = 0, \quad M_B = 0, \quad M_C = 0$$

give F_1, F_2, F_3 respectively. Thus if p_1 and q_1 denote the distances of $\mathbf{F}_1$ and **R** from *A*, we have

$$M_A = Rq_1 - F_1p_1 = 0, \quad F_1 = \frac{q_1}{p_1}R.$$

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As the moment of $\mathbf{F}_1$ must balance that of $\mathbf{R}$, $\mathbf{F}_1$ points to the left.

This problem admits of a simple graphical solution. Thus if $\mathbf{R}$ cuts *1* in the point P, draw the line *4* joining P and A and,

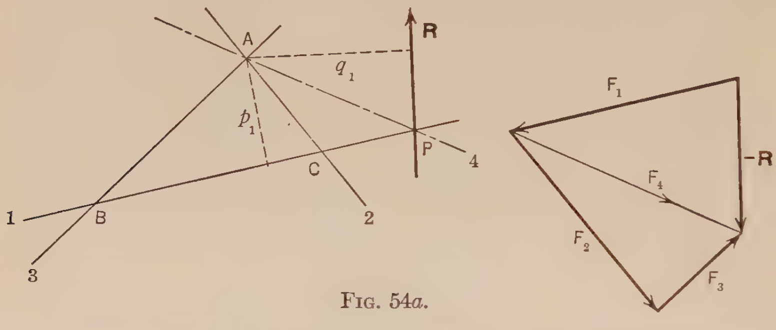


FIG. 54*a*.

with the aid of a force-diagram, replace $-\mathbf{R}$ by forces $\mathbf{F}_1$ and $\mathbf{F}_4$ along *1* and *4* (at P). Then shift $\mathbf{F}_4$ to A and replace it by the forces $\mathbf{F}_2$ and $\mathbf{F}_3$ along *2* and *3*. Now

$$-\mathbf{R} \equiv \mathbf{F}_1 + \mathbf{F}_4 \equiv \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3,$$

that is, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{R}$ form a system equivalent to zero.

If the lines *1* and *2* are parallel, the construction given above still applies (Fig. 54*b*). But since there is no point C in this case,

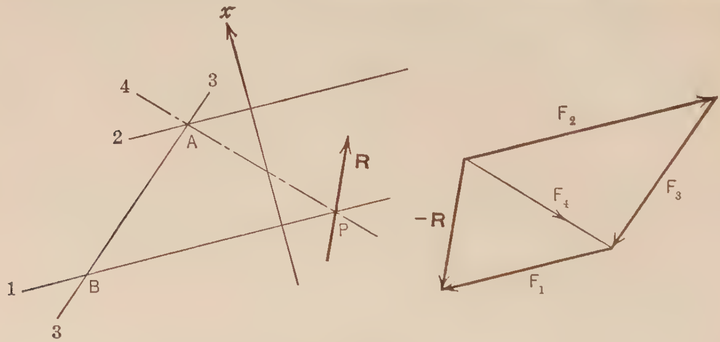


FIG. 54*b*.

the analytical solution must be modified. If x denotes an axis perpendicular to *1* and *2*, we may now use the conditions of equilibrium (§ 52, 2):

$$M_A = 0, \quad M_B = 0, \quad F_x = 0.$$

These equations give in turn F_1, F_2 and F_3.

Example. A uniform rod of weight $W = 100$ lb. rests on smooth planes at A and B and is supported by a cord CO (Fig. 54c) inclined at an angle $\alpha = 30°$ to the horizontal. The rod is in equilibrium under four forces: its weight $\mathbf{W}$, the normal reactions $\mathbf{N}_1$, $\mathbf{N}_2$ of the planes, and the tension $\mathbf{T}$ of the cord. $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{T}$ act along given lines *1*, *2*, *3*; the problem is to find their values.

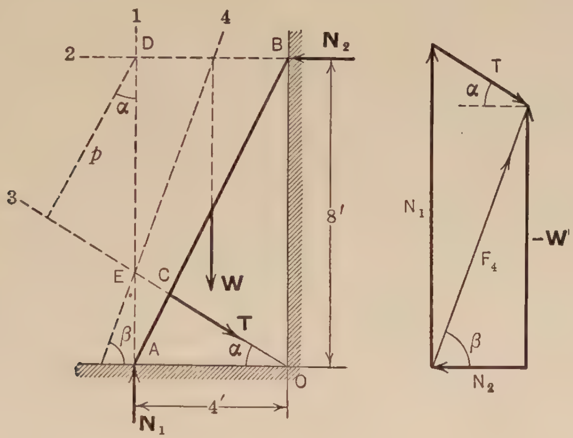


Fig. 54c.

To find T, for example, take moments about the point D where the lines *1* and *2* meet:

$$M_D = pT - 2W = 0;$$

$$p = DE \cos\alpha = (8 - 4\tan\alpha)\cos\alpha = 8\cos\alpha - 4\sin\alpha,$$

$$T = \frac{2W}{p} = \frac{100}{4\cos\alpha - 2\sin\alpha} = \frac{100}{3.464 - 1} = 40.6 \text{ lb.}$$

We may also proceed graphically, replacing $-\mathbf{W}$ by forces $\mathbf{N}_2$ and $\mathbf{F}_4$ along the lines *2* and *4*, and then replacing $\mathbf{F}_4$ by $\mathbf{N}_1$ and $\mathbf{T}$ along the lines *1* and *3*. From the force-diagram we may scale off N_1, N_2 and T, or even compute these forces after finding the angle β. This is left as an exercise for the student.

The student must not forget, however, that any set of conditions of equilibrium given in § 52 will suffice to solve our problem. Thus the equations

$$F_x = T\cos\alpha - N_2 = 0,$$
$$F_y = N_1 - W - T\sin\alpha = 0,$$
$$M_O = 2W + 8N_2 - 4N_1 = 0,$$

may be readily solved for N_1, N_2 and T. To find T, multiply the first by 8, the second by 4, and add both to the third; then

$$8\,T\cos\alpha - 4\,T\sin\alpha - 2\,W = 0, \quad T = \frac{W}{4\cos\alpha - 2\sin\alpha}$$

as before. Putting $T = 40.6$ in the first and second equations, we find

$$N_1 = 120.3 \text{ lb.}, \qquad N_2 = 35.2 \text{ lb.}$$

PROBLEMS

1. A uniform rod 8 ft. long and weighing 100 lb. rests on two smooth planes, one horizontal, the other inclined 60° to the horizontal. If the inclination of the rod is 30°, what horizontal force P must be applied at the lower end to keep the rod from slipping? Find also the reactions of the planes. Solve graphically and analytically.

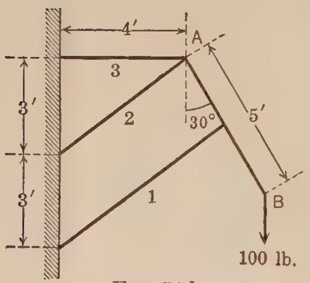


FIG. 54*d*.

2. A light rod AB, 5 ft. long, is supported by the rods *1*, *2*, *3* as shown in Fig. 54*d*. Find graphically the stresses in the rods due to the 100-lb. load at B.

55. Trusses. A *truss* is a framework of rigid bars of wood or metal (the *members* of the truss) each of which is connected to other members at two *joints* in such a manner that the entire structure is capable of supporting, without any essential change of shape, loads applied to the joints. In this chapter we shall consider trusses whose members are parallel to a single plane. In such a truss the ideal joint consists of a smooth cylindrical pin which fits into holes in the members which it connects. In order to compute the stresses in the members of a truss we assume that

(1) the members are connected by smooth pins, and

(2) all external forces acting on the truss are applied at the joints.

Under these assumptions a member AB connecting the joints A and B will be in equilibrium under the action of two forces, the resultants $\mathbf{F}_A$ and $\mathbf{F}_B$ of the forces acting on the member at A and B respectively. Hence $\mathbf{F}_A + \mathbf{F}_B \equiv 0$ and the member is subject to an *axial stress* (§ 33) numerically equal to either resultant. Such a member may always be "cut" in forming a free-body diagram (§ 36).

Actual roof and bridge trusses may differ considerably from the ideal truss described above. When the connections are all pin joints, the condition of perfect smoothness at the joints is only roughly fulfilled in practice, and frictional forces may be developed. In riveted trusses the deviation from assumption (1) is still more serious. In such trusses the members are riveted directly to each other or to a common plate, and are always subject to bending as well as axial stress. As to assumption (2), the floor-beams of a bridge or the purlins of a roof are designed to transmit the loads to the joints of the truss. Nevertheless the weights of the members themselves are not applied at the joints. If these weights are small in comparison with the other loads, no great error is made in dividing them equally between the joints or even neglecting them altogether.

56. Statically Determinate Trusses. In order that a truss may be capable of supporting loads without deformation (other than the minor changes in length of its members due to tension or compression) there must be a certain minimum number of bars con-

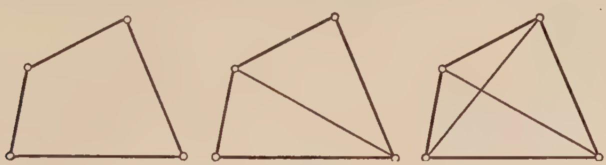


FIG. 56*a*.

necting its joints. Thus if there are four joints, four connecting bars can not form a rigid structure, five bars form one which is *just rigid*, while with six bars the structure is *over-rigid*, having one *redundant bar* (Fig. 56*a*). A structure is said to be *just rigid* if it may be deformed after any one bar is removed.

Consider a plane truss with j joints, $P_1, P_2, \ldots, P_j$. To specify their relative position take P_1 as origin of rectangular axes and draw the x-axis through P_2. The relative position of the joints will then be fixed by giving the abscissa of P_2 and both coördinates of the $j - 2$ remaining joints, $1 + 2(j - 2) = 2j - 3$ coördinates in all. The known length of each bar gives one equation between the coördinates of its joints; hence if there are n bars in the truss we will have, in general, just enough equations to locate the joints when

(1) $$n = 2j - 3.$$

Thus in the roof truss of Fig. 56b, $j = 7$ and $n = 11$, so that this relation is fulfilled.

In order that a truss as a whole be in equilibrium under its loads and the reactions of its supports, three scalar conditions of equilibrium must be fulfilled. The reactions, then, will be completely determined by these equations provided that they involve just three unknown elements — magnitudes or directions of forces. Thus in Fig. 56b, the reaction at the hinge A is unknown in magnitude and direction, while the reaction on the smooth roller B is unknown in magnitude only; these reactions may therefore be completely determined. The figure shows the graphical solution; the force **P** represents the resultant wind-pressure on one side of the roof. When the reactions may be completely determined, as in this case, they are said to be *statically determinate.* Note that this would not be true if the truss were hinged at both supports; the two reactions would then involve *four* unknown elements.

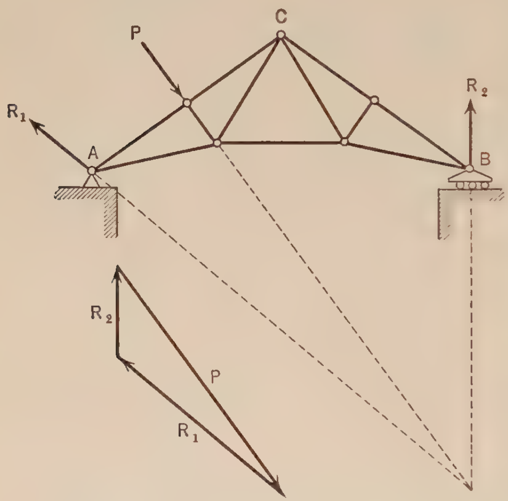


FIG. 56b.

Suppose now that we have a just rigid truss of n bars and j joints, fulfilling therefore relation (1); and let it be supported so that its reactions are statically determinate. To determine the stresses in its n bars we may regard each pin as a free body in equilibrium under the system of concurrent forces formed by the stresses in the bars connected by this pin. For example the pin C in Fig. 56b may be treated as a particle at rest under four forces. Thus each joint will furnish two scalar equations of equilibrium and the j joints will furnish $2j$ equations in all. These equations are linear in the n unknown stresses in the bars and the 3 unknown

elements in the reactions. These $n + 3$ unknowns may in general* be determined uniquely from the $2j$ equations when $n + 3 = 2j$. Since this relation is the same as (1), we have shown that when the reactions on a *just rigid* truss are statically determinate, the stresses in its members are also statically determinate, i.e. they may be found from the equations of static equilibrium.

The stresses in a statically determinate truss are entirely independent of the section areas of its members. This is not the case with an *over-rigid* truss; the equations of static equilibrium do not then suffice to find the stresses, and the elastic properties of its materials must be known in order to solve the problem. For this reason an over-rigid truss is said to be *statically indeterminate.*†

57. Stresses in Simple Structures. A *simple truss* is one which may be built up from a triangle by successively adding two new bars which meet in a new joint. Thus the truss of Fig. 57 is *simple*; note that the bar *02* is not a member of the truss and must be regarded as part of its support. On the other hand the roof truss of Fig. 60*a* is obviously not simple.

A simple truss is, in general, just rigid, and the stresses in its bars are statically determinate when the external forces acting on it are known.‡ The pin at the last joint added is in equilibrium under the known external force and the stresses in the two bars meeting there. If we draw a free-body diagram for this pin and apply the conditions of equilibrium to the three concurrent forces acting on it, the stresses in the bars may be determined. We may now imagine these bars removed from the truss, provided their action on the pins at the adjacent joints is represented by the forces just found when reversed in direction. The remaining structure again must have at least one joint where just two bars meet. The stresses in these may be found as before and the process continued until the stresses in all the bars are known.

* We disregard here an exceptional case.

† When the moduli of elasticity are known and the section areas of the members are given in advance, the stresses in a statically indeterminate truss may be determined by the *Method of Least Work.* See Parcel and Maney, *Statically Indeterminate Stresses*, (1926), Wiley.

‡ If the truss has j joints and n bars, $j - 3$ joints and $2(j - 3)$ bars must be added to the base triangle; hence

$$n = 3 + 2(j - 3) = 2j - 3,$$

in agreement with (§ 56, 1).

The directions of the unknown stresses at any joint are usually evident from inspection. However, if an error is made in assigning an arrowhead in the free-body diagram, the corresponding stress will turn out to be *negative* in the completed calculation. Thus *a negative stress indicates that the assumed direction of the force must be reversed.**

Before computing the stresses in a truss by the method above it is usually necessary to find the reactions of its supports by applying the conditions of equilibrium for coplanar forces to the truss as a whole. If the truss contains an unsupported joint of just two bars or one of three bars, two of which lie in a line, the calculation may be begun at once by considering the equilibrium of its pin. However, if all the stresses in the truss are required we may as well compute the reactions at the outset as they furnish a check on the computation.

Example. The cantilever truss of Fig. 57 is supported at the hinge *1* and by the bar *02*. The reaction **R** at *1* and the force **T** exerted by *02* on the joint *2* have three unknown elements: β, R, T. To find them we

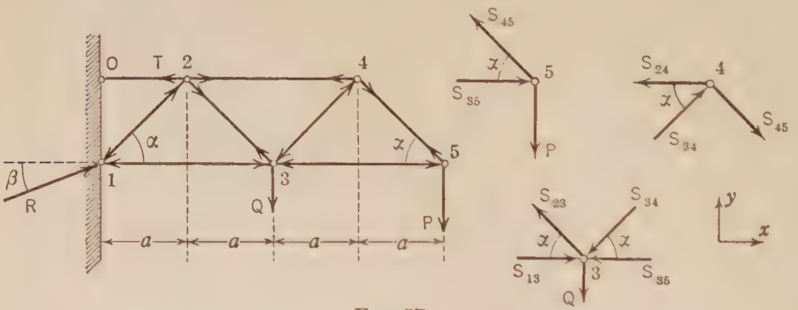


FIG. 57.

apply the conditions that insure the equilibrium of the truss as a whole under the coplanar forces **P**, **Q**, **T**, **R**:

$$F_x = R\cos\beta - T = 0, \qquad F_y = R\sin\beta - P - Q = 0,$$
$$M_1 = a\tan\alpha\, T - 2\,aQ - 4\,aP = 0.$$

From the last equation we have

(i) $$T = (4\,P + 2\,Q)\cot\alpha;$$

* If all the bars are assumed to be in tension — the arrows in the free-body diagrams all pointing away from the pins — the positive stresses in the completed calculation will be tensions, the negative, compressions.

and from the first two

(ii) $$R \cos \beta = T, \quad R \sin \beta = P + Q, \quad \tan \beta = \frac{P+Q}{T},$$

from which we may find β and R.

We now compute the stresses at the joints *5*, *4*, *3*, *2*, *1* in turn. (It will be noticed that we might have started at joint *5* without a knowledge of the reactions.) At each joint the directions of the unknown forces are supplied by inspection and then the two equations of equilibrium are written down. The first of these expresses that $F_y = 0$, the second that $F_x = 0$.

Joint 5.

$$S_{45} \sin \alpha - P = 0, \quad S_{35} - S_{45} \cos \alpha = 0;$$
$$S_{45} = \frac{P}{\sin \alpha}, \qquad S_{35} = P \cot \alpha.$$

Joint 4.

$$S_{34} \sin \alpha - S_{45} \sin \alpha = 0, \quad S_{34} \cos \alpha + S_{45} \cos \alpha - S_{24} = 0;$$
$$S_{34} = S_{45} = \frac{P}{\sin \alpha}, \qquad S_{24} = 2\,P \cot \alpha.$$

Joint 3.

$$S_{23} \sin \alpha - S_{34} \sin \alpha - Q = 0, \quad S_{13} - S_{35} - S_{34} \cos \alpha - S_{23} \cos \alpha = 0;$$
$$S_{23} = \frac{P+Q}{\sin \alpha}, \qquad S_{13} = (3\,P + Q) \cot \alpha.$$

Joint 2.

$$S_{12} \sin \alpha - S_{23} \sin \alpha = 0, \quad S_{24} + S_{12} \cos \alpha + S_{23} \cos \alpha - T = 0;$$
$$S_{12} = \frac{P+Q}{\sin \alpha}, \qquad T = (4\,P + 2\,Q) \cot \alpha.$$

Joint 1.

$$R \sin \beta - S_{12} \sin \alpha = 0, \quad R \cos \beta - S_{12} \cos \alpha - S_{13} = 0;$$
$$R \sin \beta = P + Q, \quad R \cos \beta = (4\,P + 2\,Q) \cot \alpha = T.$$

The agreement of these equations with (i) and (ii) above checks the entire calculation.

When $P = 2000$ lb., $Q = 1000$ lb., and $\alpha = 45°$, the above results give

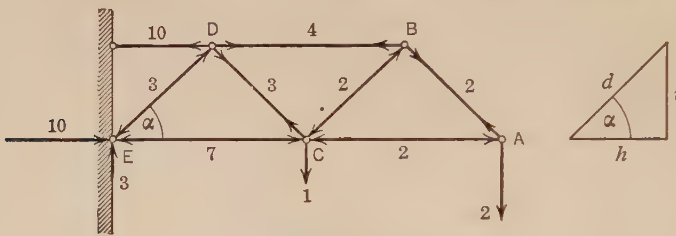
$S_{45} = 2828$, $S_{23} = 4242$, $S_{24} = 4000$ lb., *tension*,
$S_{35} = 2000$, $S_{13} = 7000$, $S_{34} = 2828$, $S_{12} = 4240$ lb. *compression*;
$T = 10{,}000$ lb. *tension*, $\beta = 16° \, 42'$, $R = 10{,}440$ lb.

PROBLEMS

Compute the reactions and stresses for the following trusses:

1. Fig. 58*b*. **2.** Fig. 58*c*. **3.** Fig. 59*d*. **4.** Fig. 59*e*.

58. Index Stresses. In a simple truss in which the diagonal numbers are all inclined at the same angle α to the horizontal, the stresses may be computed with great rapidity by the following modification of the method of joints. Consider, for example, the cantilever truss of § 57, loaded as shown in Fig. 58*a*. The loads are given in units of 1000 pounds (" kips "). The numbers written beside the diagonals AB, BC, CD, DE give the vertical components of their stresses (in kips) required for *vertical balance* at the



Loads in kips

Fig. 58*a*.

joints A, B, C, D. When the joints are taken in this order, these numbers and the associated arrows are evident on inspection; they are called the *index stresses* for the diagonals. Denoting them by I_d and the actual stresses by S_d,

$$I_d = \text{Vertical component of } S_d = S_d \sin \alpha, \quad \text{or}$$

$$S_d = \frac{I_d}{\sin \alpha} \tag{1}$$

Let us now obtain *horizontal balance* at each joint by treating the index stresses I_d as if they were the horizontal components of S_d. Taking the joints in the order A, B, C, D, we thus get the numbers written beside the horizontals, together with the associated arrows; they are called the *index stresses* for the horizontals. We denote them by I_h and the actual stresses by S_h. Since the horizontal components of S_d are really

$$S_d \cos \alpha = I_d \cot \alpha$$

instead of I_d, the actual horizontal stresses S_h will be $I_h \cot \alpha$ instead of I_h; thus

$$S_h = I_h \cot \alpha. \tag{2}$$

In the diagram of index stresses the vertical loads on a truss are taken at their actual values. Hence if a truss has vertical

members, the numbers I_v written beside them in obtaining vertical balance are the actual stresses S_v:

$$S_v = I_v. \tag{3}$$

Since α is constant throughout the truss, the same is true of the factors which convert I_d and I_h into S_d and S_h. If we draw a right triangle with its sides h, v horizontal and vertical, and its hypotenuse d inclined α to h (Fig. 58*a*), equations (1), (2), (3) may be written

$$S_d = \frac{d}{v} I_d, \quad S_h = \frac{h}{v} I_h, \quad S_v = \frac{v}{v} I_v. \tag{4}$$

For the truss given above, $\alpha = 45°$ and we may take $h = v = 1$, $d = \sqrt{2}$; hence

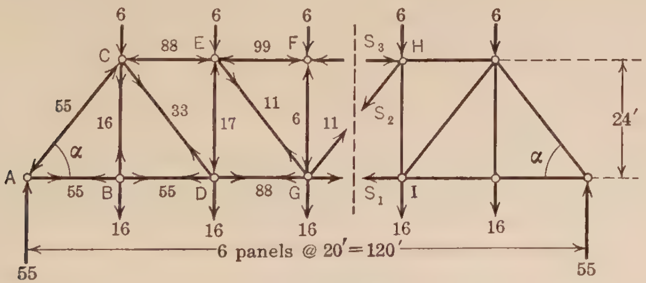
$$S_d = 1.414\, I_d, \quad S_h = I_h, \quad S_v = I_v.$$

For example $S_{AB} = 2.828$, $S_{AC} = 2$ kips.

Horizontal and vertical balance at the joint E demand that the reaction R shall have the *index components* [10, 3]. In the present case these are the actual components since both I_h and I_v represent actual forces. Hence

$$R = \sqrt{109} = 10.44 \text{ kips}, \quad \beta = \tan^{-1} .3 = 16° 42'.$$

These results agree with those of § 57.



Loads in kips

Fig. 58*b*.

Example 1. In the Pratt truss of Fig. 58*b* the loads and reactions are given in kips. As the loading is symmetrical the stresses need only be computed for one-half of the truss. The index stresses are found by taking the joints in the order *ABCDEFG*. At *G* we find that the index stress in *GH* is a tension of 11; as this agrees with the index stress in *GE* the requirements of symmetry are fulfilled and the calculation is checked.

If we cut the truss as shown we must have horizontal and vertical balance for all the external loads on the truss to the left of the section: thus,

$$88 + 11 - 99 = 0, \quad 55 + 11 - 3 \times 6 - 3 \times 16 = 0.$$

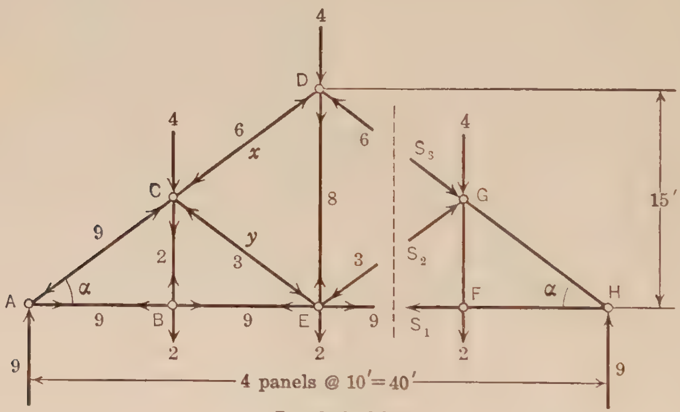
In the triangle ABC, $h = 20$ ft., $v = 24$ ft.; hence $d = 31.24$ ft. The conversion equations (4) now become

$$S_d = 1.302\, I_d, \quad S_h = 0.833\, I_h, \quad S_v = I_v.$$

For example

$$S_{CD} = 1.302 \times 33 = 43.0, \quad S_{CE} = 0.833 \times 88 = 73.3, \quad S_{CB} = 16 \text{ kips}.$$

The members of the upper chord and the three central verticals are in compression.



Loads in kips

FIG. 58c.

Example 2. The index stresses in the roof truss of Fig. 58c are found by taking the joints in the order $ABCDE$. At joint C the index stresses marked x and y are unknown. Assuming their directions as shown, we must have for horizontal and vertical balance that

$$9 - x - y = 0, \quad 9 + y - x - 4 - 2 = 0;$$

hence $x = 6$, $y = 3$. Note the check by symmetry at E and the horizontal and vertical balance of the external forces when the truss is cut as shown:

$$9 - 3 - 6 = 0, \quad 9 + 6 - 3 - 2 \times 4 - 2 \times 2 = 0.$$

In the triangle AED, $h = 20$ ft., $v = 15$ ft., $d = 25$ ft.; hence

$$S_d = \tfrac{5}{3} I_d, \quad S_h = \tfrac{4}{3} I_h, \quad S_v = I_v.$$

All the inclined members are in compression, the others in tension.

PROBLEMS

Find the index stresses and the actual stresses for the following trusses.

1. Cantilever truss, Fig. 59*d*.

2. Warren bridge truss, Fig. 59*e*.

3. Baltimore truss, Fig. 59*i* (only one-half of the truss is shown).

4. Cantilever bridge, Fig. 59*j*. [In finding the reactions make use of the fact that the sum of the moments of all the external forces to the right (or left) of the hinge *H* about *H* must vanish; for a hinge cannot sustain a bending moment. The reactions from left to right prove to be −5, 180, and 45 kips.]

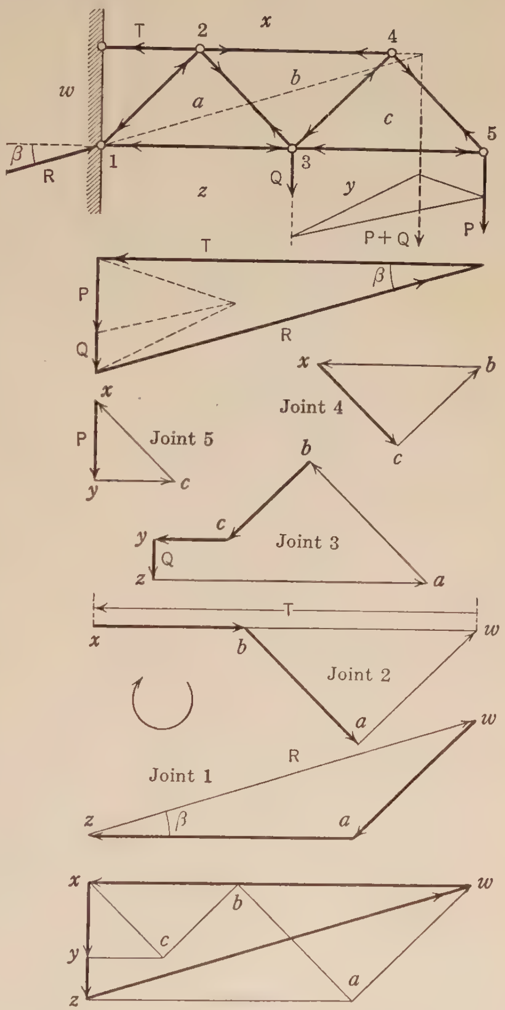


Fig. 59*a*.

59. Maxwell Diagrams. In order to avoid lengthy computations, the stresses in a *simple* truss may be determined graphically by drawing a force-polygon for each joint. The joints should be chosen in the order specified in § 57 so that there are at most two unknown stresses at any one joint. As before, it is usually necessary to find the reactions of the supports in advance; this may be done analytically, or graphically by means of a funicular polygon.

We shall apply this method to the cantilever truss of the preceding article (Fig. 59a). The external forces are represented by vectors drawn outside of the truss; these divide the outer space into four regions labeled w, x, y, z. The space inside the truss is divided into three regions a, b, c by the bars *23* and *34*. Then any force acting on the truss, whether internal or external, forms the boundary between two regions and may be denoted by the corresponding letters; for example, P, S_{45}, S_{35} are denoted by xy, xc, yc respectively in the force diagram. This is known as *Bow's notation.*

To find the reactions we first locate the resultant of the loads **P**, **Q** by the help of a funicular as shown. As this resultant **P** + **Q**, **R**, and **T** are concurrent, the direction of **R** is known; hence **R** and **T** may be found from a force-triangle.

The force-polygons are now drawn for the joints *5*, *4*, *3*, *2*, *1* in turn. *Each polygon is formed by drawing the forces at the joint in the order in which they are met in clockwise circuit about the joint, beginning with the known forces* (shown by heavy lines). Each stress appears twice in these polygons, first as an unknown, then as a known force. In order to avoid this duplication of lines and the inevitable small errors in drawing that accompany it, the five polygons may be assembled into a single figure in which each stress occurs but once. This is the *Maxwell* diagram* for the truss. In this diagram the external forces form a closed polygon and the stresses are parallel to the bars of the truss and in one-to-one correspondence with them. Of course the forces concurrent at any joint also form a closed polygon. The direction of any force is given by its segment in the diagram when the letters are read in the order in which they occur in a clockwise circuit of the joint at which the force acts.

Not every truss admits of a combined stress diagram having the properties mentioned above. But if a truss, in which the bars do not cross and the external forces act only at joints on the outer boundary, has a Maxwell diagram, it may be constructed as follows:

In the space diagram draw the external forces outside of the truss. Then the Maxwell diagram may be drawn by constructing the force-polygons for the successive joints so that the forces in each one follow

* Named after James Clerk Maxwell, the English physicist, who was the first to construct such diagrams.

in the same order in which they are met in a clockwise circuit of the joint in question, the known forces coming first.*

If two bars of a truss cross, a Maxwell diagram may often be constructed if a joint is introduced at their intersection. This leaves the truss just rigid; for if

$$n = 2j - 3 \quad \text{then} \quad n + 2 = 2(j + 1) - 3.$$

Moreover the stresses in the bars thus joined are not altered; they will be represented by the sides of a parallelogram in the diagram. This device is employed in the following example.

Example 1. In the truss of Fig. 59*b* a joint is introduced where the central diagonals meet. The reactions are found graphically by means of a funicular polygon; the ray *ow*, drawn parallel to its closing side, determines the reactions *zw* and *wx*. The diagram shows that the diagonal members are in tension, the others in compression.

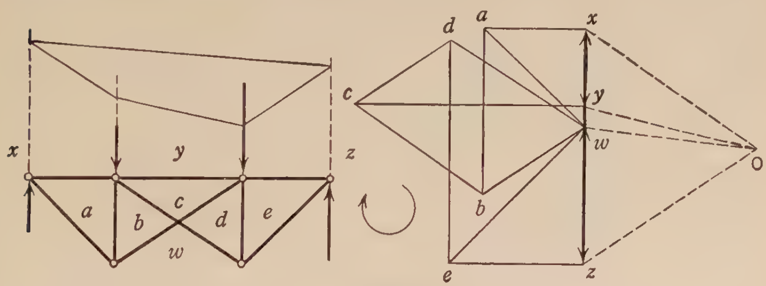


FIG. 59*b*.

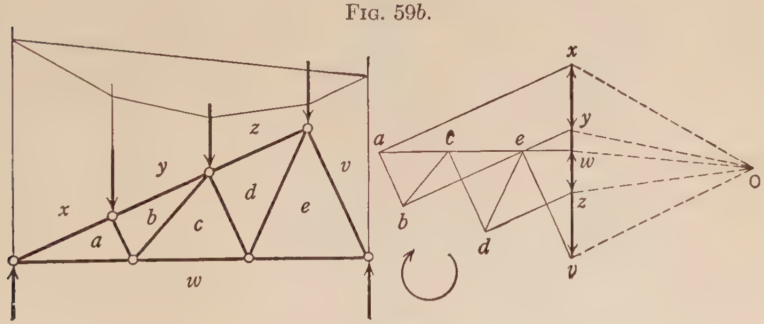


FIG. 59*c*.

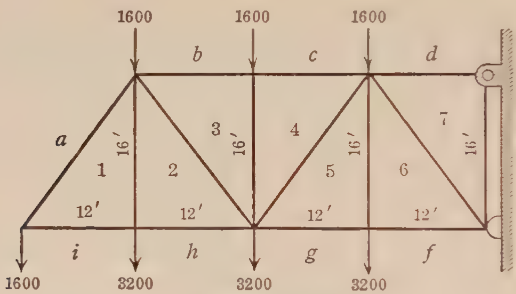
Example 2. Fig. 59*c* represents a saw-tooth roof truss and its Maxwell diagram. The ray *ow* drawn parallel to the closing side of a funicular

* It is only essential that the *same* sense shall be followed in each circuit, clockwise or the reverse.

polygon determines the reactions *vw* and *wx*. The diagram shows that the members of the lower chord and the diagonals *cb*, *ed* are in tension, the members of the upper chord and the diagonals *ba*, *dc*, *ve* are in compression.

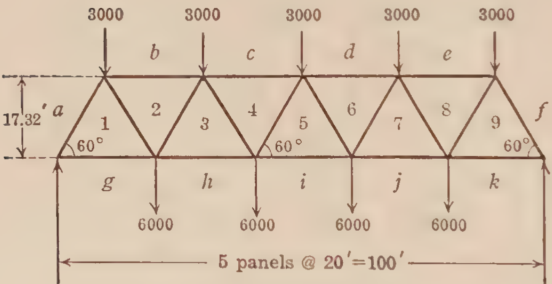
PROBLEMS

Find the stresses in the following trusses by drawing their Maxwell diagrams, using Bow's notation. The regions outside of the truss are denoted by letters, those inside by numbers.



Loads in pounds

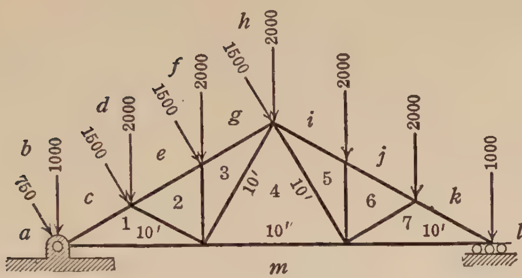
FIG. 59*d*



Loads in pounds

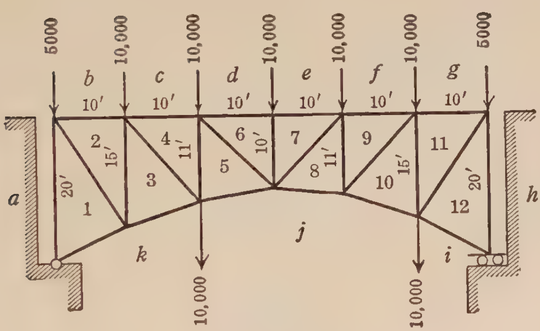
FIG. 59*e*.

1. Cantilever truss, Fig. 59*d*.
2. Warren bridge truss, Fig. 59*e*.
3. Roof truss, Fig. 59*f*.
4. Deck truss, Fig. 59*g*.
5. Crane, Fig. 59*h*.
6. Baltimore truss, Fig. 59*i*.
7. Cantilever bridge, Fig. 59*j*. (See § 58, Problem 4.)



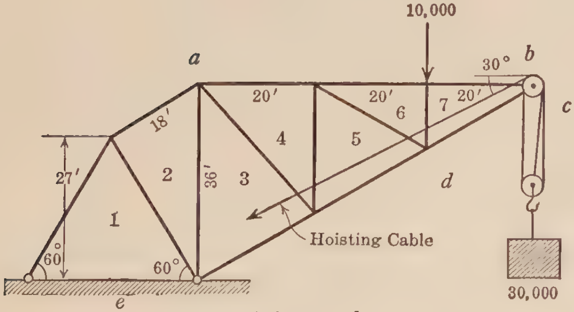
Loads in pounds

Fig. 59*f*.



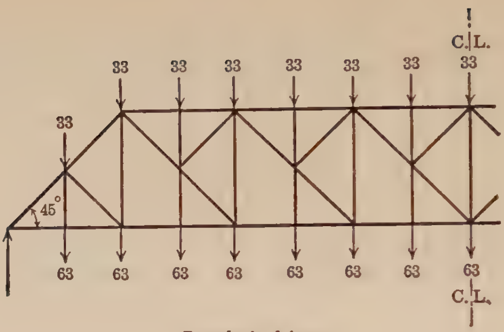
Loads in pounds

Fig. 59*g*



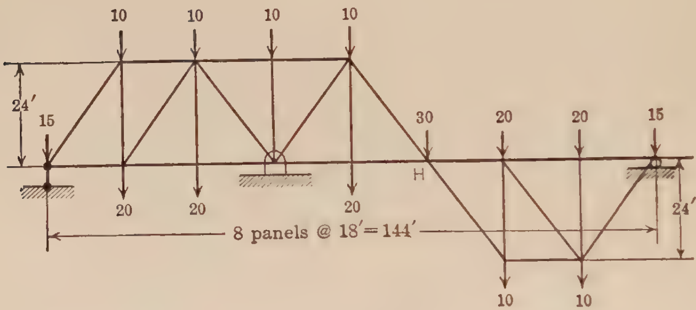
Loads in pounds

Fig. 59*h*.



Loads in kips

FIG. 59i.



Loads in kips

FIG. 59j.

60. The Method of Sections. If a truss is "cut" in two parts by any section through its bars, either part of the truss is in equilibrium under the external forces that act on it; these now include forces along the cut bars representing the action of the other part of the truss (§ 33). If the section meets, or cuts, but *three* bars, their stresses may be found by the methods of § 54. This method of computing stresses is particularly valuable when

(1) the truss is not simple and the method of joints offers difficulties; or when

(2) the stresses in certain bars only are required.

In addition the method of sections may be used to check certain stresses computed by the method of joints.

Example 1. The section through the Pratt truss of Fig. 58*b* cuts three bars whose stresses we denote by S_1, S_2, S_3. Assigning directions to these

forces as shown, we may determine them in turn from the equations

$$M_H = 0, \quad F_y = 0, \quad M_G = 0,$$

applied to the part of the truss to the right of the section. Taking the panel-length as the unit of length,

$$\begin{aligned}
M_H &= 55 \times 2 - 22 \times 1 - S_1 \tan \alpha = 0, \\
S_1 &= 88 \cot \alpha = 88 \times \tfrac{5}{6} = 73.3 \text{ kips}; \\
F_y &= 55 - 22 - 22 - S_2 \sin \alpha = 0, \\
S_2 &= \frac{11}{\sin \alpha} = \frac{11}{.768} = 14.3 \text{ kips}; \\
M_G &= 55 \times 3 - 22 \times 2 - 22 \times 1 - S_3 \tan \alpha = 0, \\
S_3 &= 99 \cot \alpha = 99 \times \tfrac{5}{6} = 82.5 \text{ kips}.
\end{aligned}$$

As the stresses are all positive the arrows have been correctly assigned. The index stresses of Fig. 58*b* lead to the same results.

Example 2. The section through the roof truss of Fig. 58*c* cuts three bars. Their stresses S_1, S_2, S_3 are determined in turn from the equations

$$M_G = 0, \quad M_H = 0, \quad M_E = 0,$$

applied to the part of the truss to the right of the section. If we take the panel-length as the unit of length, we have, for example

$$M_H = 6 \times 1 - S_2 \times 2 \sin \alpha = 0, \qquad S_2 = \frac{3}{\sin \alpha} = 3 \times \frac{5}{3} = 5 \text{ kips}.$$

Example 3. Compute the stress in the bar *17* of the roof-truss of Fig. 60*a* when subject to the wind load shown.

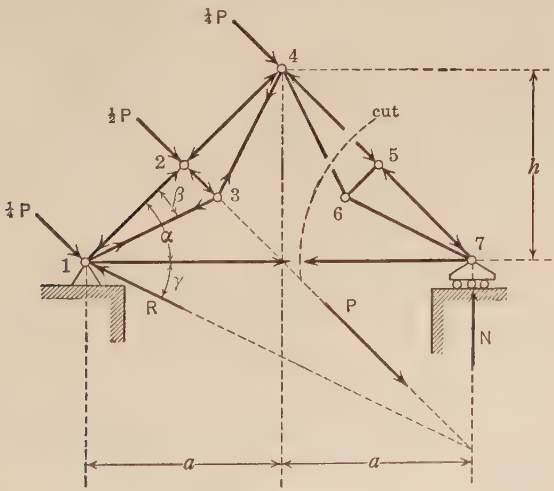


FIG. 60*a*.

If we cut the truss as shown and consider that part at the right of the section, we may obtain an equation involving only S_{17} and the reaction N

by taking moments about *4*. Moreover $\mathbf{S}_{17}$ must be a tension in order to balance the moment of $\mathbf{N}$ about 4; thus

$$Na - S_{17}h = 0, \quad S_{17} = N\frac{a}{h}.$$

To find N consider the equilibrium of the truss as a whole. Replacing the wind load by its resultant $\mathbf{P}$ acting through *2* and taking moments about *1*, we have

$$2\,aN - P\frac{\sqrt{a^2 + h^2}}{2} = 0, \quad N = \frac{P}{4}\frac{\sqrt{a^2 + h^2}}{a}.$$

Hence
$$S_{17} = \frac{P}{4}\frac{\sqrt{a^2 + h^2}}{h} = \frac{P}{4 \sin \alpha}.$$

Taking $P = 2000$ lb., $\alpha = 45°$, $\beta = 20°$, the student should now compute R, γ and the remaining stresses. The results are as follows:

$$R = 1581 \text{ lb.}, \quad \gamma = 26° \, 34', \quad N = 707 \text{ lb.}$$

Tensile Stresses: $S_{17} = 707$ lb., $S_{13} = S_{34} = 1462$ lb.
Compressive Stresses: $S_{12} = S_{24} = 1374$ lb., $S_{23} = 1000$ lb.,

$$S_{45} = S_{57} = 1000 \text{ lb.}, \quad S_{46} = S_{67} = S_{56} = 0.$$

The stresses in this truss may be computed entirely by the method of joints. In what order must the joints be taken in the calculation?

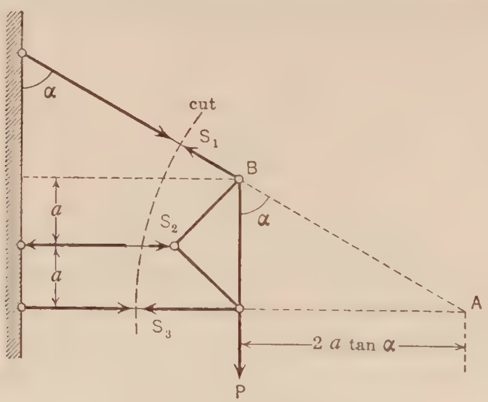


Fig. 60*b*.

Example 4. In order to find the stresses in the three bars supporting the triangle in Fig. 60*b*, take the section shown. Then, from the equilibrium of the triangle,

$$F_y = S_1 \cos \alpha - P = 0, \qquad S_1 = \frac{P}{\cos \alpha};$$
$$M_A = P \cdot 2\,a \tan \alpha - S_2 a = 0, \quad S_2 = 2\,P \tan \alpha;$$
$$M_B = S_2 a - S_3 \cdot 2\,a = 0, \qquad S_3 = P \tan \alpha.$$

PROBLEMS

1. Compute the stresses *m4*, *45*, *5i* in the roof truss of Fig. 59*f*.
2. Compute the stresses *b2*, *c4* and *d6* in the truss of Fig. 59*g*.
3. Compute the stress *a4* in the crane of Fig. 59*h*.

61. Members Subject to Non-Axial Stress. We have seen in § 55 that a bar of a structure in equilibrium under *two forces* is subject to an axial stress. Such a bar may always be "cut" provided that the twin forces of its stress are introduced as external forces at the section. But if a bar is in equilibrium under *three or more forces*, its internal forces at any cross-section do not in general reduce to an axial stress; *such a member must not be cut to form a free-body diagram.* When a member of this kind is connected to a structure by pins, a free-body diagram may be drawn for the member as a whole by introducing a reaction **R** at each pin. It is usually advantageous to replace **R** in the diagram by its horizontal and vertical projections; if **R** acts on the pin A, we shall denote the magnitudes of its projections by H_A and V_A. The direction of these projections of **R** may often be assigned by inspection. If, however, their directions are not obvious, they may be chosen at pleasure; an error in assigning an arrow is shown in the completed calculation by a negative sign in the corresponding H or V.

Example 1. Fig. 61*a* represents a crane supporting a load of 4000 lb. at the end of its boom. The weights of mast, boom and brace (BE) are 1000, 600 and 400 lb., respectively. We shall consider in turn the equilibrium of the crane as a whole, the boom, and the brace.

Crane.

$$F_x = H_A - H_D = 0, \quad F_y = V_A - 4000 - 1000 - 600 - 400 = 0,$$
$$M_A = 12\,H_D - 4000 \times 9 - 600 \times 4.5 - 400 \times 3 = 0.$$

From these equations we find

$$H_A = H_D = 3325 \text{ lb.}, \quad V_A = 6000 \text{ lb.}$$

Boom.

$$F_x = H_E - H_C = 0, \quad F_y = V_E - V_C - 4000 - 600 = 0,$$
$$M_C = 6\,V_E - 4000 \times 9 - 600 \times 4.5 = 0;$$

hence

$$H_C = H_E, \quad V_E = 6450 \text{ lb.}, \quad V_C = 1850 \text{ lb.}$$

Brace.

$$F_x = H_B - H_E = 0, \quad F_y = V_B - V_E - 400 = 0,$$
$$M_B = 6\,H_E - 6\,V_E - 400 \times 3 = 0;$$

hence

$$H_B = H_E, \quad H_E = V_E + 200 = 6650 \text{ lb.}, \quad V_B = V_E + 400 = 6850 \text{ lb.}$$

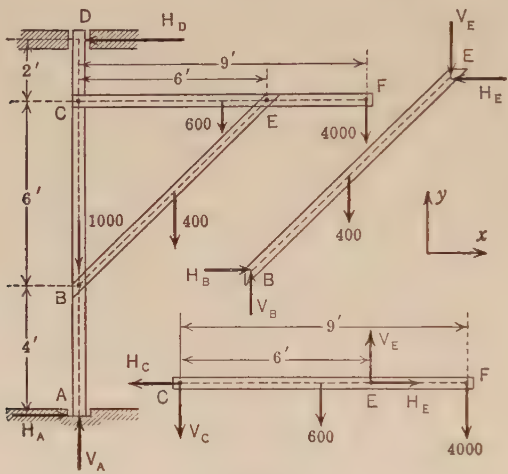


FIG. 61*a*.

The reactions on the mast are therefore

$$R_A = [3325, 6000], \quad R_D = [-3325, 0]$$

the reactions on the boom are

$$R_C = [-6650, -1850], \quad R_E = [6650, 6450];$$

and the reaction of the mast on the brace is

$$R_B = [6650, 6850].$$

The magnitude and direction of each reaction may now be computed from its components (§ 26). The components, however, are used directly in designing the members of the crane.

Example 2. A man weighing W lb. stands on a light stepladder ACB at a horizontal distance x from the hinge C (Fig. 61*b*). If the ladder stands on a smooth floor, find the tension in the tie DE and the reaction $\mathbf{R}_C$ of the ladder AC on the backstay BC.

Ladder as a whole.

$$M_B = W(x + b) - R_A(a + b) = 0, \quad R_A = \frac{x + b}{a + b} W;$$

$$M_A = R_B(a + b) - W(a - x) = 0, \quad R_B = \frac{a - x}{a + b} W.$$

Backstay.

$$F_x = H_C - T = 0, \quad F_y = R_B - V_C = 0, \quad M_C = R_B b - Tc = 0.$$

From these equations we find

$$H_C = T = \frac{b}{c} R_B, \quad V_C = R_B, \quad \mathbf{R}_C = \left[\frac{b}{c} R_B, \quad -R_B\right].$$

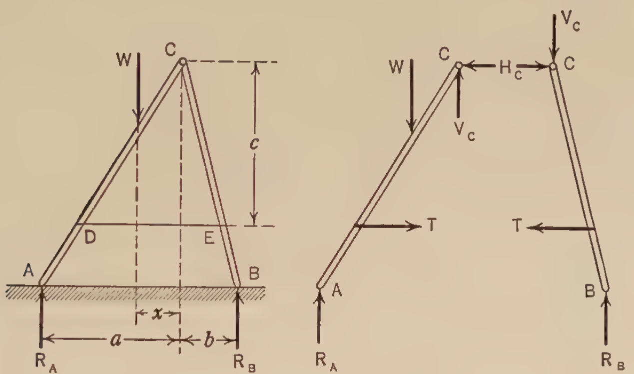


FIG. 1b.

Both T and R_C increase as the man climbs to the top of the ladder. The greatest tension (for $x = 0$) is $abW/(a + b)c$.

Since T decreases as c increases, the lower the tie is placed, the smaller will be its tension.

Example 3. Fig. 61c represents a symmetrically loaded three-hinged arch, the forces P being the resultants of all the loads on the parts *1* and *2* of the arch. By symmetry the hinge pressures at C must be horizontal; for if the pressure of *1* on *2* were $[H_C, V_C]$, the reaction of *2* on *1* would be $[-H_C, -V_C]$, and unless $V_C = 0$ the symmetry would be violated. From the free-body diagram for *1* we have

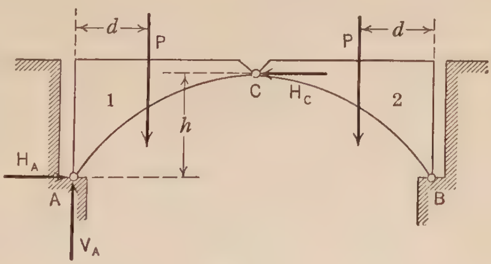


FIG. 61c.

$$H_A - H_C = 0, \quad V_A - P = 0, \quad M_A = H_C h - Pd = 0,$$

whence

$$H_A = H_C = \frac{d}{h} P, \quad V_A = P, \quad R_A = P\sqrt{1 + \frac{d^2}{h^2}}.$$

PROBLEMS

1. Solve Example 2, § 53, if the mast and boom of the crane weigh 1000 and 500 lb. respectively.

2. Solve Problem 3, § 53, if the mast, boom and brace of the crane weigh 800, 400 and 200 lb. respectively.

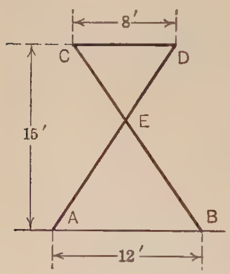


FIG. 61*d*.

3. In Fig. 61*d* the frame *ADCB* consists of three members *AD*, *DC*, *CB* hinged at *D*, *C* and *E*. If $AD = BC$ and the ends *A*, *B* rest on a smooth plane, find the components of the pin-pressure at *E* when a downward load of 1200 lb. acts on *CD*

(*a*) at its middle point;
(*b*) 2 ft. to the right of *C*;
(*c*) 5 ft. to the right of *C*;
(*d*) at *C*;
(*e*) 2 ft. to the right of *C* and 3 ft. to the left of *D* (two loads).

4. In Problem 3 prove that a downward load *P* on *CD* produces the same horizontal component of pin-pressure at *E* irrespective of its position.

5. In Fig. 61*e* the A-frame *ABCEF* consists of three members *AC*, *BC*, *EF* hinged at *A*, *C*, *E* and *F*. If the end *B* rests on a smooth plane, find the components of the pin-pressure at *C* and *E* when a downward load $P = 3000$ lb. acts on the tie *EF*

(*a*) 2 ft. to the right of *E*;
(*b*) 4 ft. to the right of *E*;
(*c*) 7 ft. to the right of *E*;
(*d*) at *F*.
(*e*) Prove that H_C is the same irrespective of the position of *P* on *EF*.

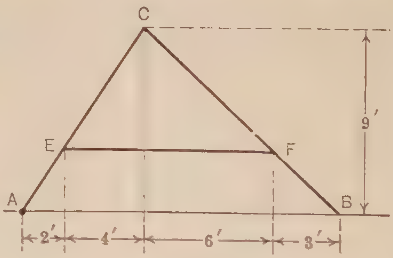


FIG. 61*e*.

6. In the A-frame of Fig. 61*e* a downward load of 3000 lb. acts on the member *AC*. Find the components of pin-pressure at *C* and *E* when the load acts

(*a*) midway between *E* and *C*;
(*b*) at *E*;
(*c*) at *C*. How is the load divided between *AC* and *BC*?

7. In the A-frame of Fig. 61*e* the member *AC* is prolonged beyond *C* an amount $CD = AC$. If a downward load of 3000 lb. acts at *D*, find the components of the pin-pressure at *C* and *E*.

8. In the A-frame of Fig. 61*e* a horizontal load of 3000 lb., to the right

acts on the member BC. Find the reactions at A, B and the components of the pin pressures at C and E when the load acts

(a) midway between C and F;
(b) at F;
(c) at C.
(d) Where must P act on BC so that $H_C = 0$?

62. Systems of Rigid Bodies. We have already dealt with systems of rigid bodies in our treatment of trusses and other framed structures. We give now a few more varied examples of such systems. No new principles or methods are employed in these problems. A free-body diagram should be drawn for each body and the equations of equilibrium should be applied to it. At the connections or surfaces of contact of two bodies, the Principle of Action and Reaction always comes into play.

Example 1. In the press represented by Fig. 62a what pressure P is exerted by the piston when a force Q is applied at the end of the lever OA in the position shown (making an angle θ with the vertical) and friction is neglected. Find also the stress S in the link BC and the total pressure N of the guides on the piston.

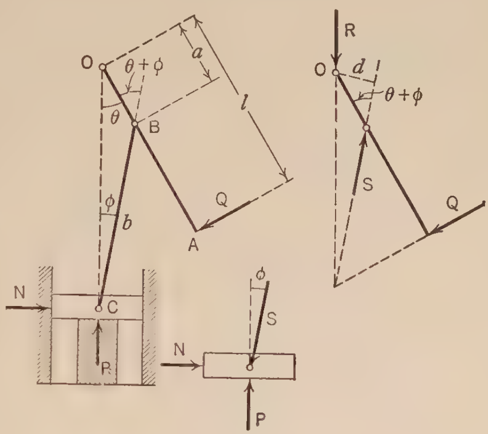


FIG. 62a.

Lever. The forces **Q**, **S**, and **R**, the reaction at the hinge, are in equilibrium; hence

$$M_O = Sd - Ql = 0, \quad S = \frac{l}{d}Q,$$

where

$$d = a \sin(\theta + \phi) \quad \text{and} \quad \sin\phi = \frac{a}{b}\sin\theta.$$

Piston. The forces **P**, **S**, and **N** are in equilibrium; hence

$$F_x = N - S\sin\phi = 0, \quad F_y = P - S\cos\phi = 0,$$

$$N = \frac{l}{d}\sin\phi\, Q, \qquad P = \frac{l}{d}\cos\phi\, Q.$$

If, for example, $a = 1$ ft., $b = 3$ ft., $l = 3$ ft., $\theta = 30°$ and $Q = 50$ lb.,

$$\sin\phi = \tfrac{1}{3}\sin 30° = 0.1667, \quad \phi = 9°\,36';$$

$$P = \frac{3\cos 9°\,36'}{\sin 39°\,36'}\,50 = 232 \text{ lb.}$$

Example 2. Fig. 62*b* represents schematically an arrangement of levers used in platform scales. From the free-body diagrams of the various parts we have the following equations.

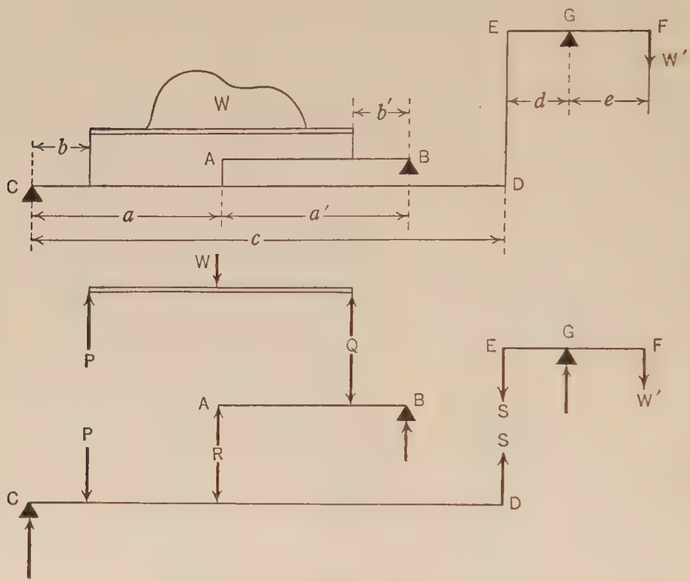


Fig. 62*b*.

Platform. $$P + Q = W.$$

Lever AB. $$M_B = Qb' - Ra' = 0, \quad R = \frac{b'}{a'}Q.$$

Lever CD. $$M_C = Sc - Pb - Ra = Sc - b\left(P + \frac{b'a}{a'b}Q\right) = 0.$$

If the scales are designed so that $a/b = a'/b'$,

$$Sc - (P + Q)b = Sc - Wb = 0, \quad S = \frac{b}{c}W,$$

so that the force S is independent of the P and Q. It is then immaterial how the load W is placed on the platform.

Lever EF. $$M_G = Sd - W'e = 0, \quad W' = \frac{d}{e}S = \frac{db}{ec}W;$$

which gives the relation between the load W and the "weight" W' needed to balance it. If we choose the dimensions so that $db/ec = 1/10$, the load W will always be ten times the weight W' that balances it.

Example 3. *Roberval's Balance.* In Fig. 62c the links AB and CD are pivoted at their centers O and P, and form with AC and BD a parallelogram in any position. We shall show that when the balance is equally loaded ($W = W'$) it will be in equilibrium in any position, no matter how the weights are placed in the scale-pans.

Balance ABCD and scale-pans. This is in equilibrium under **W**, **W′**, its own weight, and the reactions at the pins O and P. If the moment equation

(i) $M_P = W(a \cos\theta - x) - W'(a \cos\theta + x') + H_O c = 0$

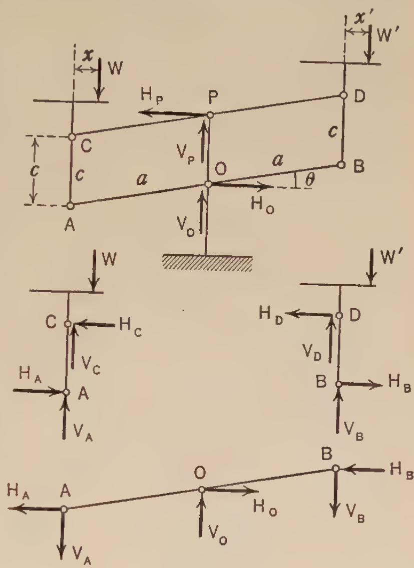


FIG. 62c.

is satisfied the reactions will adjust themselves so that the equations $F_x = 0$, and $F_y = 0$ are satisfied: $H_P = H_O$, $V_O + V_P = W + W' +$ Wt. of balance.

Link AC and pan.

$$M_C = H_A c - Wx = 0, \qquad H_A = \frac{x}{c} W.$$

Link BD and pan.

$$M_D = H_B c - W'x' = 0, \qquad H_B = \frac{x'}{c} W'.$$

Link AB.

$$F_x = H_O - H_A - H_B = 0, \quad H_O = \frac{xW + x'W'}{c}$$

Substituting this result in (i), we obtain the condition $W = W'$, for equilibrium, independently of the values of x, x' or θ.

Example 4. *Differential Pulleys.* In the differential chain hoist represented by Fig. 62*d* the upper block carries two pulleys keyed to the same axle and of slightly different radii, a and b. An endless chain passes over the pulleys as shown. The upper pulleys are grooved to receive the links of the chain so that it can not slip. In order to find the force P required to hold the weight W on the movable pulley we take a section

across the chains as shown and treat the lower and upper blocks as free bodies.

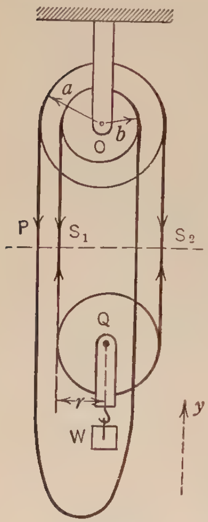


FIG. 62*d*.

Lower Block. The radius of the pulley is $r = \frac{1}{2}(a + b)$. The equations of equilibrium are

$$M_Q = S_2 r - S_1 r = 0, \quad F_y = S_1 + S_2 - W = 0;$$

hence

$$S_1 = S_2 = \tfrac{1}{2} W.$$

Upper Block. If the weight of the chain is neglected the tension in the slack chain is zero; hence

$$M_O = Pa + S_1 b - S_2 a = Pa - \tfrac{1}{2} W(a - b) = 0,$$

so that

$$P = \tfrac{1}{2} W \frac{a - b}{a}.$$

Example 5. Let us find the horizontal force P that must be applied to the wedge A in order to lift the block B carrying the weight Q when there is friction between all the rubbing surfaces (Fig. 62*e*).

Let ϕ, ϕ_1, ϕ_2 denote the angles of friction between A and B, *1* and A, *2*

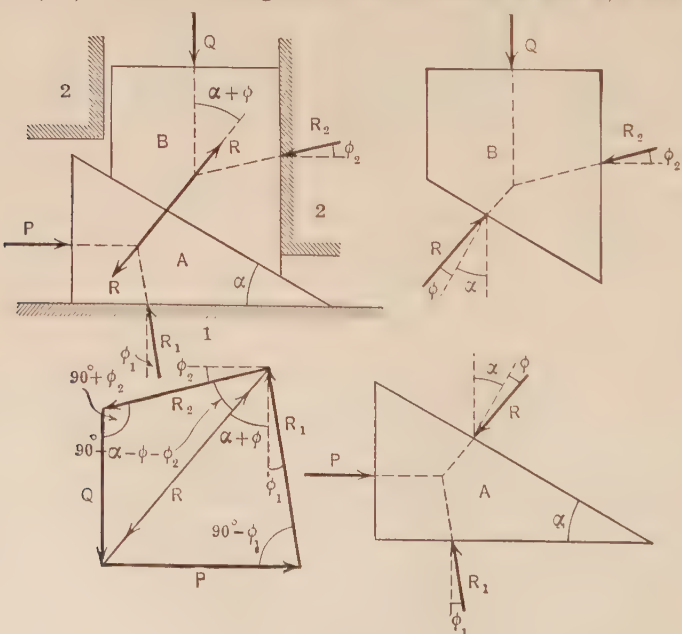


FIG. 62*e*.

and B respectively. The free-body diagrams for A and B show the forces when motion is impending; then the reactions $\mathbf{R}$, $\mathbf{R}_1$, $\mathbf{R}_2$ make angles ϕ, ϕ_1, ϕ_2 with the normals to the surfaces in contact and are inclined so that their tangential projections (the forces of friction) oppose the impending motion. In the figure the force triangles for A and B are drawn with the sides $\pm\mathbf{R}$ in common. The student should check carefully the values given for the angles. The Law of Sines applied to each triangle gives

$$\frac{P}{R} = \frac{\sin(\alpha + \phi + \phi_1)}{\cos\phi_1}, \quad \frac{R}{Q} = \frac{\cos\phi_2}{\cos(\alpha + \phi + \phi_2)}.$$

On multiplying these equations we find

$$\text{(i)} \qquad \frac{P}{Q} = \frac{\sin(\alpha + \phi + \phi_1)}{\cos(\alpha + \phi + \phi_2)} \cdot \frac{\cos\phi_2}{\cos\phi_1}.$$

Let us now find the least force P that will just support the weight Q. Since the impending motion is now the reverse of that in the problem just treated, the reactions $\mathbf{R}$, $\mathbf{R}_1$, $\mathbf{R}_2$ are now inclined to the normals at the

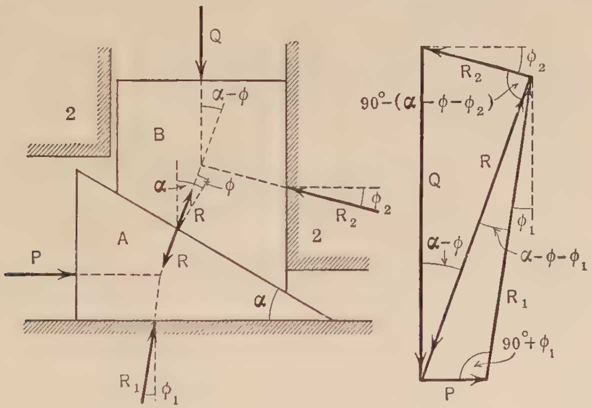


FIG. 62*f*.

same angles as before but on the opposite side (Fig. 62*f*). On applying the Law of Sines to the force-triangles for A and B we find

$$\frac{P}{R} = \frac{\sin(\alpha - \phi - \phi_1)}{\cos\phi_1}, \quad \frac{R}{Q} = \frac{\cos\phi_2}{\cos(\alpha - \phi - \phi_2)},$$

and hence

$$\text{(ii)} \qquad \frac{P}{Q} = \frac{\sin(\alpha - \phi - \phi_1)}{\cos(\alpha - \phi - \phi_2)} \cdot \frac{\cos\phi_2}{\cos\phi_1}.$$

The force-diagram shown is drawn for the case $\alpha > \phi + \phi_1$. When $\alpha = \phi + \phi_1$ we find that $\mathbf{R}_1$ just balances $\mathbf{R}$ and that $P = 0$; and when

$\alpha < \phi + \phi_1$ the impending motion of A to left demands a force **P** directed to the left. In these two cases the wedge A is *self-locking*, i.e. if the force **P** is removed, the wedge A will not be driven out by the force **Q**, no matter what its value.

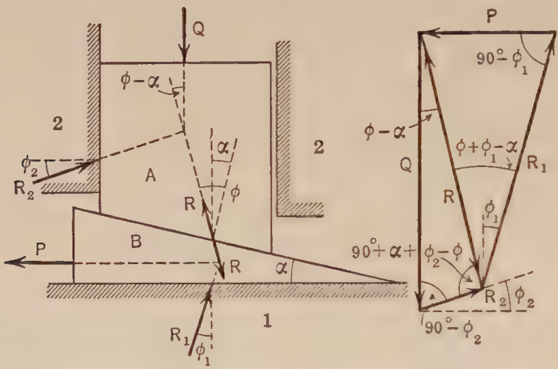


FIG. 62g.

Lastly let us find what force P directed to the left is needed to remove a self-locking wedge ($\alpha \leqq \phi + \phi_1$) when loaded as shown (Fig. 62g). From the two force-triangles (drawn for the case $\alpha < \phi$) we find

$$\frac{P}{R} = \frac{\sin(\phi + \phi_1 - \alpha)}{\cos \phi_1}, \quad \frac{R}{Q} = \frac{\cos \phi_2}{\cos(\alpha + \phi_2 - \phi)},$$

(iii) $$\frac{P}{Q} = \frac{\sin(\phi + \phi_1 - \alpha)}{\cos(\alpha + \phi_2 - \phi)} \cdot \frac{\cos \phi_2}{\cos \phi_1}.$$

The student should also draw the figure when $\phi < \alpha < \phi + \phi_1$; (iii) also holds in this case.

These results may be applied to the cotter-joint shown in Fig. 62h in which the members B and C are joined by means of the cotter-pin A. The tensile stress Q in B takes the place of the weight Q in the previous problems. If we assume that the friction at all the rubbing surfaces is the same, so that $\phi = \phi_1 = \phi_2$, we find from (i) that

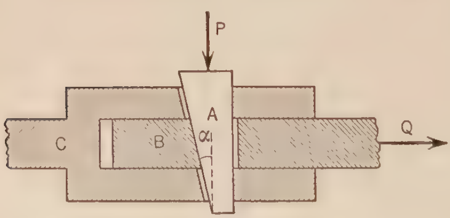


FIG. 62h.

$$P = Q \tan(\alpha + 2\phi)$$

is the force necessary to drive in the pin against the stress Q. The pin will be self-locking when $\alpha \leqq 2\phi$. In this case we find from (iii) that the

force

$$P = Q\frac{\sin (2\,\phi - \alpha)}{\cos \alpha}$$

is required to remove the pin from the joint.

PROBLEMS

1. In Fig. 62i the blocks A and B weigh 100 and 200 lb. respectively. If $\mu_{CB} = \frac{1}{3}$, $\mu_{AB} = \frac{1}{4}$, what horizontal force P applied to B will just cause slipping? What is the tension T in the rope AD at this instant?

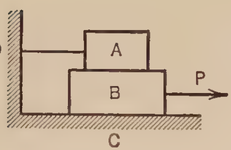


FIG. 62i.

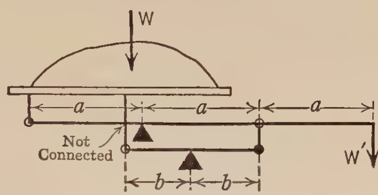


FIG. 62j.

2. With the arrangement of levers shown in Fig. 62j show that for equilibrium $W' = \frac{1}{2}W$ no matter how the load W is placed on the platform.

3. If in Fig. 62e, $Q = 1000$ lb., $\alpha = 20°$, $\mu_{1A} = 0.20$, $\mu_{AB} = 0.25$, $\mu_{2B} = 0.30$, what force P will just raise Q?

Show that the wedge A is self-locking and find the force P directed to the left that will just remove it.

4. What force P must be applied to the wedge in Fig. 62k in order to just overcome the forces of 1000 lb. on the block if $\mu_{AC} = \mu_{BC} = \mu_{AB} = 0.15$?

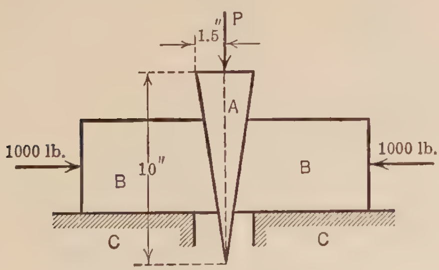


FIG. 62k.

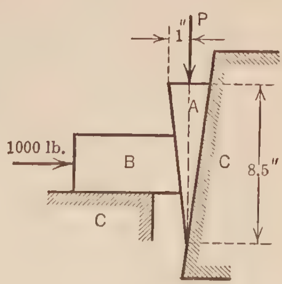


FIG. 62l.

5. Find the force P on the wedge in Fig. 62l that will just drive the blocks apart if $\mu_{AB} = \mu_{BC} = 0.1$

6. With the system of pulleys shown in (*a*) Fig. 62*m*, (*b*) Fig. 62*n*, what force P will support a weight of $W = 800$ lb. if each pulley weighs 25 lb.?

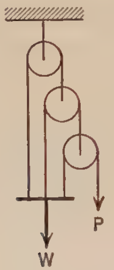


FIG. 62*m*.

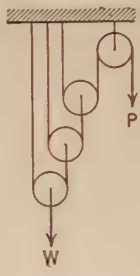


FIG. 62*n*.

63. Cone of Friction. Consider a particle P at rest on a rough plane, and let ϕ denote the angle of friction between particle and plane. The reaction of the plane on the particle can make at most the angle ϕ with the normal n to the plane at P. Hence if with P as vertex and n as axis we describe a right circular cone of semi-angle ϕ (Fig. 63*a*), the line of action of the reaction must lie on or within the cone. This cone is called the *cone of friction* at P. In plane figures it is represented by two lines making angles of ϕ on either side with the normal.

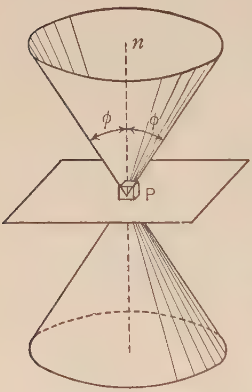


FIG. 63*a*.

Similarly if two rough surfaces may be regarded as having point contact at P, the cone of friction at P is again a cone of semi-angle ϕ described about their common normal; the reaction-line of either surface on the other must lie on or within this cone.

The use of the cone of friction in the solution of problems is shown in the following examples.

Example 1. Consider a bar AB with its ends resting on rough horizontal and vertical planes (Fig. 63*b*). The cones of friction at A and B are drawn with semi-angles ϕ, ϕ' equal to the respective angles of friction. The reactions at these points must have lines included within these cones. If the bar is in equilibrium under a force **P** and the reactions **R**, **R**′ at A and B, the line of action of **P** must pass through the point where the lines of **R** and **R**′ meet. But since all possible points of intersection of **R** and **R**′ are included in the quadrilateral $EFGH$, equilibrium is only

possible when the line of **P** crosses this quadrilateral. When it does, as in the figure, the reactions may meet on any point of the segment UV. Just which point of UV should be chosen can not be determined by the principles of Statics. In other words, the reactions **R** and **R′** are *statically indeterminate.*

The vertices E and G of the quadrilateral correspond to states of impending motion in which θ tends to increase or decrease respectively. When P passes through E or G the reactions are uniquely determined.

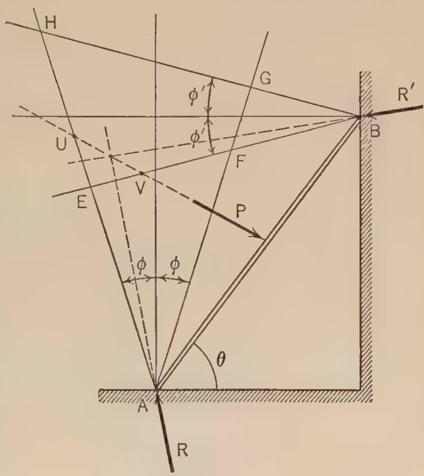


FIG. 63*b*.

Example 2. In Fig. 63*c* AB represents a ladder supported by a rough floor and wall. As a man climbs the ladder, let the resultant of his weight and that of the ladder be denoted by **W** and act at a distance x to the right of A. Let us find the value of x when the ladder begins to slip, and also the reactions at this point.

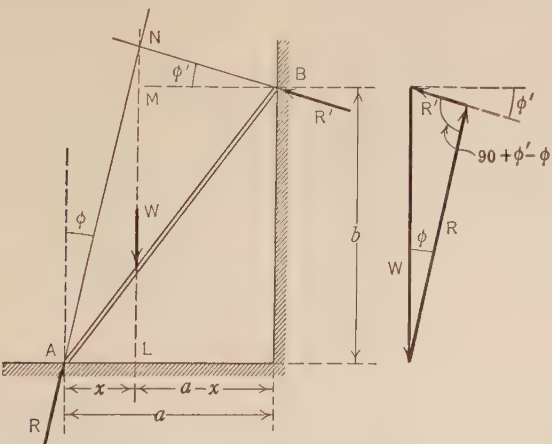


FIG. 63*c*.

At the point of slipping, the impending motions of A and B are to the left and downward. The reactions **R**, **R′** at this point are drawn as shown in order to oppose the impending motion and at an angle with the normal equal to the corresponding angle of friction. As soon as the line of **W**

moves to the right of N, the point where the limiting reactions meet, the reactions can no longer equilibrate **W** and the ladder will slip. Hence when **W** passes through N the ladder is on the point of slipping. From the figure

$$b = LN - MN = \frac{AL}{\tan\phi} - MB\tan\phi',$$

$$b = \frac{x}{\mu} - (a - x)\mu' = x\frac{1 + \mu\mu'}{\mu} - a\mu',$$

whence

$$x = \frac{b\mu + a\mu\mu'}{1 + \mu\mu'}.$$

Thus if $a = 4$ ft., $b = 8$ ft., $\mu = \frac{1}{4}$, $\mu' = \frac{1}{3}$,

$$x = \frac{2 + 1/3}{1 + 1/12} = \frac{28}{13} = 2.15 \text{ ft.}$$

From the force-triangle **W**, **R**, **R**′ we have

$$\frac{R}{\cos\phi'} = \frac{R'}{\sin\phi} = \frac{W}{\cos(\phi' - \phi)}.$$

Example 3. A bracket for supporting a lamp and shade slides on an upright cylindrical rod. (Fig. 63*d*.) If the weight **W** of lamp and shade acts at a distance x from the axis of the rod, find the least value of x before the bracket will slip.

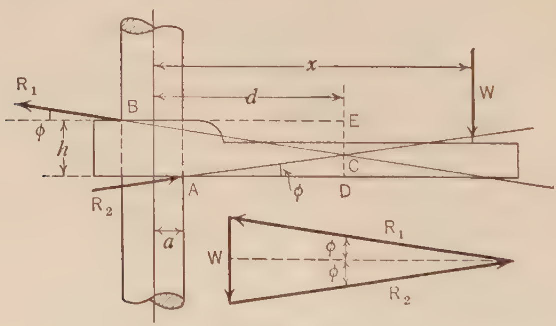


Fig. 63*d*.

Under the action of **W** the bracket will have approximately point contact with the rod at A and B. Drawing the extreme elements AC and BC of the cones of friction at A and B which correspond to impending downward motion, we see that **W** can be equilibrated by the reactions at A and B provided it does not act to left of C. Hence if the value of x for which sliding impends is denoted by d and the length of the bracket-

sleeve by h,

$$h = EC + CD = BE \tan\phi + AD \tan\phi,$$

$$h = (d+a)\mu + (d-a)\mu = 2\,d\mu, \quad d = \frac{h}{2\,\mu}.$$

Thus if $h = 2$ in., $\mu = 0.1$, $d = 2/0.2 = 10$ in.

From the force-triangle

$$\frac{W}{2} = R_1 \sin\phi, \quad R_1 = R_2 = \frac{W}{2\sin\phi}.$$

PROBLEMS

1. Solve Example 2 from the three scalar equations of equilibrium.

2. Do the same for Example 3.

3. A man weighing 150 lb. climbs to the top of the ladder of Fig. 63*c*. If the ladder weighs 50 lb. and the coefficients of friction at floor and wall are $\frac{1}{2}$ and $\frac{1}{3}$, find the least possible inclination of the ladder to the horizontal.

4. Of three equal cylinders A, B, C with parallel axes, A and B lie in contact on a rough plane, while C rests on top of them. Show that the angles of friction involved must at least equal 15° in order to preserve equilibrium.

64. Journal Friction. Fig. 64*a* represents a journal of radius r in a cylindrical bearing of slightly greater radius. Suppose that the journal, when on the point of revolving in the direction shown, under the action of certain impressed forces, is in contact with the bearing along the line through A. The reaction **R** of the bearing on the journal will then be inclined to the common normal AO to the surfaces at an angle ϕ, the angle of friction, so that the friction **F** opposes the impending motion. If OB is drawn perpendicular to **R**, we have

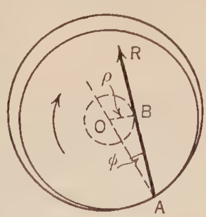


Fig. 64*a*.

$$\rho = OB = OA \sin\phi = r \sin\phi;$$

hence for impending motion the reaction **R** is tangent to a circle of radius $\rho = r\sin\phi$. This circle is called the *circle of friction.* When the bearing is lubricated ϕ is small and $\sin\phi$ is nearly equal to $\tan\phi$ or μ; in this case the radius of the friction circle is very nearly μr.

If a bar having pin connections at two points *1* and *2* is in equilibrium under two forces transmitted through the pins (this im-

plies that the weight of the bar is neglected), these forces must balance and therefore act along the same line. If pin-friction is considered, the limiting positions of this line are given by the four common tangents to the friction circles at *1* and *2* (Fig. 64*b*). If μ is small at the joints, the radii of these circles are also small and it is customary to assume that the pin reactions act along the line *12* through the centers of the pins. This was done, for example, in dealing with the stresses in a truss.

FIG. 64*b*.

Example 1. If the shaft of the bell-crank shown in Fig. 64*c* has the radius $r = 1$ in. and the coefficient of axle-friction is $\mu = 0.2$, what force P will just set the crank in motion against a resistance of $Q = 100$ lb.?

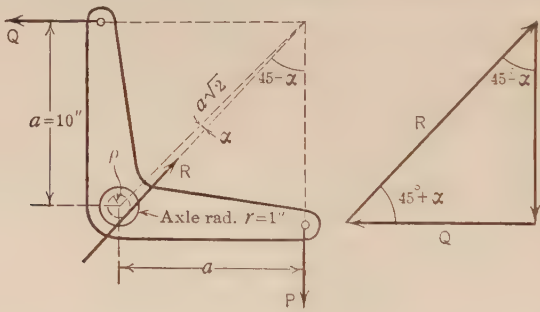


FIG. 64*c*.

The reaction **R** of the bearing must pass through the point where the lines of **P** and **Q** meet. Since **R** is also tangent to the friction circle and must exert a counterclockwise moment about the axis of the shaft in order to resist the impending clockwise rotation of the crank, **R** must act as shown. From the force-triangle

$$\frac{P}{\sin (45° + \alpha)} = \frac{Q}{\sin (45° - \alpha)} .$$

Taking $\rho = r\mu$ as the radius of the friction circle,

$$\sin \alpha = \frac{r\mu}{a\sqrt{2}} = \frac{1 \times 0.2}{10 \times \sqrt{2}} = 0.01414, \quad \alpha = 50',$$

and hence

$$P = \frac{100 \sin 45° \, 50'}{\sin 44° \, 10'} = 103 \text{ lb.}$$

Is the approximation $\rho = r\mu$ justified in this problem?

Example 2. Fig. 64*d* represents an arrangement for raising a weight W by turning the crank of the windlass below. What force P must be applied at the end of the crank in order to just start a weight of 500 lb. moving upward when $a = 12$, $b = 24$, $c = 10$, $d = 25$, $r_1 = 1.5$, $r_2 = 1$ in., $\mu_1 = 0.1$, $\mu_2 = 0.15$?

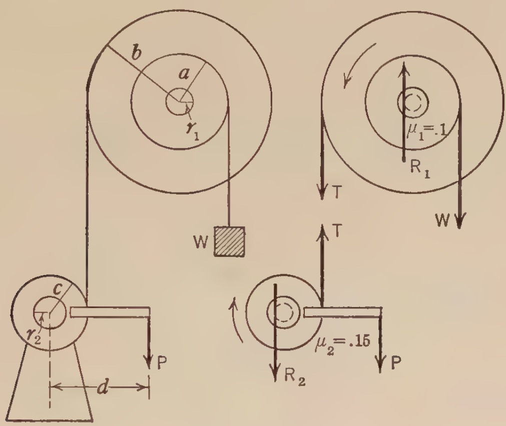


Fig. 64*d*.

Let T be the tension in the rope. The reaction $\mathbf{R}_1$ must act upward to balance $\mathbf{T}$ and $\mathbf{W}$. $\mathbf{R}_1$ is tangent to the friction circle of radius $\rho_1 = r_1\mu_1$ to the left to oppose the impending counterclockwise rotation. In the free-body diagram of the pulley, take moments about a point of $\mathbf{R}_1$; then

$$T(b - \rho_1) - W(a + \rho_1) = 0, \quad T = \frac{a + \rho_1}{b - \rho_1} W.$$

Again since P is clearly less than T, the reaction $\mathbf{R}_2$ must act downward to balance $\mathbf{T}$ and $\mathbf{P}$. $\mathbf{R}_2$ is tangent to the friction circle of radius $\rho_2 = r_2\mu_2$ to the left to oppose the impending clockwise rotation. In the free-body diagram of the windlass take moments about a point of $\mathbf{R}_2$; then

$$T(c + \rho_2) - P(d + \rho_2) = 0,$$

$$P = \frac{c + \rho_2}{d + \rho_2} T = \frac{(c + \rho_2)(a + \rho_1)}{(d + \rho_2)(b - \rho_1)} W.$$

With the numerical values given above we have

$$\rho_1 = 1.5 \times 0.1 = 0.15 \text{ in.}, \quad \rho_2 = 1 \times 0.15 = 0.15 \text{ in.},$$

$$P = \frac{10.15 \times 12.15}{25.15 \times 23.85} 500 = 102.8 \text{ lb.}$$

With no axle friction $P = 100$ lb.

PROBLEMS

1. A weight W hangs from a rope that passes over a pulley of radius a mounted on a horizontal axle of radius r. Show that the least force, applied downward at the other end of the rope, that will raise the weight is

$$P = \frac{a + r \sin \phi}{a - r \sin \phi} W,$$

where ϕ is the angle of axle friction.

Compute P when $a = 1$ ft., $r = 1.5$ in., $\mu = 0.2$, and $W = 100$ lb.

2. Find the tension T in the rope in Fig. 64*d* when $\mu_1 = 0.4$. The other numerical data are as given in Example 2. May we take $\rho_1 = \mu_1 r_1$ in this case?

3. A flywheel weighing 2000 lb. is mounted on a horizontal shaft which turns in 3-inch bearings. What torque T applied to the shaft will just set it in motion if its center of gravity is on its axis and $\mu = 0.05$?

4. In Fig. 64*e* find the force P that will just cause the 1000-lb. weight W to slide up the plane. The pulleys A, B, of radii 12 and 6 in., are keyed to an axle of radius 2 in. At the bearing $\mu = 0.15$, on the plane $\mu' = 0.25$.

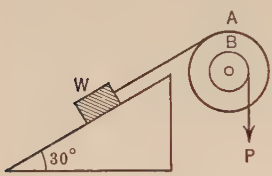


FIG. 64*e*.

65. Summary, Chapter IV. If $\mathbf{F}'$ is the projection of a force $\mathbf{F}$ on a plane normal to the axis s, the *moment of* $\mathbf{F}$ *about* s is defined as the product of the magnitude of $\mathbf{F}'$ and its perpendicular distance from s, taken positive or negative in agreement with the sense of rotation about s indicated by $\mathbf{F}'$. The moment of $\mathbf{F}$ about an axis s is equal to the component of the vector $\mathbf{r}\times\mathbf{F}$ on this axis, where $\mathbf{r}$ is any vector from a point on s to a point on the force's line of action.

The sum of the moments of a set of *concurrent* forces about any axis is equal to the moment of their resultant about this axis. If two systems of forces are equivalent, each system has the same force-sum and moment-sum about any given axis.

A pair of forces equal in magnitude, opposite in direction, and having different (parallel) lines of action is called a *couple.* A couple has the same moment about all parallel axes in the same direction. The moment of a couple about any axis normal to its plane is numerically equal to the magnitude of either force times the perpendicular distance between the forces.

If a system of coplanar forces has the force-sum $\mathbf{F}$ and the moment-sum M_A about the point A, it may be reduced to

(1) a *single force* if $\mathbf{F} \neq 0$;
(2) a *couple* if $\mathbf{F} = 0, \quad M_A \neq 0$;
(3) *zero* if $\mathbf{F} = 0, \quad M_A = 0$.

These reductions may be carried out graphically with the aid of a *funicular polygon.*

The conditions $\mathbf{F} = 0$, $M_A = 0$ are necessary and sufficient for the equilibrium of a system of coplanar forces. These conditions are equivalent to any one of the following sets of scalar equations:

$$F_x = 0, \quad F_y = 0, \quad M_A = 0;$$
$$F_x = 0, \quad M_A = 0, \quad M_B = 0 \quad (x \text{ not } \perp AB);$$
$$M_A = 0, \quad M_B = 0, \quad M_C = 0 \quad (A, B, C \text{ not in a line}).$$

Three non-parallel coplanar forces are in equilibrium only when they form a closed triangle and their lines of action meet in a point.

The stresses in the members of a *simple* truss may be found systematically by the *method of joints;* in this method the reactions at the supports are first computed and the equations of equilibrium ($F_x = 0$, $F_y = 0$) set up for each joint-pin of the truss. When the diagonals of a truss are all inclined at the same angle to the horizontal, this method may be greatly shortened by the use of *index stresses.* The method of joints may also be carried out graphically by drawing the force-polygon for each joint. These force-polygons, when combined into a single diagram, in which each stress is represented by a single segment, constitute the *Maxwell diagram* for the truss.

In the *method of sections* the truss is divided into two parts by a section cutting just three bars; the equations of equilibrium for either part will then determine the stresses in the members cut.

In making sections through a structure only members subject to *axial stress* should be cut. If a pin-connected member is subject to non-axial stress, a free-body diagram should be drawn for the member as a whole; in this case it is best to replace the reactions at the pins by their horizontal and vertical projections.

In dealing with the equilibrium of systems of rigid bodies, the free-body diagram of each body should be drawn. The Principle

of Action and Reaction comes into play where the bodies exert contact forces on each other.

If a journal of radius r has line contact with a cylindrical bearing just as rotation is impending, the reaction of the bearing on the journal is tangent to the *circle of friction* of radius $r \sin \phi$ (ϕ = the angle of friction) described about the center of the journal.

PROBLEMS

1. The center of gravity of a safety-valve lever weighing 16 lb. is 12 in. from the hinge; the valve is 4 in. in diameter, weighs 8 lb. and its stem is 4 in. from the hinge. What steam pressure will open the valve if the lever carries an 80-lb. weight 28 in. from the hinge?

2. A uniform beam 20 ft. long and weighing 200 lb. rests with its middle point on top of a rough circular cylinder 4 ft. in diameter. Find the greatest weight that can be hung from one end of the beam without causing it to slip off if $\mu = 0.3$.

3. Fig. 65a shows a frame supporting a pulley 2 ft. in diameter. A rope, fastened to the frame at F, passes over the pulley and supports a weight of 400 lb. Find the stress in AE.

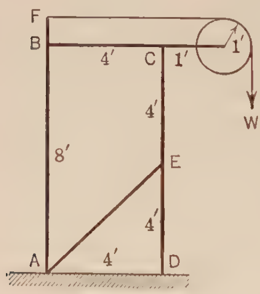


FIG. 65a.

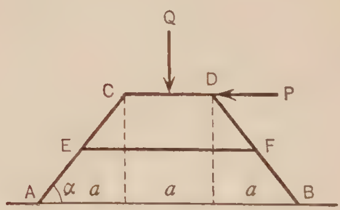


FIG. 65b.

4. In the trapezoidal frame of Fig. 65b the end A is hinged while B rests on a smooth plane. If $AE = EC = DF = FB$, show that

(a) the horizontal load P at D produces a tension of $\frac{2}{3} P$ in the tie EF;

(b) the vertical load Q at the center of CD produces a tension of $Q \cot \alpha$ in EF.

5. Four equal and uniform bars of weight W are jointed at their ends to form a parallelogram $ABCD$. If this frame is stiffened by a rod BD of negligible weight and suspended from A, find the pin pressures at A and C and the stress in BD, given that AB is inclined α to the vertical.

6. What head h of water can be supported by the swing valve shown in Fig. 65c? Water weighs 62.5 lb./ft.[3]

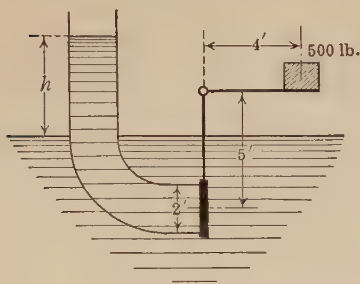


FIG. 65c.

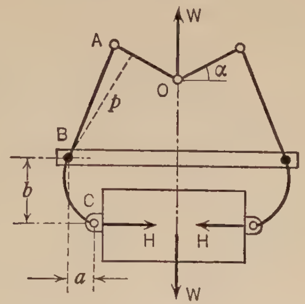


FIG. 65d.

7. In the grip for lifting stones, shown schematically in Fig. 65d, the link ABC is pivoted at A, B and C. Show that the horizontal gripping forces

$$H = \frac{1}{2} W\left(\frac{p}{b \sin \alpha} - \frac{a}{b}\right)$$

and that the coefficient of friction at the grips must equal or exceed the reciprocal of the quantity in parenthesis.

What is the least value of μ when $\alpha = 30°$, $p = 3\,b$, $a = \frac{1}{2}\,b$? What is the ratio H/W in this case?

8. Two circular cylinders of radii a, b are pressed together by means of a closed belt passing around them. If the belt tension is T, show that the pressure between the cylinders is

$$P = 4\,T\frac{\sqrt{ab}}{a + b}.$$

CHAPTER V

STATICS IN THREE DIMENSIONS

66. Moment of a Force about a Point. In § 44 we proved the

THEOREM. *The moment of a force* $\mathbf{F}$ *about an axis* s *is equal to the component of the vector* $\mathbf{r}\times\mathbf{F}$ *on this axis, where* $\mathbf{r}$ *is any vector from a point on the axis to a point on the force's line of action.* Thus if $\mathbf{e}$ is a unit vector in the direction of s,

$$(1) \qquad M_s = \mathbf{e}\cdot\mathbf{r}\times\mathbf{F} = \mathbf{e}\cdot\overrightarrow{AP}\times\mathbf{F}$$

where A is any point on s and P any point on the force's line of action. From (1) we can compute the moment of $\mathbf{F}$ about any axis through A as soon as the vector $\overrightarrow{AP}\times\mathbf{F}$ is known for this point. For this reason $\overrightarrow{AP}\times\mathbf{F}$ is called the *moment of* $\mathbf{F}$ *about the point* A.

Definition. The moment of a force $\mathbf{F}$ *about a point* A *is the vector* $\overrightarrow{AP}\times\mathbf{F}$, *where* P *is any point on the action-line of* $\mathbf{F}$. It is obvious that shifting a force along its line of action does not alter its moment about a point.

The moment of a force about a point A may be regarded as a vector indicating the turning effect of the force on a rigid body having the point A fixed; its magnitude is the measure of this effect, while its direction gives the instantaneous axis and sense of rotation it tends to produce.

If the force $\mathbf{F}$ is not zero, its moment about A will vanish only when $\overrightarrow{AP}$ is zero or is parallel to $\mathbf{F}$; in either case $\mathbf{F}$ passes through A. Therefore:

The moment of a force about a point vanishes when, and only when, its line of action passes through the point.

If we denote the moment of $\mathbf{F}$ about A by $\mathbf{M}_A$, we may write (1) as

$$(2) \qquad M_s = \mathbf{e}\cdot\mathbf{M}_A,$$

it being understood that A is a point on s. We restate this important result as follows:

The moment of a force $\mathbf{F}$ *about a* POINT A *is the* VECTOR $\mathbf{M}_A = \overrightarrow{AP}\times\mathbf{F}$; *the moment of* $\mathbf{F}$ *about any* AXIS s *through* A *is the* SCALAR *component of* $\mathbf{M}_A$ *on* s.

Example. In the example of § 44 the moment of $\mathbf{F}$ about A is $[7, -1, -4]$. If axes are drawn through A in the direction of the x-, y-, and z-axes, the moments of $\mathbf{F}$ about these axes are respectively $7, -1, -4$.

67. Theorem of Moments. *The sum of the moments of a system of concurrent forces about any point is equal to the moment of their resultant about this point* (cf. § 46).

Proof. Let the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots$, be concurrent at the point P; then if $\mathbf{r} = \overrightarrow{AP}$, the sum of their moments about A is

$$\mathbf{r}\times\mathbf{F}_1 + \mathbf{r}\,\mathbf{F}_2 + \cdots = \mathbf{r}\times(\mathbf{F}_1 + \mathbf{F}_2 + \cdots) = \mathbf{r}\times\mathbf{R}$$

where $\mathbf{R}$ denotes the resultant. As $\mathbf{R}$ acts through P, $\mathbf{r}\times\mathbf{R}$ is the moment of $\mathbf{R}$ about A.

68. Force-sum and Moment-sum. If $\mathbf{F}_1, \mathbf{F}_2, \ldots$ are a system of forces acting at the points $P_1, P_2, \ldots$, the vectors

$$(1) \qquad \mathbf{F} = \sum \mathbf{F}_i, \qquad \mathbf{M}_A = \sum \overrightarrow{AP_i}\times\mathbf{F}_i$$

are called the *force-sum* and *moment-sum about* A of the system. The importance of these vectors is due to the following

THEOREM 1. *If* $\mathbf{F}$ *and* $\mathbf{M}_A$ *are the force-sum and moment-sum about* A *for a system of forces, and* $\mathbf{F}'$ *and* $\mathbf{M}_A'$ *are the corresponding sums for an equivalent system, then*

$$\mathbf{F} = \mathbf{F}', \quad \textit{and} \quad \mathbf{M}_A = \mathbf{M}_A'.$$

Proof. In the reduction of one system to the other by means of Principles A and B the force-sum is not altered (§ 28). Moreover the moment-sum about A is neither altered by applying Principle A (Theorem of Moments, § 67) nor by applying Principle B (cf. § 46).

Let us now compare the moment-sums of a given system of forces about two points A, B. We have

$$\mathbf{M}_B = \sum \overrightarrow{BP_i}\times\mathbf{F}_i = \sum(\overrightarrow{BA} + \overrightarrow{AP_i})\times\mathbf{F}_i = \overrightarrow{BA}\times\sum\mathbf{F}_i + \sum\overrightarrow{AP_i}\times\mathbf{F}_i,$$

or in view of (1),

$$(2) \qquad \mathbf{M}_B = \mathbf{M}_A + \overrightarrow{BA}\times\mathbf{F}.$$

If, in particular, the system consists of a single force **F**, this relation shows how its moment varies from point to point.

From (2) we see that the moment-sum is the same for all points of a line parallel to **F**. But $\mathbf{M}_B = \mathbf{M}_A$ *for any choice of* B only when $\mathbf{F} = 0$; hence we have

THEOREM 2. *The moment-sum of a system of forces is the same for all points when, and only when, the force-sum is zero.*

69. Moment of a Couple. Since the force-sum of a couple is zero,

A couple has the same moment-sum about all points of space (Theorem 2, § 68).

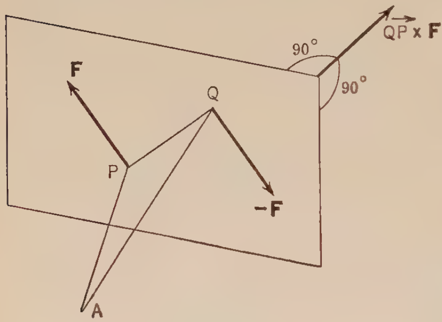


FIG. 69.

The moment-sum of a couple is simply called its *moment*. To show directly that the moment of a couple is the same about all points, let us compute it for the point A. Thus if P, Q are any points on the lines of action of **F** and $-\mathbf{F}$ (Fig. 69),

$$\mathbf{M}_A = \overrightarrow{AP}\times\mathbf{F} + \overrightarrow{AQ}\times(-\mathbf{F}) = (\overrightarrow{AP} - \overrightarrow{AQ})\times\mathbf{F} = \overrightarrow{QP}\times\mathbf{F},$$

a result entirely independent of the choice of A. This result is most simply obtained by taking moments about Q.

The moment $\overrightarrow{QP}\times\mathbf{F}$ of the couple is a free vector perpendicular to its plane in the direction in which a right-hand screw would advance when turned in the sense indicated by the couple; its magnitude is the product of F into the *arm* of the couple (§ 47).

Example. In Fig. 70*d* the forces **U**, **V** and **T**, **S** form couples. On taking moments about O we find that their moments are

$$\overrightarrow{OP}\times\mathbf{U} = (4\,\mathbf{i} + 3\,\mathbf{j})\times 2\,\mathbf{k} = 6\,\mathbf{i} - 8\,\mathbf{j},$$

$$\overrightarrow{OQ}\times\mathbf{S} = 2\,\mathbf{k}\times(-3\,\mathbf{j} - 4\,\mathbf{i}) = 6\,\mathbf{i} - 8\,\mathbf{j}.$$

Thus the couples **U**, **V** and **T**, **S** have equal moments. If the units of force and length are pounds and feet, the magnitude of the moments is $\sqrt{36 + 64} = 10$ lb.-ft. In **U**, **V** the forces are 2 lb., the arm 5 ft.; in **T**, **S** the forces are 5 lb., the arm 2 ft.

70. Reduction of Forces Acting on a Rigid Body. In § 48 we proved that a system of coplanar forces acting on a rigid body could always be reduced to a single force, to a couple, or to zero. We shall now consider the reduction of any system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots$ acting at the points $P_1, P_2, \ldots$ of a rigid body.

Choose a plane not passing through any of the points P_i and take three points A, B, C, not in a straight line, in this plane. Then the force $\mathbf{F}_1$ at P_1 can be expressed as the sum of three forces acting along P_1A, P_1B, P_1C (§ 9). Replace $\mathbf{F}_1$ by these three forces acting at P_1 (Theorem A) and shift them along their lines of action until they act at A, B, C respectively (Prin. B). In the same way replace each of the forces $\mathbf{F}_2, \mathbf{F}_3, \ldots$ by forces acting at A, B, C. Now combine all the forces at A into a single force $\mathbf{L}$, their vector sum (Prin. A); similarly combine the forces at B and C into their vector sums $\mathbf{M}$ and $\mathbf{N}$. The original system is then reduced to three forces $\mathbf{L}, \mathbf{M}, \mathbf{N}$ acting at A, B, C respectively.

If one of these forces, as $\mathbf{L}$, lies in the plane ABC it may be expressed as the sum of two forces acting along AB and AC (§ 8). If we replace $\mathbf{L}$ by these forces, shift them along their lines of action until they act at B and C, and then combine them with $\mathbf{M}$ and $\mathbf{N}$ respectively, the system is reduced to two forces acting at B and C.

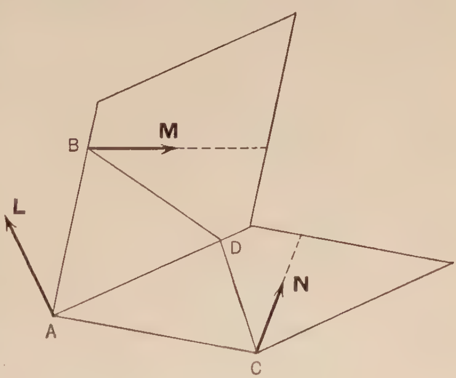


FIG. 70*a*.

The system, however, may be reduced to two forces even when $\mathbf{L}, \mathbf{M}, \mathbf{N}$ all lie outside of the plane ABC. For pass planes through A and $\mathbf{M}$, A and $\mathbf{N}$, and let AD be their line of intersection (Fig. 70*a*). Take any point D on this line and draw BD and CD. Replace $\mathbf{M}$ by two forces acting at B along the lines BA and BD (§ 8) and shift them along these lines until they act at A and D respectively. Similarly replace $\mathbf{N}$ by two forces acting at C along the lines CA and CD, and shift them also until they act at A and D. Finally replace the three forces at A (one being $\mathbf{L}$) and the two forces at D by their resultants $\mathbf{F}'$ and $\mathbf{F}''$. The

original system is then reduced to *two* forces $\mathbf{F}'$, $\mathbf{F}''$ acting at A and D.

In this reduction the point A can be chosen at pleasure. For the points P_i on the action-lines of the forces can always be taken so that A is not one of their number, and the plane ABC then chosen so as to avoid all the points P_i.

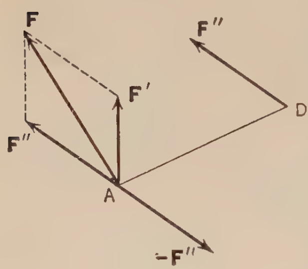


Fig. 70*b*.

Let us now introduce the two forces $\mathbf{F}''$ and $-\mathbf{F}''$ at the point A (Theorem A) and then replace $\mathbf{F}'$ and $\mathbf{F}''$ at A by their resultant $\mathbf{F} = \mathbf{F}' + \mathbf{F}''$ (Fig. 70*b*). The system is now reduced to the force $\mathbf{F}$ at A and the couple formed by $-\mathbf{F}''$ at A and $\mathbf{F}''$ at D.

We summarize these results in

Theorem 1. *A system of forces applied to a rigid body may always be reduced to either of the following equivalent systems:*

(1) *Two forces, one of which acts through any given point;*

(2) *A single force, acting through a given point, and a couple.*

In the latter case the force-sum is equal to the single force and the moment-sum about A is equal to the moment of the couple. Hence from § 68, Theorem 1 we have the

Theorem 2. *When a system of forces is reduced to a single force through the point A and a couple, the force is equal to the force-sum of the system and the moment of the couple is equal to the moment-sum of the system about A.*

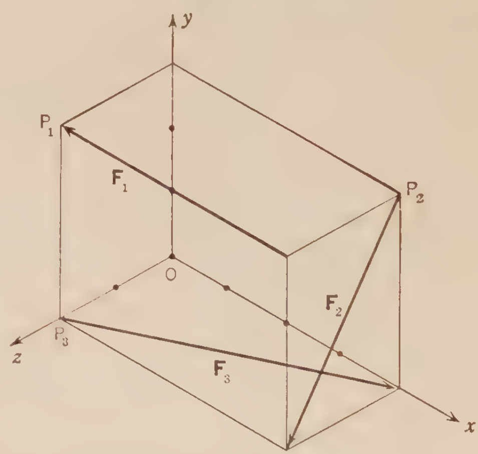


Fig. 70*c*.

Example. Let us reduce the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ of Fig. 70*c* to a force through O and a couple.

From the figure

$$\mathbf{F}_1 = -4\,\mathbf{i}, \quad \mathbf{F}_2 = -3\,\mathbf{j} + 2\,\mathbf{k}, \quad \mathbf{F}_3 = 4\,\mathbf{i} - 2\,\mathbf{k}.$$

Hence the force at O and the moment of the couple will be

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = -3\,\mathbf{j},$$

$$\begin{aligned}\mathbf{M}_A &= \overrightarrow{OP_1}\times\mathbf{F}_1 + \overrightarrow{OP_2}\times\mathbf{F}_2 + \overrightarrow{OP_3}\times\mathbf{F}_3 \\ &= (3\,\mathbf{j} + 2\,\mathbf{k})\times(-4\,\mathbf{i}) + (4\,\mathbf{i} + 3\,\mathbf{j})\times(-3\,\mathbf{j} + 2\,\mathbf{k}) + 2\,\mathbf{k}\times(4\,\mathbf{i} - 2\,\mathbf{k}) \\ &= 12\,\mathbf{k} - 8\,\mathbf{j} - 12\,\mathbf{k} - 8\,\mathbf{j} + 6\,\mathbf{i} + 8\,\mathbf{j} \\ &= 6\,\mathbf{i} - 8\,\mathbf{j}.\end{aligned}$$

If the units are pounds and feet, the magnitude of this moment is 10 lb.-ft. The forces **F**, **U**, **V** of Fig. 70*d* accomplish the desired reduction. The student may show that either **U**, **V** or **S**, **T** form a couple of moment $\mathbf{M}_A$. Of course these are but two of an infinite number of couples having this moment.

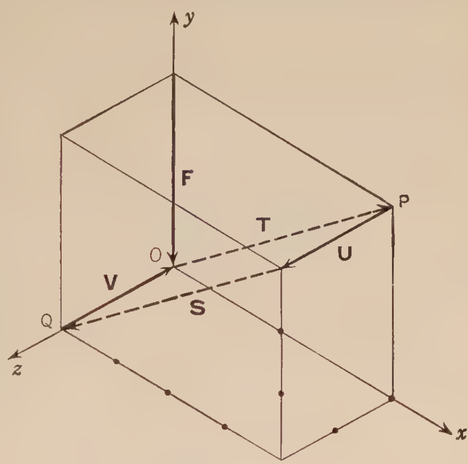


FIG. 70*d*.

71. Reduction in Special Cases. If a system of forces can be reduced to a couple, the force-sum $\mathbf{F} = 0$. Conversely if $\mathbf{F} = 0$ and $\mathbf{M}_A \neq 0$ the system can be reduced to a couple. For if we reduce the system to two forces, $\mathbf{F}'$ at A, $\mathbf{F}''$ at B, then

$$\mathbf{F} = \mathbf{F}' + \mathbf{F}'' = 0, \quad \mathbf{F}'' = -\mathbf{F}',$$

and the forces form a couple of moment $\mathbf{M}_A$.

THEOREM 1. *A system of forces applied to a rigid body can be reduced to a couple when, and only when,*

$$\mathbf{F} = 0 \quad \textit{and} \quad \mathbf{M}_A \neq 0.$$

If a system of forces can be reduced to a single force **R**, its *resultant*, we have by Theorem 1 of § 68,

$$\mathbf{F} = \mathbf{R}, \quad \mathbf{M}_A = \overrightarrow{AP}\times\mathbf{R}$$

where P is a point on the action-line of **R**. Hence

$$\mathbf{F}\cdot\mathbf{M}_A = \mathbf{R}\cdot\overrightarrow{AP}\times\mathbf{R} = 0 \qquad (\S\ 18).$$

Conversely, if $\mathbf{F}\cdot\mathbf{M}_A = 0$ and $\mathbf{F} \neq 0$, the system can be reduced to a single force. For let us reduce the system to two forces, $\mathbf{F}'$ at A, $\mathbf{F}''$ at B. Then

$$\mathbf{F} = \mathbf{F}' + \mathbf{F}'', \quad \mathbf{M}_A = \overrightarrow{AB}\times\mathbf{F}'',$$
$$\mathbf{F}\cdot\mathbf{M}_A = \mathbf{F}'\cdot\overrightarrow{AB}\times\mathbf{F}'' = 0,$$

and from § 18 we see that the forces $\mathbf{F}'$ and $\mathbf{F}''$ are coplanar. They may therefore be combined into a single force (§ 48, Case 1).

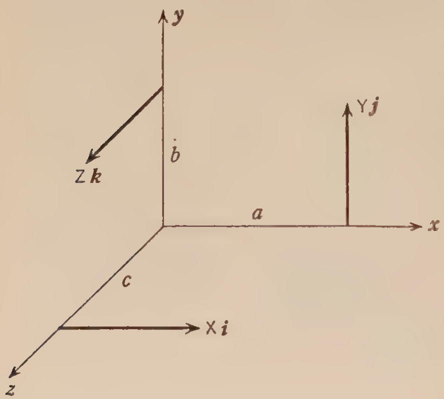


FIG. 71.

THEOREM 2. *A system of forces applied to a rigid body can be reduced to a single force when, and only when,*

$$\mathbf{F} \neq 0 \quad \text{and} \quad \mathbf{F}\cdot\mathbf{M}_A = 0.$$

Example. Let us find under what circumstances the forces $Y\mathbf{j}$, $Z\mathbf{k}$, $X\mathbf{i}$, acting at the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, will have a resultant (Fig. 71).

The force-sum and moment-sum about O are

$$\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k},$$
$$\mathbf{M}_O = a\mathbf{i}\times Y\mathbf{j} + b\mathbf{j}\times Z\mathbf{k} + c\mathbf{k}\times X\mathbf{i} = bZ\mathbf{i} + cX\mathbf{j} + aY\mathbf{k}.$$

The system will have a resultant only when

$$\mathbf{F}\cdot\mathbf{M}_O = bZX + cXY + aYZ = 0.$$

If we divide this equation by XYZ, this condition becomes

$$\frac{a}{X} + \frac{b}{Y} + \frac{c}{Z} = 0.$$

If $\mathbf{r}$ is any vector from O to the line of action of the resultant, the equations of this line are given by $\mathbf{r}\times\mathbf{F} = \mathbf{M}_O$.

For example if $X = Y = Z$ and $a = 1$, $b = 2$, $c = -3$, the condition above is satisfied. The forces have the resultant $\mathbf{F} = X(\mathbf{i} + \mathbf{j} + \mathbf{k})$ whose line of action is given by $\mathbf{r}\times\mathbf{F} = \mathbf{M}_O$, or

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ X & X & X \end{vmatrix} = X(2\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}),$$

that is, by any two of the equations

$$y - z = 2, \quad z - x = -3, \quad x - y = 1.$$

72. Equilibrium of a Rigid Body. Consider a system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots$ acting at the points $P_1, P_2, \ldots$ of a rigid body. If the system is equivalent to zero it follows from Theorem 1 of § 68 that both the force-sum $\mathbf{F}$ and the moment-sum $\mathbf{M}_A$ about any point A must vanish.

Conversely suppose that $\mathbf{F} = 0$ and $\mathbf{M}_A = 0$. Let the system be reduced to two forces, $\mathbf{F}'$ at A, $\mathbf{F}''$ at B; then

$$\mathbf{F} = \mathbf{F}' + \mathbf{F}'' = 0, \quad \mathbf{M}_A = \overrightarrow{AB}\times\mathbf{F}'' = 0.$$

The second equation shows that either

(*a*) $\mathbf{F}'' = 0$, or (*b*) $\mathbf{F}''$ acts along AB.

Then from the first we see that

(*a*) $\mathbf{F}' = 0$, or (*b*) $\mathbf{F}'$ acts along AB.

In case (*b*), $\mathbf{F}'$ and $\mathbf{F}''$ reduce to zero (Theorem B, § 27). Hence in both cases the system of forces is equivalent to zero.

In view of Principle C (§ 30) and Theorem D (§ 31) we have proved the following fundamental

THEOREM. *A rigid body is in equilibrium when, and only when, the vector sum of the external forces acting on it is zero, and the vector sum of the moments of these forces about any point is zero*:

$$\mathbf{F} = \sum \mathbf{F}_i = 0, \quad \mathbf{M}_A = \sum \overrightarrow{AP_i}\times\mathbf{F}_i = 0.\text{*} \tag{1}$$

These vector equations are equivalent to six scalar equations. Thus if we take components of $\mathbf{F}$ and $\mathbf{M}_A$ on rectangular axes x, y, z through A we obtain

$$\begin{aligned} F_x &= 0, & F_y &= 0, & F_z &= 0; \\ M_x &= 0, & M_y &= 0, & M_z &= 0.\dagger \end{aligned} \tag{2}$$

For the component of $\mathbf{M}_A$ on any axis s through A is equal to the moment-sum M_s about this axis; thus if $\mathbf{e}$ is the unit vector in the direction of s we have (§ 66)

$$\mathbf{e}\cdot\mathbf{M}_A = \sum \mathbf{e}\cdot\overrightarrow{AP_i}\times\mathbf{F}_i = M_s.$$

* When $\mathbf{F} = 0$, $\mathbf{M}_A = 0$ we see from (§ 68, 2) that the moment sum $\mathbf{M}_B$ about any other point B is also zero.

† If the forces all lie in the xy-plane, the equations $F_z = 0$, $M_x = 0$, $M_y = 0$ reduce to $0 = 0$. The three remaining equations are precisely the conditions (1) of § 52 for the equilibrium of coplanar forces.

Let us project the forces $\mathbf{F}_i$ of a system in equilibrium on a plane π perpendicular to s. We thus obtain a system of coplanar vectors $\mathbf{F}_i'$. Since the polygon of forces $\mathbf{F}_i$ is closed, its projection on π is also closed; that is, $\mathbf{F}' = \sum \mathbf{F}_i' = 0$. Again, the moment of $\mathbf{F}_i$ about s is equal to the moment of $\mathbf{F}_i'$ about s (§ 43). Hence if M_s' denotes the moment-sum of the vectors $\mathbf{F}_i'$ about s, $M_s' = M_s = 0$. Now the equations

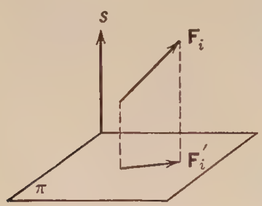


FIG. 72*a*.

$$\mathbf{F}' = 0, \quad M_s' = 0$$

show that the vectors $\mathbf{F}_i'$ may be regarded as a system of coplanar forces in equilibrium (§ 52). Therefore:

If the forces acting on a rigid body are in equilibrium, their projections on any plane may also be regarded as a system of forces in equilibrium.

The six equations (2) suffice, in general to determine six unknown quantities. Thus if a rigid body is anchored to the ground by means of six links, the stresses in these links may be found by solving equations (2) provided that the determinant of these linear equations is not zero.*

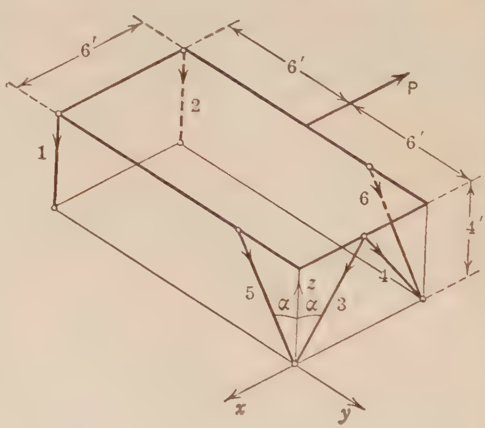


FIG. 72*b*.

Example 1. A light rectangular plate 12 ft. long and 6 ft. wide is supported by six links (Fig. 72*b*); of these *1* and *2* are vertical and 4 ft. high, and *3*, *4*, *5*, *6* make an angle α with the vertical. Find the stresses in these links due to the horizontal force $P = 300$ lb.

Assume that the stresses in the links are all tensile. On chosing

* This determinant is always zero when more than three links meet in a point or lie in a plane; or more generally, whenever a line may be drawn that meets all the links.

rectangular axes as shown we have the following equations for the equilibrium of the plate:

(i) $F_x = S_3 \sin \alpha - S_4 \sin \alpha - P = 0,$

(ii) $F_y = S_5 \sin \alpha + S_6 \sin \alpha = 0,$

(iii) $F_z = -S_1 - S_2 - (S_3 + S_4 + S_5 + S_6) \cos \alpha = 0,$

(iv) $M_x = 12\, S_1 + 12\, S_2 = 0,$

(v) $M_y = -6\, S_2 - 6\, S_6 \cos \alpha - 6\, S_4 \cos \alpha - 4\, P = 0,$

(vi) $M_z = -6\, S_6 \sin \alpha - 6\, P = 0.$

The stresses S_i may be found by solving these equations. Since $\sin \alpha = 3/5$, $\cos \alpha = 4/5$, we have from (vi) and from (ii):

$$S_6 = -\frac{P}{\sin \alpha} = -300 \times \tfrac{5}{3} = -500 \text{ lb.},$$

$$S_5 = -S_6 = 500 \text{ lb.}$$

Since $S_1 + S_2 = 0$ from (iv), we have from (iii) and (i)

$$S_3 + S_4 = 0, \quad S_3 - S_4 = 500;$$

hence $$S_3 = 250 \text{ lb.}, \quad S_4 = -250 \text{ lb.}$$

From (v)

$$S_2 = -(S_6 + S_4) \cos \alpha - \tfrac{2}{3} P = 750 \times \tfrac{4}{5} - 200 = 400 \text{ lb.},$$

and from (iv), $S_1 = -400$ lb. The positive stresses are tensile, the negative compressive:

$$S_2 = 400 \text{ lb.}, \qquad S_3 = 250 \text{ lb.}, \qquad S_5 = 500 \text{ lb.},$$
$$S_1 = -400 \text{ lb.}, \quad S_4 = -250 \text{ lb.}, \quad S_6 = -500 \text{ lb.},$$

Example 2. If the plate in Example 1 has a concentrated downward load of 400 lb. at its center, we will have

$$S_1 = S_2, \quad S_3 = S_4, \quad S_5 = S_6,$$

due to symmetry. It is natural to assume, now, that the stresses are compressive. The student should draw the free-body diagram for the plate. Owing to the relations above, only three more equations are needed to compute the stresses. These may be chosen as follows:

$$M_z = 6\, S_6 \sin \alpha = 0, \qquad S_6 = 0;$$
$$M_x = 6 \times 400 - 12\,(S_1 + S_2) = 0, \qquad S_1 = 100 \text{ lb.};$$
$$M_y = 6\, S_2 + 6\, S_4 \cos \alpha - 3 \times 400 = 0, \qquad S_4 = 125 \text{ lb.}$$

PROBLEMS

1. Find the stresses in the six links supporting the light rectangular plate of Fig. 72*c* when acted on by the two forces $P = 300$ lb. as shown.

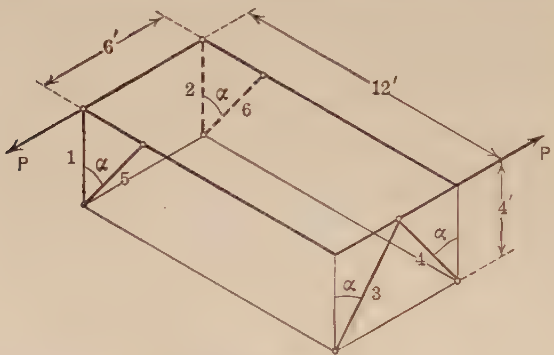


FIG. 72*c*.

2. Find the stresses in these links when the plate supports a downward load of 400 lb. at its center.

3. A mechanic exerts a turning moment of 90 lb.-in. on the handle of a screwdriver whose point is 3/8 in. wide and beveled to an angle of 8°. If the screwdriver engages the screw at the upper edges only of a square slot, find the vertical force tending to raise it out of the slot.

4. An iron sphere 10 cm. in diameter rests in a circular hole 8 cm. in diameter in a horizontal table. Find the reaction R of the table per unit of length. Iron weighs 7.80 grams per cubic centimeter

5. A soap bubble of radius r is under pressures of p and p_0 per unit area inside and outside respectively. If we imagine the bubble divided into two halves by a great circle, show that the tension across its circumference is $\frac{1}{2}\,r(p - p_0)$ per unit of length.

6. Four equal smooth spheres of weight W are placed in contact so as to form a pyramid — three resting on a smooth table, one on top of these. Find

(*a*) the horizontal force H that must be applied to the lower balls to maintain equilibrium;
(*b*) the pressure P of the upper ball on each ball beneath;
(*c*) the reaction V of the table on each lower ball.

[The centers of the spheres are at the vertices of a regular tetrahedron whose upper edges are inclined at an angle $\cos^{-1}\sqrt{\frac{2}{3}}$ to the vertical. Draw a free-body diagram for each ball.]

7. The base plate of a crane is bolted to the floor by four bolts at the corners $ABCD$ of a 2-ft. square whose center is O. If the crane load of 2000 lb. acts downward through a point P, such that $OP = 10$ ft. and OP is perpendicular to AB and CD, find the stresses in the bolts.

73. Body with a Fixed Axis. Consider a rigid body capable of turning about a fixed axis. If the body cannot move along the axis (like a screw in a nut), it will be in contact with its supports or *bearings* along portions of surfaces of revolution. If friction at the bearings is neglected, the reactions there will be normal to the surface of contact; their lines of action will therefore either cut the fixed axis or be parallel to it. In either case the reactions have no moment about the axis. Suppose, now, that the body is in equilibrium under certain impressed forces and the reactions. If we take moments about the fixed axis the reactions will not enter in the equation; hence we obtain the general

LAW OF THE LEVER. *If a rigid body, free to turn about a fixed axis in frictionless bearings, is in equilibrium, the sum of the moments of the impressed forces about this axis is zero.*

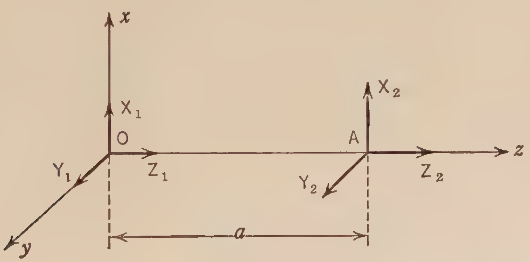


FIG. 73*a*.

Let us take the fixed axis as the x-axis of a system of rectangular coördinates. If we assume that the reactions at the bearings have the resultants $[X_1, Y_1, Z_1]$, $[X_2, Y_2, Z_2]$ acting at the points O and A of the z-axis (Fig. 73a), the equations of equilibrium are:

(i) $$F_x = F_x' + X_1 + X_2 = 0,$$
(ii) $$F_y = F_y' + Y_1 + Y_2 = 0,$$
(iii) $$F_z = F_z' + Z_1 + Z_2 = 0,$$
(iv) $$M_x = M_x' - aY_2 = 0,$$
(v) $$M_y = M_y' + aX_2 = 0,$$
(vi) $$M_z = M_z' = 0.$$

Here F_x' denotes the sum of the x-components of the impressed forces, M_x' the sum of the moments of these forces about the x-axis. When the impressed forces are known, we can compute X_2 and Y_2 from (v) and (iv), and then X_1 and Y_1 from (i) and (ii). But as (iii) is the only equation containing Z_1, Z_2, these components can not be computed separately; in other words, *the components of the reactions on the axis of rotation are statically indeterminate.*

Example 1. Fig. 73*b* represents a weight of $W = 400$ lb. being raised by means of a force of P lb. applied normal to the crank of a

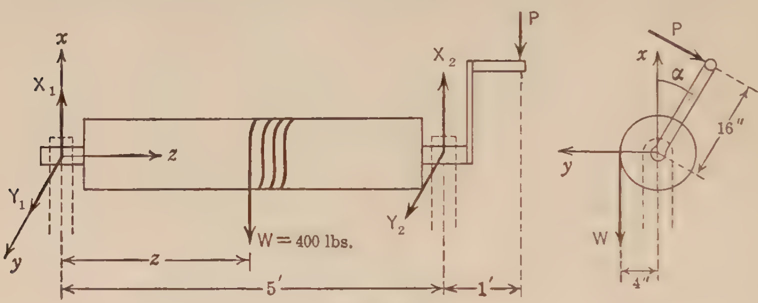


FIG. 73*b*.

windlass. If the bearings are smooth and cannot exert any reaction parallel to the axis of the windlass, the equations of equilibrium are:

(i) $F_x = X_1 + X_2 - P \sin \alpha - 400 = 0,$
(ii) $F_y = Y_1 + Y_2 - P \cos \alpha = 0,$
(iii) $F_z = 0,$
(iv) $M_x = 6\,P \cos \alpha - 5\,Y_2 = 0,$
(v) $M_y = 5\,X_2 - 400\,z - 6\,P \sin \alpha = 0,$
(vi) $M_z = \frac{1}{3}\,400 - \frac{4}{3}\,P = 0.$

From (vi), $P = 100$ lb. Then from (v), (iv), (i), (ii) we find in turn

$$X_2 = 80\,z + 120 \sin \alpha, \qquad Y_2 = 120 \cos \alpha,$$
$$X_1 = 80(5 - z) - 20 \sin \alpha, \qquad Y_1 = -20 \cos \alpha.$$

We see that Y_1 varies between ± 20 and Y_2 between ± 120 for every revolution of the crank. For a given z, X_1 has its least value.

$$X_1 = 380 - 80\,z \quad \text{when} \quad \alpha = 90°;$$

and X_2 has its least value

$$X_2 = 80\,z - 120 \quad \text{when} \quad \alpha = 270°.$$

Hence X_1 will pass through negative values when $z > 4.75$ ft., and X_2 will pass through negative values when $z < 1.5$ ft. If $1.5 < z <$

4.75, X_1 and X_2 will always remain positive; in this case the bearings for the windlass may be open at the top.

Example 2. *Static Balance of a Shaft.* A number of eccentric weights W_i are mounted on a shaft free to turn in frictionless bearings (Fig. 73*c*). This shaft is said to be in *static balance* when it is in equilibrium, as regards rotation in any position. The condition for equi-

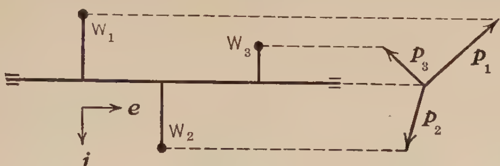


FIG. 73*c*.

librium is that the sum of the moments of the weights about the axis of the shaft shall be zero no matter how the shaft is turned. If $\mathbf{p}_i$ is a normal vector from the axis to the center of gravity of the body W_i, $\mathbf{e}$ a unit vector along the axis, and $\mathbf{j}$ a unit vector directed downwards along the plumb-line, the sum of the moments of the weights about the axis is (§ 66, 1)

$$\sum \mathbf{e}\cdot\mathbf{p}_i\times W_i\mathbf{j} = \mathbf{j}\times\mathbf{e}\cdot\sum W_i\mathbf{p}_i.$$

If the shaft is vertical, $\mathbf{j}\times\mathbf{e} = 0$ and this moment-sum is always zero. This of course is obvious since the forces of gravity are all parallel to the axis.

If the shaft is not vertical, $\mathbf{j}\times\mathbf{e}$ is a horizontal vector; the moment-sum will then vanish for all positions of the shaft only when $\sum W_i\mathbf{p}_i = 0$.

A shaft, whose axis is not vertical, is in static balance when, and only when, the vectors $W_i\mathbf{p}_i$ form a closed polygon.

If the vectors $W_i\mathbf{p}_i$ do not form a closed polygon, the shaft can be balanced by adding a weight W whose center of gravity has the vector eccentricity $\mathbf{p}$ determined by

$$W\mathbf{p} + \sum W_i\mathbf{p}_i =$$

Tne use of two or more vectors to close the polygon gives a method of balancing by adding two or more weights.

Numerical Example. A shaft has weights of 4 lb. and 6 lb. whose eccentricities are $3\,\mathbf{i}$ and $2\,\mathbf{j}$ respectively ($\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors). To balance these with a single weight W we must have

$$W\mathbf{p} + 12\,\mathbf{i} + 12\,\mathbf{j} = 0.$$

Thus, if we take $W = 3$ lb., $\mathbf{p} = -4(\mathbf{i} + \mathbf{j})$; then $p = 4\sqrt{2}$ and $\mathbf{p}$ is 135° from both $\mathbf{i}$ and $\mathbf{j}$.

PROBLEMS

1. A rectangular iron door weighing W lb. is held open at an angle α to the horizontal by a force of P lb. applied perpendicular to the door at one corner (Fig. 73d). If the y-axis lies in the plane of the door as shown and the z-axis is drawn in the direction of P, show that $P = \frac{1}{2} W \cos \alpha$ and that the reaction components have the values

$$Y_1 = Y_2 = \tfrac{1}{2} W \sin \alpha,$$
$$Z_1 = \tfrac{1}{2} W \cos \alpha,$$
$$Z_2 = 0.$$

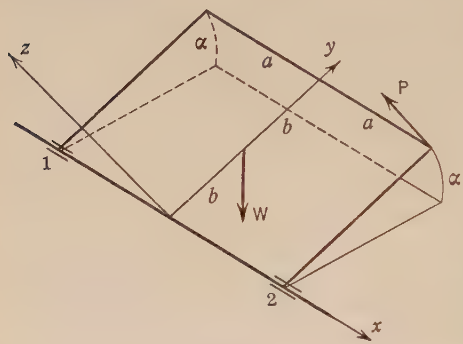


Fig. 73d.

2. A shaft carries two weights of 10 and 5 lb. whose centers of gravity A, B are 5 and 6 in. respectively from the axis O. If $AOB = 60°$, what weight will balance these at a point C 10 in. from O? Find the angle AOC.

74. Equivalent Systems. We shall now prove the converse of Theorem 1 of § 68.

Theorem 1. *If for two systems of forces, S and S', acting on a rigid body*

$$\mathbf{F} = \mathbf{F}' \quad \textit{and} \quad \mathbf{M}_A = \mathbf{M}_A',$$

the systems are equivalent.

Proof. Let $-S$ denote the system formed from S by reversing the direction of each of its forces while keeping their points of application unchanged. Then the system $\{S, -S\} \equiv 0$. Moreover the force-sum and moment-sum of the system $\{-S, S'\}$ are $-\mathbf{F} + \mathbf{F}' = 0$, $-\mathbf{M}_A + \mathbf{M}_A' = 0$; hence from § 72 (paragraph 2), $\{-S, S'\} \equiv 0$. We may now reduce S to S' by adding to S the system $\{-S, S'\} \equiv 0$, obtaining $\{S, -S, S'\}$, and then reducing $\{S, -S\}$ to zero. The systems S and S' are therefore equivalent.

Theorem 1 of § 68 and the one just proved are combined in the following

Equivalence Theorem. *A necessary and sufficient condition for the equivalence of two systems of forces acting on a rigid body*

is that their force-sums and moment-sums about any given point are equal, each to each.

In particular

Two couples acting on a rigid body are equivalent when, and only when, their (vector) moments are equal.

For the force-sums of the couples are necessarily equal, both being zero.

Example. The couples **U**, **V** and **S**, **T** in Fig. 70*d* are equivalent.

75. Resultant of Parallel Forces. In § 29 we saw that a system of parallel forces $\mathbf{F}_1, \mathbf{F}_2, \ldots$ had a resultant when, and only when, the force-sum was not zero. We propose, now, to locate the line of action of this resultant.

If $\mathbf{e}$ is a unit vector parallel to the forces $\mathbf{F}_i$ we may write $\mathbf{F}_i = F_i\mathbf{e}$, where the scalars F_i are positive or negative according as $\mathbf{F}_i$ has the same direction as $\mathbf{e}$ or the opposite. The force-sum of the system is

$$\mathbf{F} = (\sum F_i)\mathbf{e}.$$

If $\mathbf{F}_i$ acts at the point P_i and $\mathbf{r}_i = \overrightarrow{AP_i}$, the moment-sum about A is

$$\mathbf{M}_A = \sum \mathbf{r}_i \times (F_i\mathbf{e}) = (\sum F_i\mathbf{r}_i) \times \mathbf{e}.$$

Now from (§ 12, 2) we have

$$\mathbf{r}^* = \frac{\sum F_i\mathbf{r}_i}{\sum F_i}, \tag{1}$$

where $\mathbf{r}^*$ is the position vector $\overrightarrow{AP^*}$ of the centroid P^* of the points P_i with the associated numbers F_i. Hence we may write

$$\mathbf{M}_A = (\sum F_i)\, \mathbf{r}^* \times \mathbf{e} = \mathbf{r}^* \times (\sum F_i)\, \mathbf{e} = \mathbf{r}^* \times \mathbf{F}.$$

This result states that $\mathbf{M}_A$ is equal to the moment about A of the single force $\mathbf{F}$ acting through P^*. In other words, the force $\mathbf{F}$ at P^* has the same force-sum and moment-sum about A as the given system of parallel forces. Hence by the Equivalence Theorem of § 74 we have the

THEOREM. *The resultant of a system of parallel forces* $F_i\mathbf{e}$ *acting at the points* P_i *and having a force-sum* $\mathbf{F} \neq 0$ *is a force* $\mathbf{F}$ *equal to this sum and acting at the centroid of the points* P_i *associated with the numbers* F_i.

Note that the position of P^* depends only on the points P_i and the numbers F_i and not on the direction of the vector **e**. Hence the resultant of the parallel forces $F_i\mathbf{e}$ acting at the points P_i passes through P^* no matter what direction **e** may have. On account of this property, P^* is called the *center* of all such systems of parallel forces.

We see also from (1) that P^* is not changed when the magnitudes of all the forces are changed in the same ratio, that is, when all the numbers F_i are multiplied by the same constant.

If the coördinates of P_i are (x_i, y_i, z_i), we see from (1) that P^* has the coördinates

$$x^* = \frac{\sum F_i x_i}{\sum F_i}, \quad y^* = \frac{\sum F_i y_i}{\sum F_i}, \quad z^* = \frac{\sum F_i z_i}{\sum F_i}.$$

Example. Let us find the resultant of the following system of forces acting at the points of the xy-plane written below:

Forces:	3 **e**,	−5 **e**,	−4 **e**,	2 **e**,
Points:	(1, 1),	(3, 0),	(−2, −1) ,	(−1, 2).

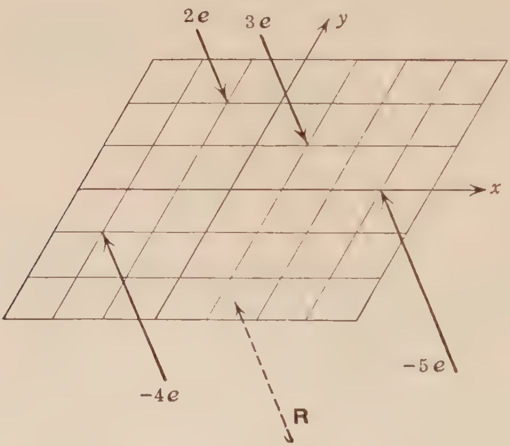


Fig. 75.

The resultant is a force $\mathbf{R} = -4\,\mathbf{e}$ acting through the point

$$x^* = \frac{3 - 15 + 8 - 2}{-4} = 1.5, \qquad y^* = \frac{3 + 0 + 4 + 4}{-4} = -2.75$$

of the xy-plane (Fig. 75). This result is independent of the direction of **e**.

76. Center of Gravity. The forces of gravity on a set of particles on the earth may be regarded as a system of parallel forces, at least if the particles are not widely separated. Hence, from § 75, the resultant gravity of the particles $P_1, P_2, \ldots$ of weights $W_1, W_2, \ldots$ acts through the point P^* given by

$$\mathbf{r}^* = \frac{\sum W_i \mathbf{r}_i}{\sum W_i}.$$

The position vectors $\mathbf{r}_i = \overrightarrow{OP_i}$ and $\mathbf{r}^* = \overrightarrow{OP^*}$ may be referred to any origin O. The point P^* is called the *center of gravity* of the set of particles:

Any body may be conceived to be divided into a large number of small parts or *elements*, each of which may be approximately treated as a particle. If W is the weight of the body, ΔW the weight of any element, and $\mathbf{r}$ the position vector of any point of this element, the center of gravity of this set of elements has the position vector

$$\frac{\sum \mathbf{r}\, \Delta W}{W}.$$

If we increase the number of elements indefinitely in any way so that their volumes all approach zero as a limit, the expression above approaches a limit $\mathbf{r}^*$, which, in the notation of the Integral Calculus, is denoted by

$$\mathbf{r}^* = \frac{\int \mathbf{r}\, dW}{W}. \tag{1}$$

This expression for $\mathbf{r}^*$ $(\overrightarrow{OP^*})$ locates the *center of gravity* P^* of the body. The resultant gravity of the body acts through its center of gravity *no matter what position it has relative to the earth.*

If the body is homogeneous its weight w per unit volume will be everywhere the same. If V denotes the volume of the body, we have in this case $W = wV$, $\Delta W = w\, \Delta V$, and $\mathbf{r}^*$ is the limiting value of

$$\frac{\sum \mathbf{r}\, \Delta W}{W} = \frac{\sum \mathbf{r}\, \Delta V}{V},$$

that is

$$\mathbf{r}^* = \frac{\int \mathbf{r}\, dV}{V}. \tag{2}$$

The element ΔV is supposed to be infinitesimal in all dimensions so that the vector $\mathbf{r}$ to one of its points approaches a definite limit as $\Delta V \to 0$. Thus with rectangular coördinates $\Delta V = \Delta x\, \Delta y\, \Delta z$, and the integral in (2) is computed as a triple integral.

If we put

$$\mathbf{r}^* = x^*\mathbf{i} + y^*\mathbf{j} + z^*\mathbf{k}, \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

in (2), we obtain three scalar equations for the coördinates of the center of gravity:

$$(3) \qquad Vx^* = \int x\, dV, \quad Vy^* = \int y\, dV, \quad Vz^* = \int z\, dV,$$

where $dV = dx\, dy\, dz$. We shall show in the next article that in certain cases these triple integrals may be evaluated as single integrals.

If a homogeneous body has a plane of symmetry or a line of symmetry, its center of gravity will lie on this plane or line.

Proof. If the body has a plane of symmetry, choose it as the xy-plane. Then to each element dV at (x, y, z) we have a corresponding element at $(x, y, -z)$; such a pair contributes $z\, dV + (-z)dV = 0$ to $\int z\, dV$. Hence

$$Vz^* = \int z\, dV = 0 \text{ and } z^* = 0.$$

If the body has an axis of symmetry, choose it as the z-axis. Then to each element dV at (x, y, z) we have a corresponding element at $(-x, -y, z)$. Such a pair contributes $x\, dV + (-x)\, dV = 0$ to $\int x\, dV$. Hence

$$Vx^* = \int x\, dV = 0 \quad \text{and} \quad x^* = 0.$$

Similarly $y^* = 0$. Thus P^* lies on the z-axis.

77. Center of Gravity: *Continuation.* If a body of weight W can be divided into a number of parts of weight $W_1, W_2, \ldots$ whose centers of gravity are known, the center of gravity of the whole is readily found. Thus if $\mathbf{r}_i^*$ locates the center of gravity of the i th part, we have from (§ 76, 1)

$$W\mathbf{r}^* = \int \mathbf{r}\, dW = \int \mathbf{r}\, dW_1 + \int \mathbf{r}\, dW_2 + \cdots = \sum \int \mathbf{r}\, dW_i,$$

where $\int \mathbf{r}\, dW_i$ is an integral over the i th part only. But since

$$W_i\mathbf{r}_i^* = \int \mathbf{r}\, dW_i$$

from (§ 76, 1), the above equation becomes

$$W\mathbf{r}^* = \sum W_i\mathbf{r}_i^*. \tag{1}$$

When the body is homogeneous the weights are proportional to the volumes and (1) may be written

$$V\mathbf{r}^* = \sum V_i\mathbf{r}_i^*. \tag{1$'$}$$

Suppose now that a homogeneous body is divided into a number of elements ΔV that are infinitesimal in but one dimension. For example the elements ΔV may be thin slices of the body made by a series of parallel planes. Then if $\bar{\mathbf{r}}$ is the position vector to the center of gravity of ΔV, we have from (1)′

$$V\mathbf{r}^* = \sum \bar{\mathbf{r}}\Delta V;$$

and, on passing to the limit $\Delta V \to 0$, we obtain

$$V\mathbf{r}^* = \int \bar{\mathbf{r}}\, dV. \tag{2}$$

This result has the same form as (§ 76, 2); now, however, dV need only be infinitesimal in *one dimension** and $\bar{\mathbf{r}}$ *denotes the position vector to the center of gravity of the element* dV.

From (2) we obtain the scalar equations

$$Vx^* = \int \bar{x}\, dV, \quad Vy^* = \int \bar{y}\, dV, \quad Vz^* = \int \bar{z}\, dV, \tag{3}$$

where $(\bar{x}, \bar{y}, \bar{z})$ is the center of gravity of dV.

The condition for the static balance of a shaft with eccentric weights is simply that the center of gravity of all the weights lie on the shaft-axis; for then the resultant weight has no turning moment about the axis in any position. It is easily shown that the equation $\sum W_i\mathbf{p}_i = 0$ of § 73, Example 2 imposes this requirement.

* Of course dV may be infinitesimal in two or three dimensions; in the latter case $\bar{\mathbf{r}} = \mathbf{r}$.

Example 1. Locate the center of gravity of a homogeneous right circular cone of height h.

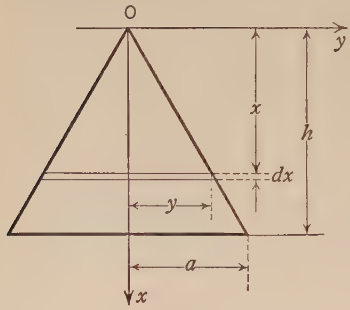


FIG. 77*a*.

Let the cone be generated by revolving the triangle of Fig. 77*a* about the x-axis. Since the center of gravity lies in the axis of the cone (a line of symmetry) we need only find its distance from the vertex. If we take a thin slice parallel to the base as the element of volume,

$$dV = \pi y^2\, dx,$$

and

$$V = \pi \int_0^h y^2\, dx = \pi \frac{a^2}{h^2} \int_0^h x^2\, dx = \tfrac{1}{3}\pi a^2 h;$$

for, by similar triangles,

$$\frac{y}{a} = \frac{x}{h} \qquad \text{or} \qquad y = \frac{a}{h}x.$$

Since $\bar{x} = x$, we have from (3)

$$Vx^* = \pi \int_0^h xy^2\, dx = \pi \frac{a^2}{h^2} \int_0^h x^3\, dx = \tfrac{1}{4}\pi a^2 h^2.$$

Therefore

$$x^* = \frac{\tfrac{1}{4}\pi a^2 h^2}{\tfrac{1}{3}\pi a^2 h} = \tfrac{3}{4}h.$$

Example 2. Locate the center of gravity of a homogeneous hemisphere of radius a.

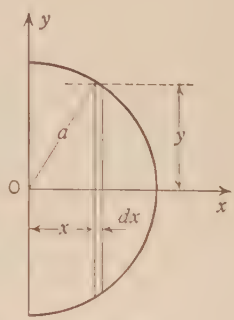


FIG. 77*b*.

Let the hemisphere be generated by revolving the semicircle of Fig. 77*b* about the x-axis. Since the center of gravity lies on the x-axis (a line of symmetry) we need only find its distance from O. If we take a thin slice parallel to the base as the element of volume, $dV = \pi y^2\, dx$, and

$$V = \pi \int_0^a y^2\, dx = \pi \int_0^a (a^2 - x^2)\, dx = \tfrac{2}{3}\pi a^3.$$

Since $\bar{x} = x$, we have from (3)

$$Vx^* = \pi \int_0^a xy^2\, dx = \pi \int_0^a (a^2 x - x^3)\, dx = \tfrac{1}{4}\pi a^4.$$

Therefore

$$x^* = \frac{\tfrac{1}{4}\pi a^4}{\tfrac{2}{3}\pi a^3} = \tfrac{3}{8}a.$$

Example 3. Locate the center of gravity of one-half of a homogeneous right circular cylinder of radius a and height h.

In Fig. 77c, the half-cylinder is shown in plan and elevation; its volume is $V = \frac{1}{2}\pi a^2 h$. Since the center of gravity lies on the x-axis (a line of symmetry) we need only find its distance from O. If we take a thin slice parallel to the plane $x = 0$ as the element of volume, $dV = 2\, y dx \cdot h$ and

$$Vx^* = 2\, h \int_0^a xy\, dx = 2\, h \int_0^a x\sqrt{a^2 - x^2}\, dx = \tfrac{2}{3} ha^3.$$

Therefore

$$x^* = \frac{\frac{2}{3} ha^3}{\frac{1}{2}\pi a^2 h} = \frac{4}{3}\frac{a}{\pi}.$$

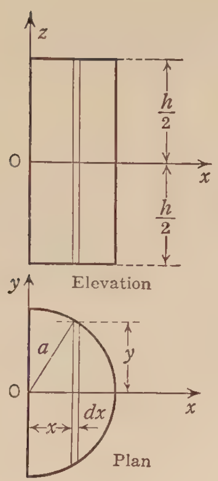


Fig. 77c.

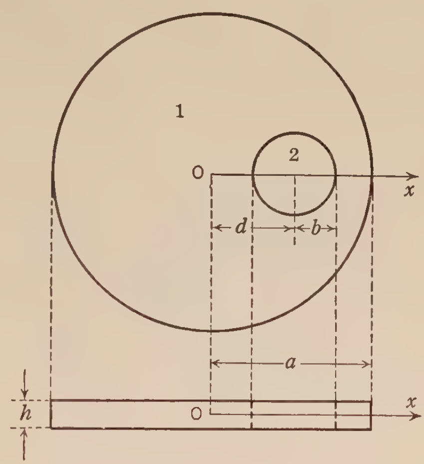


Fig. 77d.

Example 4. Locate the center of gravity of a steel disk of radius a in which a hole of radius b has been drilled at a distance d from the center (Fig. 77d).

The center of gravity lies on the line of symmetry — the x-axis in Fig. 77d. To find its distance from O we regard the *solid* disk of radius a as made up of the given disk with the hole (part *1*) and a disk of radius b (part *2*). From (1)′ we have

$$Vx^* = V_1 x_1^* + V_2 x_2^*$$

where V and x^* refer to the solid disk of radius a, V_1 and x_1^* to the disk with a hole. Since

$$V = \pi a^2 h, \quad V_1 = \pi(a^2 - b^2)h, \quad V_2 = \pi b^2 h,$$
$$x^* = 0, \quad x_1^* \text{ (unknown)}, \quad x_2^* = d,$$

we have

$$0 = \pi(a^2 - b^2)hx_1^* + \pi b^2 h\, d,$$

$$x_1^* = -\frac{b^2}{a^2 - b^2}\, d.$$

Thus if $a = 6$, $b = 2$, $d = 3$ in.,

$$x_1^* = -\frac{4}{36 - 4} \times 3 = -\tfrac{3}{8} \text{ in.}$$

The negative sign shows that the center of gravity lies to the left of O.

78. Centroids. We have seen in the preceding articles that when a body is homogeneous, its center of gravity depends only on its geometrical boundaries. Since the term *center of gravity* can be appropriately applied only to bodies which have *gravity*, that is, to actual physical bodies, the point P^* defined by

$$(1) \qquad V\mathbf{r}^* = \int \mathbf{r}\, dV$$

is often called the *centroid* of the geometrical solid over which the volume integral is taken. Thus the centroid of a geometrical solid coincides with the center of gravity of a homogeneous body having the same boundaries.

We define the centroid of a portion of a geometrical *surface* or *curve* by equations analogous to (1). Thus if A is the area of a surface and dA the element of area, its centroid has the position vector

$$(2) \qquad A\mathbf{r}^* = \int \mathbf{r}\, dA.$$

Similarly if s is the length of a curve and ds is the element of arc, its centroid has the position vector

$$(3) \qquad s\mathbf{r}^* = \int \mathbf{r}\, ds.$$

The centroids thus defined coincide very nearly with the centers of gravity of homogeneous physical bodies in the form of thin shells or wires. Just as in § 77, we may show that if dA is infinitesimal in but one dimension, $\mathbf{r}$ must be replaced by $\bar{\mathbf{r}}$, the position vector of the centroid of dA. Moreover if a solid, a surface, or an arc has a plane or line of symmetry, its centroid will lie on this plane or line.

Example 1. Locate the centroid of a triangle.

We choose as element of area a thin strip parallel to the base (Fig. 78a); then $dA = l\,dy$ and $\bar{\mathbf{r}} = \overrightarrow{OP}$, where P is the mid-point of the strip. Now, from similar triangles,

$$l = \frac{y}{h}b, \qquad \overrightarrow{OP} = \frac{y}{h}\overrightarrow{OM},$$

where M is the mid-point of the base; hence

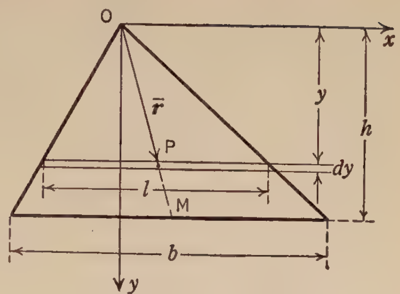


FIG. 78a.

$$dA = \frac{b}{h}y\,dy, \quad A = \frac{b}{h}\int_0^h y\,dy = \tfrac{1}{2}bh;$$

$$A\mathbf{r}^* = \int \bar{\mathbf{r}}\,dA = \frac{b}{h^2}\overrightarrow{OM}\int_0^h y^2\,dy = \tfrac{1}{3}bh\,\overrightarrow{OM},$$

$$\mathbf{r}^* = \frac{\tfrac{1}{3}bh}{\tfrac{1}{2}bh}\overrightarrow{OM} = \tfrac{2}{3}\overrightarrow{OM}.$$

Thus the centroid of a triangle is the point of intersection of its medians. It coincides, also, with the centroid of its vertices when associated with equal numbers (§ 12, Example 3).

Example 2. Locate the centroid of the arc of a semicircle of radius a.

The centroid lies on the line of symmetry, which in Fig. 78b has been chosen as the x-axis. To find x^* we have from (3)

$$sx^* = \int x\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos\theta\,a d\theta = a^2\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2a^2;$$

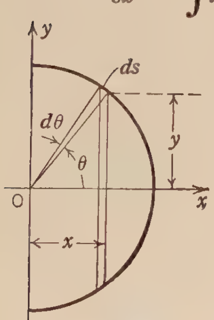


FIG. 78b.

and since $s = \pi a$,

$$x^* = \frac{2a^2}{\pi a} = \frac{2}{\pi}a.$$

Example 3. Locate the centroid of the surface of a hemisphere of radius a.

Let the hemisphere be generated by revolving the semicircle of Fig. 78b about the x-axis. Choose as element of area the zone bounded by two planes perpendicular to the x-axis; then

$$dA = 2\pi y\,ds = 2\pi a\sin\theta\cdot a\,d\theta, \quad \bar{x} = x = a\cos\theta;$$

$$A = 2\pi a^2\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta = -2\pi a^2\cos\theta\Big|_0^{\frac{\pi}{2}} = 2\pi a^2;$$

$$Ax^* = \int x\,dA = 2\,\pi a^3 \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta = \pi a^3 \sin^2\theta \Big|_0^{\frac{\pi}{2}} = \pi a^3,$$

$$x^* = \frac{\pi a^3}{2\,\pi a^2} = \tfrac{1}{2}\,a.$$

PROBLEMS

1. Find the centroid of the segment bounded by the parabola $y^2 = b^2x/a$, the x-axis and the line $x = a$.
2. Find the centroid of the portion of the ellipse $x^2/a^2 + y^2/b^2 = 1$ lying in the first quadrant.
3. A trapezoid of height h has its parallel sides of length b_1 and b_2. Show that the distance of its centroid from b is

$$y^* = \tfrac{1}{3}\,h \cdot \frac{b_1 + 2\,b_2}{b_1 + b_2}.$$

[Divide the trapezoid into two triangles. Then

$$Ay^* = A_1y_1^* + A_2y_2^*.]$$

4. Find the centroid of a quadrilateral with vertices at (0, 0), (12, 0), (9, 10), (0, 8).
5. Find the centroid of a regular square pyramid of height h.

79. Square-threaded Screw. If the shaded rectangle in Fig. 79a revolves uniformly about the axis z in its plane and at the same time moves parallel to z at a constant rate, it will generate the thread of a *square-threaded screw.* The distance p that the rectangle advances while making a complete revolution is called the *pitch* of the screw. The distance r of the center of the rectangle from the axis is the *mean radius* of the screw; and this center describes a helix of *pitch-angle* α given by

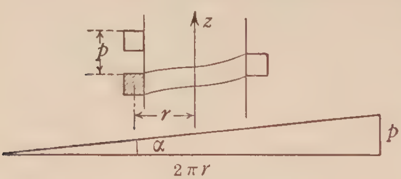


FIG. 79a.

$$\tan\alpha = \frac{p}{2\,\pi r}.$$

Fig. 79b represents a jack-screw for raising or lowering heavy weights. Let us find the force P that must be applied to the lever at a distance l from the axis to just start the weight W upward. (In the figure the screw is right-handed and the force P acts ver-

tically downward on the plane of the paper.) The nut will react on the screw along the under surfaces of the thread. Let us assume that the reactions may be replaced by a continuous distribution of forces along the helix of mean radius r. Then if $\mathbf{R}\,ds$ denotes the reaction on an element of arc ds of this helix and ϕ is the angle of friction between nut and screw, $\mathbf{R}$ will be normal to the radius

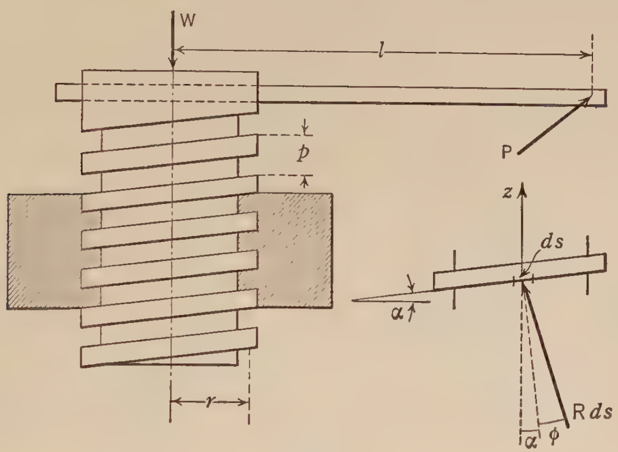


FIG. 79*b*.

r of the element ds and inclined at an angle $\alpha + \phi$ to the axis of the screw. To find the relation between W and P we need only two of the six equations of equilibrium, namely:

$$F_z = \int R \cos(\alpha + \phi)\, ds - W = 0,$$

$$M_z = Pl - \int rR \sin(\alpha + \phi)\, ds = 0,$$

the integrals extending over that part of the mean helix in contact with the nut. From these equations we have

$$W = \cos(\alpha + \phi) \int R\, ds, \quad Pl = r \sin(\alpha + \phi) \int R\, ds;$$

and on eliminating the integrals we find

$$Pl = Wr \tan(\alpha + \phi). \tag{1}$$

The turning moment Pl needed to lift W decreases with α and with ϕ.

In order to prevent the screw from running down under the load W on the moment

$$Pl = Wr \tan(\alpha - \phi) \tag{2}$$

must be applied; for, since the direction of impending motion is reversed, $\mathbf{R}\,ds$ is now inclined at an angle $\alpha - \phi$ to the axis. The moment Pl is positive as long as $\alpha > \phi$. But if $\alpha < \phi$ the moment is negative; that is, the screw is self-locking and requires a negative moment about its axis to loosen it. Referring to the figure, P must then act vertically upward on the plane of the paper.

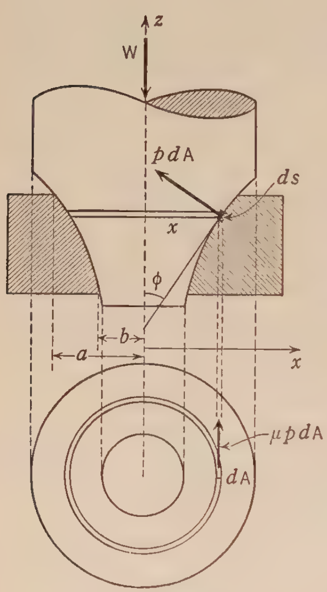


FIG. 80*a*.

It is assumed in the above treatment that the load W will revolve with the screw. If, however, the load is stationary, the frictional moment developed at the head of the screw must be taken into account in the equation $M_z = 0$. The method of computing this frictional moment is considered in the next article.

80. Pivot Friction. Fig. 80*a* represents a section of a pivot in a step bearing. Let p denote the intensity of normal pressure (per unit of area) of the bearing on the pivot at any point. Then the axial component of the pressure on the element dA of the bearing surface is $p \sin \phi \, dA$. If the pivot is subject to the downward load W, equilibrium requires that

$$\int p \sin \phi \, dA - W = 0.$$

If a moment M about the z-axis is applied to the shaft so that revolution is impending, the friction on the element dA is horizontal and of magnitude $\mu p \, dA$. The moment of this friction about the z-axis is $x\mu p \, dA$; hence for equilibrium

$$\int \mu p x \, dA - M = 0.$$

By symmetry p will be constant along any horizontal circle on the pivot. Hence we may choose a thin horizontal zone as element of area: $dA = 2\,\pi x\, ds$. The first equation then gives

$$(1) \qquad 2\,\pi \int_b^a px\, dx = W$$

since $\sin\phi\, ds = dx$; and the second becomes

$$(2) \qquad 2\,\pi\mu \int_{x=b}^{x=a} px^2\, ds = M,$$

where we have assumed that μ is constant over the bearing surface.*

As to the normal pressure p, the simplest assumption is that

(*a*) p is constant over the bearing surface.

Under this assumption we have from (1)

$$(1a) \qquad 2\,\pi p \int_b^a x\, dx = \pi(a^2 - b^2)\, p = W, \qquad p = \frac{W}{\pi(a^2 - b^2)};$$

hence p is equal to the load divided by the horizontal projection of the bearing surface. On substituting this value of p in (2) we find

$$(2a) \qquad M = \frac{2\,\mu W}{a^2 - b^2} \int_{x=b}^{x=a} x^2\, ds.$$

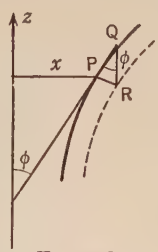


FIG. 80*b*.

Assumption (*a*) may be roughly correct for a new pivot. It is improbable, however, that p will remain constant after the bearing has been worn down after use. For when the pressure is constant, the wear will be greatest where the velocity of rubbing is greatest, that is, farthest from the axis, and this unequal wear will tend to increase p toward the center. Thus for well-worn pivots it is customary to abandon assumption (*a*) and assume instead that the pressure distribution is such that the wear in the direction of the axis is constant. To formulate this mathematically, we suppose that the *normal wear* (PR in Fig. 80*b*)

* We shall see later that these equations may also be applied when the pivot is revolving uniformly. Then μ denotes the so-called coefficient of kinetic friction. As the value of this coefficient depends to some extent on the velocity at the surface element in question, and the velocity varies as the distance from the axis of the pivot, μ is not strictly constant. The μ in (2) should therefore be regarded as a mean value.

varies directly as the pressure p and the distance x from the axis; hence the *axial wear* $QR = PR/\sin\phi$, must vary as $px/\sin\phi$. The assumption of constant axial wear is therefore expressed by

$$(b) \qquad \frac{px}{\sin\phi} = \text{constant}, \quad \text{or} \quad p = \frac{C\sin\phi}{x}.$$

On substituting this value of p in (1) we obtain the equation

$$(1b) \qquad 2\pi C\int_b^a \sin\phi\, dx = W$$

which may be used to compute C. And from (2) we find

$$2\pi\mu C\int_{x=b}^{x=a} x\sin\phi\, ds = 2\pi\mu C\int_b^a x\, dx = M$$

whence

$$(2b) \qquad M = \pi\mu C(a^2 - b^2).$$

We shall now apply these results to pivots of various forms, using assumptions (a) and (b) in turn.

Example 1. *Flat Pivot.* Consider first a *hollow* pivot (Fig. 80c). When p is constant we have from (2a)

$$(3) \qquad M = \frac{2\mu W}{a^2 - b^2}\int_b^a x^2\, dx = \frac{2}{3}\frac{a^3 - b^3}{a^2 - b^2}\mu W.$$

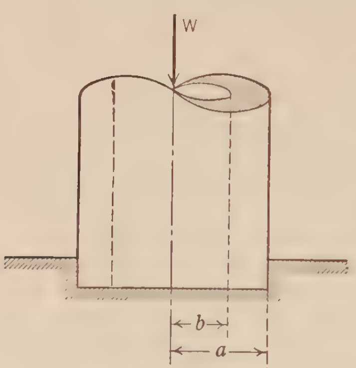


FIG. 80c.

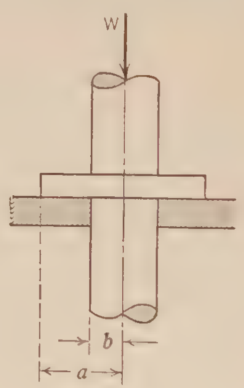


FIG. 80d.

For uniform axial wear we have from (1b), with $\phi = 90°$,

$$2\pi C\int_b^a dx = W, \quad C = \frac{W}{2\pi(a - b)};$$

then from (2b)

$$(4) \qquad M = \pi\mu\cdot\frac{W}{2\pi(a - b)}(a^2 - b^2) = \tfrac{1}{2}(a + b)\mu W.$$

The above results also apply to the collar bearing of Fig. 80d.

For a *solid* flat pivot we need only put $b = 0$ in the above results; thus

$$M = \tfrac{2}{3}\,a\mu W \quad \text{or} \quad \tfrac{1}{2}\,a\mu W$$

in the respective cases.

Example 2. *Conical Pivot.* The angle $\phi = \alpha$ (Fig. 80*e*) and $ds = dx/\sin\alpha$.

When p is constant we have from (2*a*)

$$(5) \qquad M = \frac{2\,\mu W}{a^2 - b^2}\int_b^a \frac{x^2\,dx}{\sin\alpha} = \frac{2}{3}\,\frac{a^3 - b^3}{a^2 - b^2}\,\frac{\mu W}{\sin\alpha}.$$

For uniform axial wear we have from (1*b*) and (2*b*)

$$(6) \qquad M = \tfrac{1}{2}\,(a + b)\,\frac{\mu W}{\sin\alpha}.$$

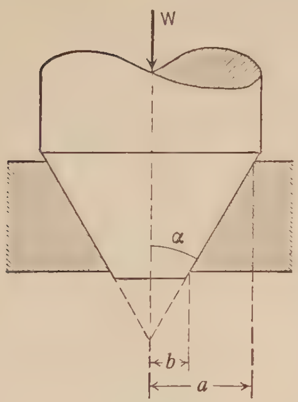


FIG. 80*e*.

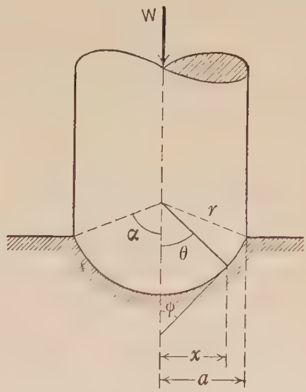


FIG. 80*f*.

Example 3. *Spherical Pivot.* The radius of the pivot is r. From Fig. 80*f* we have

$$a = r\sin\alpha, \quad b = 0; \quad x = r\sin\theta, \quad ds = r\,d\theta.$$

When p is constant we have from (2*a*)

$$M = \frac{2\,\mu W}{r^2\sin^2\alpha}\int_0^\alpha r^2\sin^2\theta \cdot r\,d\theta = \mu W r\,\frac{\alpha - \sin\alpha\cos\alpha}{\sin^2\alpha}.$$

For uniform axial wear we have from (1*b*), with $\sin\phi = \cos\theta$,

$$2\,\pi C\int_0^\alpha \cos\theta \cdot r\cos\theta\,d\theta = \pi r C\,(\alpha + \sin\alpha\cos\alpha) = W,$$

and hence from (2*b*),

$$M = \frac{\pi\mu W}{\pi r(\alpha + \sin\alpha\cos\alpha)}\,r^2\sin^2\alpha = \mu W r\,\frac{\sin^2\alpha}{\alpha + \sin\alpha\cos\alpha}.$$

PROBLEMS

1. A solid 6 in. vertical shaft and its load weigh 15,000 lb. If its lower end is supported by a flat step bearing for which $\mu = 0.02$, find the frictional moment when (*a*) the pressure is constant, (*b*) the axial wear is constant.

2. Solve the above problem when the shaft is hollow and 3 in. in inside diameter.

3. A propeller shaft, 4 in. in diameter, has two collar bearings of 8-in. outside diameter. Find the frictional moment if the thrust on the shaft is 50,000 lb. and $\mu = 0.03$, when (*a*) the pressure is constant, (*b*) the axial wear is constant.

4. A solid 4-in. vertical shaft and its load weigh 10,000 lb. If $\mu = 0.01$, compute the frictional moment, under the assumption of constant axial wear, when the end forms:

(1) a flat pivot;
(2) a truncated conical pivot with $a = 2$, $b = \frac{1}{2}$ in., $\alpha = 30°$;
(3) a hemispherical pivot.

81. Friction Clutches. In the design of friction clutches, such as are used in connecting the motor of an automobile with the transmission, the normal pressure p is assumed to be constant. In the case of disk clutches, formula (§ 80, 3) gives the maximum torque that can be transmitted, where W denotes the axial spring pressure on the clutch. In multiple-disk clutches a number n of annular disks engage. There will then be $n - 1$ friction surfaces and the right member of (§ 80, 3) must be multiplied by $n - 1$. In practice $a - b$ is small compared with a and the mean torque radius may be taken as $\frac{1}{2}(a + b)$ instead of $\frac{2}{3}(a^3 - b^3)/(a^2 - b^2)$.* The maximum torque transmitted by n disks is thus practically equal to

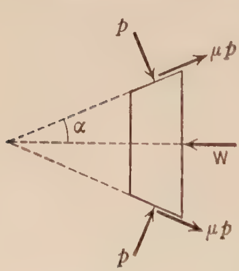


Fig. 81.

$$(1) \qquad M = (n - 1)\, r\mu W$$

where r is the mean disk radius, W the spring pressure.

In the case of cone clutches, formula (§ 80, 5) is not correct since only circumferential friction was considered in deriving it. If p is the normal pressure intensity between the cones, the friction opposing disclutching is μp per unit area (Fig. 81);

* In fact the difference of these quantities is $\frac{1}{6}(a - b)^2/(a + b)$.

and for equilibrium

$$\int (p \sin \alpha + \mu p \cos \alpha)\, dA = W, \quad pA = \frac{W}{\sin \alpha + \mu \cos \alpha}.$$

The friction opposing rotational slipping is μpA; hence, if the mean torque radius is again taken as $r = \frac{1}{2}(a + b)$, we have

$$M = r \cdot \mu p A = \frac{r\mu W}{\sin \alpha + \mu \cos \alpha}. \tag{2}$$

PROBLEMS

1. Find the requisite spring pressure on a cone clutch of 7-in. mean radius and angle $\alpha = 12\frac{1}{2}°$ in order to transmit a torque of 2000 lb.-in. $\mu = 0.2$. If $p = 12$ lb./in.^{2} is the allowable normal pressure, find the width of the friction surface.

2. Find the number of annular metal disks, of mean radius 5 in., required in a lubricated multiple-disk clutch in order to transmit a torque of 2000 lb.-in. with a spring pressure of 250 lb. on the disks. $\mu = 0.06$.

82. Summary, Chapter V. The *moment of a force* **F** *about a point A* is defined as the vector $\overrightarrow{AP} \times \mathbf{F}$, where P is any point on the action-line of **F**. The moment of **F** about any *axis* through A is equal to the component of $\overrightarrow{AP} \times \mathbf{F}$ on this axis.

A system of forces acting on a rigid body can always be reduced to a force acting through a given point A, and a couple; the force is a vector equal to force-sum **F** of the system, and the moment of the couple is equal to the moment-sum $\mathbf{M}_A$ of the system about A.

A system of forces acting on a rigid body can be reduced to (1) a single force, (2) a single couple, (3) to zero when, and only when, the following conditions are fulfilled:

(1) $\mathbf{F} \neq 0, \quad \mathbf{F} \cdot \mathbf{M}_A = 0$: Single Force;
(2) $\mathbf{F} = 0, \quad \mathbf{M}_A \neq 0$: Couple;
(3) $\mathbf{F} = 0, \quad \mathbf{M}_A = 0$: Zero.

In case (2) the moment-sum is the same for any choice of the point A; the vector $\mathbf{M}_A$ is then called the *moment of the couple.*

The conditions (3) are necessary and sufficient for the equilibrium of the rigid body. The vector equations (3) are equivalent to the six scalar equations:

$$F_x = 0, \quad F_y = 0, \quad F_z = 0; \quad M_x = 0, \quad M_y = 0, \quad M_z = 0.$$

If the forces acting on a rigid body are in equilibrium, their projections on any plane may also be treated as a system of forces in equilibrium.

If a rigid body, free to turn about a fixed axis in frictionless bearings, is in equilibrium, the moment-sum of the impressed forces about this axis is zero (Law of the Lever).

A necessary and sufficient condition for the equivalence of two systems of forces acting on a rigid body is that their force-sums be equal and that their moment-sums about any given point be equal. In particular, all couples having the same vector moment are equivalent.

A system of *parallel forces* $F_i\mathbf{e}$ acting at the points P_i and having a force-sum $\mathbf{F} \neq 0$ has a resultant, equal to $\mathbf{F}$, which acts through the centroid of the points P_i associated with the numbers F_i. The position vector $\mathbf{r}^*$ of this centroid is given by

$$\mathbf{r}^* \sum F_i = \sum F_i \mathbf{r}_i.$$

The resultant gravity of a body of weight W and volume V always acts through its *center of gravity* P^*, given by

$$W\mathbf{r}^* = \int \mathbf{r}\, dW \quad \text{or by} \quad V\mathbf{r}^* = \int \mathbf{r}\, dV$$

if the body is homogeneous. If dV is not infinitesimal in all its dimensions, $\mathbf{r}$ must be taken as the vector to the center of gravity of dV. If the body consists of a number of parts of weight W_i whose centers of gravity are given by $\mathbf{r}_i^*$, then

$$W\mathbf{r}^* = \sum W_i \mathbf{r}_i^*.$$

CHAPTER VI

VECTOR CALCULUS

83. Derivative of a Vector. Let $\mathbf{u}(t)$ denote a vector which varies with the scalar variable t in some definite manner. For the sake of definiteness, let us regard t as the time measured from a certain instant chosen as zero. To obtain a clear idea of the manner in which $\mathbf{u}$ varies with the time, let its initial point be held fast. This, of course, is no restriction as $\mathbf{u}$ may always be shifted parallel to itself. Then, as the time increases, the end point of $\mathbf{u}$ traces a certain curve Γ in space.

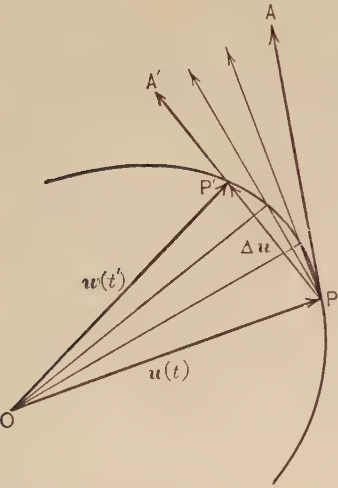


FIG. 83.

At the instants t and $t' = t + \Delta t$ let

$$\overrightarrow{OP} = \mathbf{u}(t), \qquad \overrightarrow{OP'} = \mathbf{u}(t').$$

The vectorial change in $\mathbf{u}$ during the interval Δt is

$$\Delta\mathbf{u} = \mathbf{u}(t') - \mathbf{u}(t) = \overrightarrow{OP'} - \overrightarrow{OP} = \overrightarrow{PP'}.$$

The average vectorial change per unit of time is therefore $\Delta\mathbf{u}/\Delta t$; this is a vector ($\overrightarrow{PA'}$ in Fig. 83) having the same direction as $\Delta\mathbf{u}$ and $1/\Delta t$ times as long. If $\overrightarrow{PA'}$ approaches a limiting vector $\overrightarrow{PA}$ as Δt approaches zero, we call this limiting vector the *derivative of* $\mathbf{u}$ *with respect to* t, and denote it by $d\mathbf{u}/dt$. The equation defining this derivative is therefore

$$(1) \qquad \frac{d\mathbf{u}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{u}(t + \Delta t) - \mathbf{u}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{u}}{\Delta t}.$$

As Δt approaches zero, P' describes the arc $P'P$ of the curve Γ, and the limiting direction of the chord $\overrightarrow{PP'}$, and hence of the vector $\overrightarrow{PA'}$, is the direction of tangency at P. The limiting vector

$\overrightarrow{PA} = d\mathbf{u}/dt$ is therefore tangent to Γ at P; it points in the instantaneous direction in which P moves as t increases. In view of the interpretation of $\Delta\mathbf{u}/\Delta t$ given above, we may say that $d\mathbf{u}/dt$ *represents the instantaneous rate at which the end-point of* $\mathbf{u}$ *changes its position, relative to its initial point.*

Equation (1) defines $d\mathbf{u}/dt$ when t is *any* scalar variable. Since $\Delta\mathbf{u}/\Delta t$ has the same direction as $\Delta\mathbf{u}$ when Δt is positive, the opposite when Δt is negative, $d\mathbf{u}/dt$ is a vector tangent to Γ in the direction of increasing t. We restate this important result as follows:

If the vector $\mathbf{u}(t) = \overrightarrow{OP}$ *varies with* t *so that* P *describes the curve* Γ *when* O *is held fast, the derivative* $d\mathbf{u}/dt$, *for any value of* t, *is a vector tangent to* Γ *at* P *in the direction of increasing* t.

Example 1. If $\mathbf{u} = \overrightarrow{OP}$ is a variable vector of constant direction, P will move on a straight line when O is held fast; hence $d\mathbf{u}/dt$, being tangent to this line, will be parallel to $\mathbf{u}$.

Example 2. If $\mathbf{u} = \overrightarrow{OP}$ is a variable vector of constant length, P will describe a curve Γ on the surface of a sphere when O is held fast; hence $d\mathbf{u}/dt$, being tangent to Γ at P, will be perpendicular to the radius OP of the sphere. In brief: *If* $|\mathbf{u}|$ *is constant,* $d\mathbf{u}/dt$ *is perpendicular to* $\mathbf{u}$.

If $\mathbf{u}$ is a constant vector, that is, constant in both length and direction,

$$\Delta\mathbf{u} = 0, \qquad \frac{\Delta\mathbf{u}}{\Delta t} = 0, \quad \text{and} \quad \therefore \quad \frac{d\mathbf{u}}{dt} = 0.$$

The derivative of a constant vector is zero.

When $\mathbf{u}$ is a function of a scalar variable s, and s in turn a function of t, a change of Δt in t will produce a change Δs in s and therefore a change $\Delta\mathbf{u}$ in $\mathbf{u}$. On passing to the limit $\Delta t \to 0$ in the identity

$$\frac{\Delta\mathbf{u}}{\Delta t} = \frac{\Delta\mathbf{u}}{\Delta s}\frac{\Delta s}{\Delta t}$$

we obtain the familiar rule for differentiating a function of a function:

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{u}}{ds}\frac{ds}{dt}. \tag{2}$$

Just as in the Calculus we define the *second derivative* of a vector as the derivative of its derivative, and write

$$\frac{d^2\mathbf{u}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{u}}{dt}\right).$$

84. Derivatives of Sums and Products. Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be two vectors that are functions of a scalar t. When t increases by an amount Δt, let $\Delta\mathbf{u}$, $\Delta\mathbf{v}$, and $\Delta(\mathbf{u} + \mathbf{v})$ denote the vectorial changes in $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$. Then

$$\begin{aligned} \mathbf{u} + \mathbf{v} + \Delta(\mathbf{u} + \mathbf{v}) &= \mathbf{u} + \Delta\mathbf{u} + \mathbf{v} + \Delta\mathbf{v}, \\ \Delta(\mathbf{u} + \mathbf{v}) &= \Delta\mathbf{u} + \Delta\mathbf{v}, \\ \frac{\Delta(\mathbf{u} + \mathbf{v})}{\Delta t} &= \frac{\Delta\mathbf{u}}{\Delta t} + \frac{\Delta\mathbf{v}}{\Delta t}; \end{aligned}$$

and passing to the limit $\Delta t \to 0$,

$$(1) \qquad \frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}.$$

Consequently, *the derivative of the sum of two vectors is equal to the sum of their derivatives.* This result may be generalized to the sum of any number of vectors.

Consider next the product $f(t)\mathbf{u}(t)$ of a scalar and a vector function of t. When t increases by an amount Δt, let Δf, $\Delta\mathbf{u}$, and $\Delta(f\mathbf{u})$ denote the increments of f, $\mathbf{u}$, and $f\mathbf{u}$ respectively. Then

$$\begin{aligned} f\mathbf{u} + \Delta(f\mathbf{u}) &= (f + \Delta f)(\mathbf{u} + \Delta\mathbf{u}) \\ &= f\mathbf{u} + f\Delta\mathbf{u} + \Delta f\,\mathbf{u} + \Delta f\Delta\mathbf{u}, \end{aligned}$$

since the multiplication of vectors by scalars is distributive (§ 6, 7, 8). Hence

$$\begin{aligned} \Delta(f\mathbf{u}) &= f\Delta\mathbf{u} + \Delta f\,\mathbf{u} + \Delta f\,\Delta\mathbf{u}, \\ \frac{\Delta(f\mathbf{u})}{\Delta t} &= f\,\frac{\Delta\mathbf{u}}{\Delta t} + \frac{\Delta f}{\Delta t}\,\mathbf{u} + \Delta f\,\frac{\Delta\mathbf{u}}{\Delta t}. \end{aligned}$$

Passing to the limit $\Delta t \to 0$, and noting that $\Delta f \to 0$, we have

$$(2) \qquad \frac{d}{dt}(f\mathbf{u}) = f\,\frac{d\mathbf{u}}{dt} + \frac{df}{dt}\,\mathbf{u},$$

a rule formally the same as that given in the Calculus for the derivative of a product.

Important special cases of (2) arise when either f or $\mathbf{u}$ is constant:

$$\frac{d}{dt}(c\mathbf{u}) = c\frac{d\mathbf{u}}{dt}, \quad \frac{d}{dt}(f\mathbf{c}) = \frac{df}{dt}\mathbf{c}. \tag{3}$$

If the components of the general space vector

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

are functions of the t

$$\frac{d\mathbf{u}}{dt} = \frac{du_x}{dt}\mathbf{i} + \frac{du_y}{dt}\mathbf{j} + \frac{du_z}{dt}\mathbf{k}.$$

The components of the derivative of a vector are therefore the derivatives of its components.

Passing now to the products $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are vector functions of t, we may prove in the same manner as above that

$$(4), (5) \quad \frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}, \quad \frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\times\mathbf{v}.$$

The proofs, depend essentially on the distributive laws (§ 15, 2, 3) and (§ 17, 2, 3). Note that in (5) the order of the factors must be preserved.

Since $\mathbf{u}^2 = \mathbf{u}\cdot\mathbf{u}$, we have from (4)

$$\frac{d}{dt}\mathbf{u}^2 = 2\,\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}.$$

In particular if $\mathbf{u}$ is a variable vector of constant length,

$$\mathbf{u}^2 = \text{constant}, \quad \mathbf{u}\cdot\frac{d\mathbf{u}}{dt} = 0, \quad \text{and} \quad \frac{d\mathbf{u}}{dt} \perp \mathbf{u}. \tag{6}$$

If $|\mathbf{u}|$ *is constant,* $d\mathbf{u}/dt$ *is perpendicular to* $\mathbf{u}$ (cf. § 83, Example 2).

If $\mathbf{e}$ is a positive unit vector along the axis s, the component of $\mathbf{u}$ on s is $\mathbf{e}\cdot\mathbf{u}$ (§ 14, 3). Hence, from (4),

$$\frac{d}{dt}(\text{comp}_s\,\mathbf{u}) = \mathbf{e}\cdot\frac{d\mathbf{u}}{dt} = \text{comp}_s\frac{d\mathbf{u}}{dt}.$$

The derivative of the component of a vector on an axis is equal to the component of its derivative on this axis.

85. Unit Tangent Vector. Let $\mathbf{r} = \overrightarrow{OP}$ be the position vector of a point P moving along a curve; and let s denote the arc AP measured from a fixed point A of the curve, reckoned posi-

tive in one direction along the curve, negative in the other. From the definition of the derivative, we have

$$\frac{d\mathbf{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \mathbf{r}}{\Delta s} = \lim_{P' \to P} \frac{\overrightarrow{PP'}}{\text{arc } PP'}.$$

As P' approaches P, the ratio of the chord PP' to the arc PP' approaches unity. The magnitude of $\Delta\mathbf{r}/\Delta s$ therefore increases toward 1 as a limit. Moreover $\Delta\mathbf{r}/\Delta s$ has the same direction as $\Delta\mathbf{r}$ if Δs is positive, the opposite if Δs is negative. The limiting vector $d\mathbf{r}/ds$ is thus a unit vector * $\mathbf{T}$ tangent to the curve at P and pointing in the direction of increasing arcs:

$$(1) \qquad \frac{d\mathbf{r}}{ds} = \mathbf{T}.$$

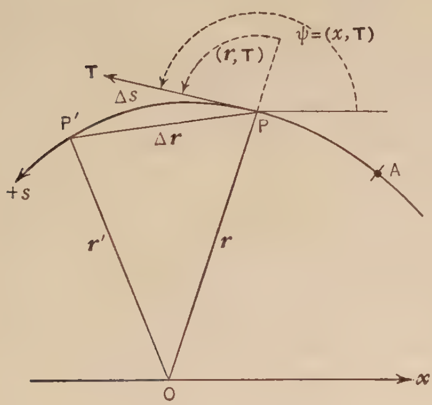


FIG. 85a.

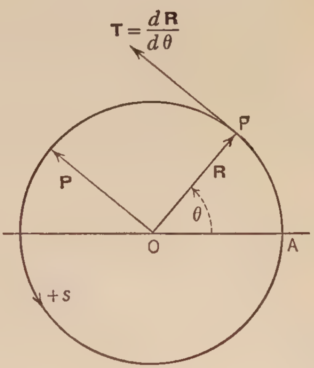


FIG. 85b.

An important special case arises when $\mathbf{r}$ is a *unit* vector revolving in a plane. If we imagine this vector, $\mathbf{R} = \overrightarrow{OP}$, always drawn from the same initial point O, its end-point will describe a circle of unit radius (Fig. 85b), and $s = \theta$, where θ denotes the angle *AOP expressed in radians.* Hence

$$\frac{d\mathbf{R}}{d\theta} = \frac{d\mathbf{R}}{ds} = \mathbf{T} = \mathbf{P},$$

where $\mathbf{P}$ is a unit vector perpendicular to $\mathbf{R}$ in the direction of increasing angles. If $\mathbf{k}$ is a unit vector normal to the plane,

* We shall use small capitals in heavy type to denote unit vectors, except in the case of **i**, **j**, **k**, **e**.

associated with the positive sense of θ according to the rule of the right-hand screw, $\mathbf{P} = \mathbf{k}\times\mathbf{R}$. Thus

$$(2) \qquad \frac{d\mathbf{R}}{d\theta} = \mathbf{P} \quad \text{or} \quad \frac{d\mathbf{R}}{d\theta} = \mathbf{k}\times\mathbf{R}.$$

In Fig. 85*b*, **k** points upward from the paper.

The derivative of a unit vector, revolving in a plane, with respect to the angle that it makes with a fixed direction, is another unit vector perpendicular to the first in the direction of increasing angles.

Since $\mathbf{P}$ is a unit vector revolving at the same rate as $\mathbf{R}$,

$$(3) \qquad \frac{d\mathbf{P}}{d\theta} = -\mathbf{R} \quad \text{or} \quad \frac{d\mathbf{P}}{d\theta} = \mathbf{k}\times\mathbf{P};$$

for $-\mathbf{R}$ is a unit vector perpendicular to $\mathbf{P}$ in the direction of increasing angles.

Example 1. If θ is measured counterclockwise from the x-axis,

$$\mathbf{R} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta, \quad \mathbf{P} = \mathbf{i}\cos(\theta + \tfrac{1}{2}\pi) + \mathbf{j}\sin(\theta + \tfrac{1}{2}\pi).$$

In view of (2) the corresponding components of $d\mathbf{R}/d\theta$ and $\mathbf{P}$ must be equal; we thus deduce the formulas

$$\frac{d}{d\theta}\cos\theta = \cos(\theta + \tfrac{1}{2}\pi) = -\sin\theta, \qquad \frac{d}{d\theta}\sin\theta = \sin(\theta + \tfrac{1}{2}\pi) = \cos\theta.$$

Example 2. If $P(x, y)$ is a variable point on a plane curve,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}, \quad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}.$$

Hence

$$(4), (5) \qquad \mathbf{i}\cdot\mathbf{T} = \cos(x, \mathbf{T}) = \frac{dx}{ds}, \quad \mathbf{j}\cdot\mathbf{T} = \sin(x, \mathbf{T}) = \frac{dy}{ds}.$$

If (r, θ) are the polar coördinates of P, we write $\mathbf{r} = r\mathbf{R}$ where $\mathbf{R}$ is a unit vector; then

$$\mathbf{T} = \frac{dr}{ds}\mathbf{R} + r\frac{d\mathbf{R}}{ds} = \frac{dr}{ds}\mathbf{R} + r\frac{d\mathbf{R}}{d\theta}\frac{d\theta}{ds} = \frac{dr}{ds}\mathbf{R} + r\frac{d\theta}{ds}\mathbf{P},$$

$$(6), (7) \qquad \mathbf{R}\cdot\mathbf{T} = \cos(\mathbf{r}, \mathbf{T}) = \frac{dr}{ds}, \quad \mathbf{P}\cdot\mathbf{T} = \sin(\mathbf{r}, \mathbf{T}) = r\frac{d\theta}{ds}.$$

Example 3. If r_1, r_2 are the distances of a point P on an ellipse from the foci, $r_1 + r_2 = \text{const}$. On differentiating this equation with respect to s we have, in view of (6),

$$\frac{dr_1}{ds} + \frac{dr_2}{ds} = (\mathbf{R}_1 + \mathbf{R}_2)\cdot\mathbf{T} = 0.$$

As $\mathbf{R}_1 + \mathbf{R}_2$ is perpendicular to $\mathbf{T}$, the normal to the ellipse at P has the direction of $\mathbf{R}_1 + \mathbf{R}_2$. The normal therefore bisects the angle between the focal radii.

PROBLEMS

1. If r and x are the distances of a point on a parabola from the focus and directrix, $r - x = 0$. Show that $(\mathbf{R} - \mathbf{i})\cdot\mathbf{T} = 0$ and interpret the equation. [Cf. Example 3.]

2. An equiangular spiral cuts all vectors from its pole O at the same angle α. If r, θ are the polar coördinates of a variable point on the spiral, show that

(*a*) $ds/dr = \sec \alpha$ and $s_2 - s_1 = (r_2 - r_1) \sec \alpha.$

(*b*) $\dfrac{1}{r}\dfrac{dr}{d\theta} = \cot \alpha$ and $\ln \dfrac{r_2}{r_1} = (\theta_2 - \theta_1) \cot \alpha.$

[Apply (6) and (7).]

3. If $f(x, y) = 0$ is the Cartesian equation of a plane curve, show that the normal is parallel to

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}. \qquad \text{[Apply (4) and (5).]}$$

4. If $f(r, \theta) = 0$ is the polar equation of a plane curve, show that the normal is parallel to

$$\frac{\partial f}{\partial r}\mathbf{R} + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{P}. \qquad \text{[Apply (6) and (7).]}$$

5. The ellipse $r_1 + r_2 = c$ and hyperbola $r_1 - r_2 = c'$ have the same foci. Show that they intersect at right angles.

6. If O_1, O_2 are the poles and O_1O_2 the initial line of a system of bipolar coördinates, show that the two families of circles

$$r_1/r_2 = \text{constant}, \quad \theta_2 - \theta_1 = \text{constant}$$

cut at right angles.

86. Curvature. If $\mathbf{T}$ is the unit tangent vector to the space curve Γ at P (Fig. 86*a*), $|\mathbf{T}| = 1$ and $d\mathbf{T}/ds$ is perpendicular to $\mathbf{T}$ (§ 84). A directed line through P in the direction of $d\mathbf{T}/ds$ is therefore a normal of the curve;* it is called the *principal normal* of Γ at P. If $\mathbf{N}$ is a unit vector in the direction of the principal normal, we may write

$$(1) \qquad \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \quad \text{where} \quad \kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

The positive number κ is called the *curvature* of Γ at P.

* Any line through P perpendicular to $\mathbf{T}$ is called a *normal* of the curve at P. Thus there are infinitely many normals at P; they all lie in the *normal plane* at P.

Consider, for example, a circle of radius r (Fig. 86b). If s denotes the arc AP, taken positive in the counterclockwise direction,

$$\mathbf{T} = \mathbf{P} \quad \text{and} \quad \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{P}}{ds} = \frac{d\mathbf{P}}{d\theta}\frac{d\theta}{ds} = (-\mathbf{R})\frac{1}{r};$$

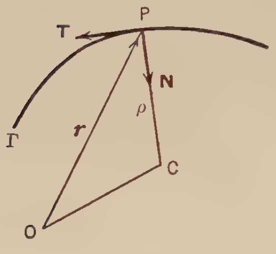


FIG. 86a.

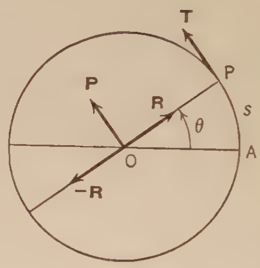


FIG. 86b.

for $d\mathbf{P}/d\theta = -\mathbf{R}$ (§ 85, 3) and $\theta = s/r$, $d\theta/ds = 1/r$. Thus $-\mathbf{R}$ gives the direction of the principal normal at $\mathbf{P}$ and $1/r$ is the curvature. Therefore *the curvature of a circle is everywhere equal to the reciprocal of its radius.*

Let us return now to our general curve Γ whose curvature at P is κ. This is also the curvature of a circle of radius $1/\kappa$; for this reason $1/\kappa$ is called the *radius of curvature* of Γ at P. If we denote the radius of curvature by ρ, $\rho = 1/\kappa$.

The *center of curvature* of Γ at P is a point C on the principal normal at a distance ρ from C in the direction of $\mathbf{N}$; hence

$$\text{(2)} \qquad \overrightarrow{PC} = \rho\mathbf{N} \quad \text{and} \quad \overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{PC} = \mathbf{r} + \rho\mathbf{N}.$$

A directed line through P in the direction of the unit vector $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ is called the *binormal* of Γ at P. The binormal is thus perpendicular to both tangent and principal normal at P. At every point of a curve the three unit vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ form a right-handed set giving the positive directions along the tangent, principal normal, and binormal respectively. Therefore

$$\text{(3)} \qquad \mathbf{T}\times\mathbf{N} = \mathbf{B}, \quad \mathbf{N}\times\mathbf{B} = \mathbf{T}, \quad \mathbf{B}\times\mathbf{T} = \mathbf{N}.$$

By way of summary, we restate these important definitions: *At any point P of a curve the derivative of the unit tangent vector with respect to the arc ($d\mathbf{T}/ds$) gives the direction of the principal normal; and the numerical value κ of this derivative is called the*

curvature at P. The reciprocal of the curvature, $\rho = 1/\kappa$, is called the radius of curvature. The point C given by $\overrightarrow{PC} = \rho\mathbf{N}$ is the center of curvature.

To compute the radius of curvature of a curve Γ, let us suppose that $\mathbf{r}$ is given as a function of a scalar variable t — the time, for example. Then

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt},$$

$$\frac{d^2\mathbf{r}}{dt^2} = \left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right)\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} = \mathbf{N}\kappa\left(\frac{ds}{dt}\right)^2 + \mathbf{T}\frac{d^2s}{dt^2};$$

$$\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2} = \mathbf{T}\times\mathbf{N}\,\kappa\left(\frac{ds}{dt}\right)^3 = \mathbf{B}\frac{1}{\rho}\left(\frac{ds}{dt}\right)^3.$$

If we equate the numerical values of the vectors forming the members of this equation and observe that

$$|ds/dt| = |d\mathbf{r}/dt|,$$

we obtain

$$\rho = \frac{\left|\dfrac{d\mathbf{r}}{dt}\right|^3}{\left|\dfrac{d\mathbf{r}}{dt}\times\dfrac{d^2\mathbf{r}}{dt^2}\right|} \quad\text{or}\quad \rho = \frac{|\dot{\mathbf{r}}|^3}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}, \tag{4}$$

where the dots denote differentiation with respect to t.

For example, if the rectangular coördinates x, y along a plane curve are given as functions of t,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j},\quad \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j},\quad \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j},$$

$$|\dot{\mathbf{r}}|^2 = \dot{x}^2 + \dot{y}^2,\quad \dot{\mathbf{r}}\times\ddot{\mathbf{r}} = (\dot{x}\ddot{y} - \dot{y}\ddot{x})\mathbf{k},$$

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}.$$

If the Cartesian equation of the curve is $y = f(x)$, we write $x = t$, $y = f(t)$; then $\dot{x} = 1$, $\ddot{x} = 0$, and

$$\rho = \frac{(1 + y'^2)^{\frac{3}{2}}}{|y''|}$$

where the primes denote derivatives with respect to x.

When the polar equation, $r = f(\theta)$ of the curve is known, we let θ play the part of t in (4). Denoting derivatives with respect to θ by primes, we have from (§ 85, 2, 3),

$$\mathbf{r} = r\mathbf{R},\quad \mathbf{r}' = r'\mathbf{R} + r\mathbf{P},\quad \mathbf{r}'' = (r'' - r)\mathbf{R} + 2\,r'\mathbf{P};\ \text{and}$$

$$\rho = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{|r^2 + 2\,r'^2 - r'r''|}.$$

87. Plane Curves. Since the vector $d\mathbf{T}/ds$ is always directed toward the concave side of a plane curve, the same is true of $\mathbf{N}$. The unit vector $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ along the binormal is perpendicular to the plane of the curve; but on passing a point of inflection $\mathbf{B}$ reverses its direction. However, along any arc of the curve without points of inflection, $\mathbf{B}$ will have a constant direction; and since

$$\mathbf{N} = \mathbf{B}\times\mathbf{T}, \qquad \frac{d\mathbf{N}}{ds} = \mathbf{B}\times\frac{d\mathbf{T}}{ds} = \mathbf{B}\times\kappa\mathbf{N} = -\kappa\mathbf{T}.$$

The equations

(1), (2) $$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \quad \text{and} \quad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T}$$

are fundamental in the differential geometry of plane curves.

Let ψ denote the angle that $\mathbf{T}$ makes with a fixed direction in its plane. Then, from the Theorem of § 85, $d\mathbf{T}\, d\psi = \pm\mathbf{N}$, the sign being chosen so that the vector points in the direction of increasing ψ. Consequently

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{d\psi}\frac{d\psi}{ds} = \pm\mathbf{N}\frac{d\psi}{ds} \quad \text{and} \quad \kappa = \left|\frac{d\psi}{ds}\right|.$$

If ψ increases with s, $d\psi/ds$ is positive and

(3) $$\kappa = \frac{d\psi}{ds}, \quad \rho = \frac{ds}{d\psi}.^*$$

Example 1. As a point P describes a plane curve Γ, the center of curvature C describes its *evolute* Γ' (Fig. 87*a*). If $\mathbf{r} = \overrightarrow{OP}$, $\mathbf{r}' = \overrightarrow{OC}$, and s, s' denote the arcs P_0P and C_0C along the curves,

$$\mathbf{r}' = \mathbf{r} + \rho\mathbf{N} \qquad (§ 86, 2),$$

$$\frac{d\mathbf{r}'}{ds} = \frac{d\mathbf{r}'}{ds'}\frac{ds'}{ds} = \frac{d\mathbf{r}}{ds} + \rho\frac{d\mathbf{N}}{ds} + \frac{d\rho}{ds}\mathbf{N},$$

or
$$\mathbf{T}'\frac{ds'}{ds} = \mathbf{T} - \rho\kappa\mathbf{T} + \frac{d\rho}{ds}\mathbf{N} = \frac{d\rho}{ds}\mathbf{N}.$$

* In the differential geometry of plane curves it is better to take $\mathbf{B}$ as a *constant* vector perpendicular to the plane and then define $\mathbf{N} = \mathbf{B}\times\mathbf{T}$. The curvature κ is defined by (1) and may be positive or negative. Equation (2) holds as before. The positive direction of ψ is chosen so that a right-hand screw, given a positive rotation in the plane, will advance in the direction of $\mathbf{B}$. Then $\mathbf{N}$ is always 90° in advance of $\mathbf{T}$, and

$$\frac{d\mathbf{T}}{d\psi} = \mathbf{N}, \qquad \frac{d\mathbf{T}}{ds} = \mathbf{N}\frac{d\psi}{ds}, \qquad \kappa = \frac{d\psi}{ds}.$$

With these definitions $\mathbf{N}$ need not point toward the center of curvature. In mechanics, however, it is better to adopt the definitions given in the text; then κ is positive and $\mathbf{N}$ always points toward the center of curvature.

Since both $\mathbf{T}'$ and $\mathbf{N}$ are unit vectors,

$$\mathbf{T}' = \mathbf{N} \quad \text{and} \quad \frac{ds'}{ds} = \frac{d\rho}{ds}.$$

The first equation shows that a tangent to the evolute is normal to Γ. If we integrate the second from C_0 to C,

$$s' = \text{arc } C_0C = \rho - \rho_0;$$

hence an arc of the evolute is equal to the difference of the radii of curvature which correspond to its end-points. These properties show that Γ may be traced by the end of taut string unwound from the evolute, the string being always tangent to Γ' and its free portion equal to ρ. From this point of view, Γ is called the *involute* of Γ'.

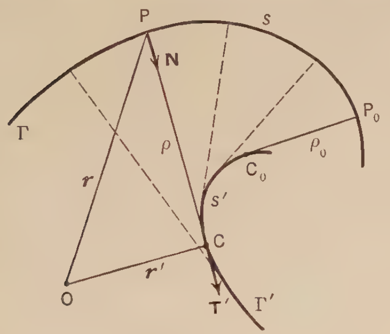


FIG. 87*a*.

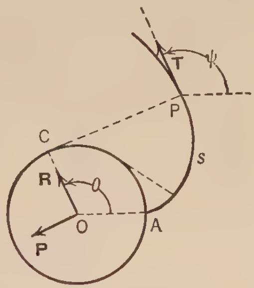


FIG. 87*b*.

Example 2. The *involute of a circle* is generated by a point P of a taut string unwound from a circle (Fig. 87*b*). If P was originally at A on the circle, $CP = \text{arc } CA = a\theta$, where a is the radius of the circle; hence

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP} = a\mathbf{R} - a\theta\mathbf{P},$$

$$\frac{d\mathbf{r}}{d\theta} = \frac{d\mathbf{r}}{ds}\frac{ds}{d\theta} = a\frac{d\mathbf{R}}{d\theta} - a\mathbf{P} - a\theta\frac{d\mathbf{P}}{d\theta},$$

or $$\mathbf{T}\frac{ds}{d\theta} = a\theta\mathbf{R} \qquad (\S\ 85,\ 2,\ 3).$$

Since both $\mathbf{T}$ and $\mathbf{R}$ are unit vectors,

(i, ii) $$\mathbf{T} = \mathbf{R} \quad \text{and} \quad \frac{ds}{d\theta} = a\theta.$$

Equation (i) shows that the tangent to the involute at P is parallel to the radius OC of the circle. Hence the angle $\psi = \theta$ and from (3), $\rho = ds/d\theta$. Equation (ii) now shows that $\rho = a\theta = CP$; therefore C is the center of curvature of the involute at P. Moreover on integrating (ii) we find that the arc $AP = s = \frac{1}{2}a\theta^2$.

PROBLEMS

1. Show that the tangent to the equiangular spiral of § 85, Problem 2, at the point $P(r, \theta)$ makes an angle of $\psi = \theta + \alpha$ with the initial line and

$$\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta} = \frac{r}{\sin\alpha} \qquad (\S\ 85, 7).$$

Hence prove that the center of curvature is the point where the perpendicular to OP at O cuts the normal at P.

2. If $\mathbf{r}$ is the position vector along a plane curve Γ, $\mathbf{r}' = \mathbf{r} + c\mathbf{N}$ is the position vector along a *parallel curve* Γ' at a normal distance c from Γ. Prove that

$$\mathbf{T}'\frac{ds'}{ds} = (1 - c\kappa)\,\mathbf{T} \quad \text{and hence} \quad \mathbf{T}' = \mathbf{T}, \quad \frac{ds'}{ds} = 1 - c\kappa.$$

Show also that $\rho' = \rho - c$ and $s' = s - c\psi$ when both s and s' are measured from a common normal.

3. A *cycloid* is traced by a point P on a circle of radius b that rolls without slipping along a straight line (Fig. 87c). Then

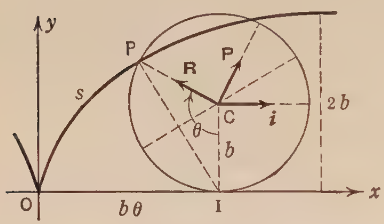


FIG. 87c.

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OI} + \overrightarrow{IC} + \overrightarrow{CP} = b\theta\mathbf{i} + b\mathbf{j} + b\mathbf{R}$$

where $\mathbf{R}$ is a unit vector. Prove that

$$\mathbf{T} = b\frac{d\theta}{ds}(\mathbf{i} + \mathbf{P}), \qquad 1 = 2\,b \sin \tfrac{1}{2}\,\theta \frac{d\theta}{ds},$$

and that $\mathbf{T}$ is perpendicular to IP.

Since the angle $\psi = (\mathbf{j}, \mathbf{T}) = \frac{1}{2}\,\theta$, show that

$$\rho = 4\,b \sin \tfrac{1}{2}\,\theta = 2\,PI, \quad s = \text{arc } OP = 4\,b\,(1 - \cos \tfrac{1}{2}\,\theta).$$

88. Integral of a Vector. If $\mathbf{u}(t)$ and $\mathbf{U}(t)$ are vector functions of the variable t such that

$$\text{(1)} \qquad \frac{d\mathbf{U}}{dt} = \mathbf{u},$$

then $\mathbf{U}$ is called an *integral of* $\mathbf{u}$ *with respect to* t and is written

$$\text{(2)} \qquad \mathbf{U} = \int \mathbf{u}\, dt.$$

In other words, both (1) and (2) state the same fact: the derivative of $\mathbf{U}$ with respect to t is $\mathbf{u}$. The process of finding a function

which has a given derivative is called *integration.* In (2) the function $\mathbf{u}$ — the *integrand* — is *integrated* to produce $\mathbf{U}$.

If $\mathbf{U}$ is any function which satisfies (1), then $\mathbf{U} + \mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector, will also satisfy (1). Hence the integral $\mathbf{U}$ is indefinite to the extent of an additive vector constant — the *constant of integration.* For this reason the function $\mathbf{U}$ denoted by (2) is called an *indefinite integral.*

Just as in the Calculus we may show that

$$\int (\mathbf{u} + \mathbf{v})\, dt = \int \mathbf{u}\, dt + \int \mathbf{v}\, dt,$$

$$\int c\mathbf{u}\, dt = c \int \mathbf{u}\, dt, \quad \int \mathbf{c}u\, dt = \mathbf{c} \int u\, dt.$$

If $\mathbf{u}$ is expressed in terms of its components (§ 11, 1), we have

$$\int \mathbf{u}\, dt = \mathbf{i} \int u_x\, dt + \mathbf{j} \int u_y\, dt + \mathbf{k} \int u_z\, dt.$$

The integration of vector functions is thus reduced to the integration of scalar functions.

89. Definite Integral. The definite integral $\int_a^b \mathbf{u}(t)\, dt$ of the vector function $\mathbf{u}(t)$ is defined as a limit of a sum just as in the ordinary Calculus. If $\mathbf{u}(t)$ is expressed in terms of its components, we have

$$\int_a^b \mathbf{u}\, dt = \mathbf{i} \int_a^b u_x\, dt + \mathbf{j} \int_a^b u_y\, dt + \mathbf{k} \int_a^b u_z\, dt.$$

Thus vector definite integrals may be reduced to scalar definite integrals. Owing to this fact many of the properties of scalar definite integrals may be extended at once to the vector case; for example

$$\int_a^b \mathbf{u}\, dt = -\int_b^a \mathbf{u}\, dt.$$

Moreover if $\mathbf{U}(t)$ is an indefinite integral of $\mathbf{u}(t)$ we have the fundamental result

$$\text{(1)} \qquad \int_a^b \mathbf{u}(t)\, dt = \mathbf{U}(b) - \mathbf{U}(a).$$

If the upper limit in (1) is the variable of integration,

$$\int_a^t \mathbf{u}(t)\, dt = \mathbf{U}(t) - \mathbf{U}(a)$$

and hence from (§ 88, 1)

$$(2) \qquad \frac{d}{dt}\int_a^t \mathbf{u}(t)\,dt = \frac{d\mathbf{U}(t)}{dt} = \mathbf{u}(t).$$

90. Summary, Chapter VI. The derivative $d\mathbf{u}/dt$ of a vector $\mathbf{u}$ with respect to a scalar variable t is defined as the limit of $\Delta\mathbf{u}/\Delta t$ as Δt approaches zero. If $\mathbf{u} = \overrightarrow{OP}$ and P describes the curve Γ when O is held fast, $d\mathbf{u}/dt$ is a vector tangent to Γ at P in the direction of increasing t. In particular, if the magnitude of $\mathbf{u}$ is constant, $d\mathbf{u}/dt$ is perpendicular to $\mathbf{u}$.

The derivative of a constant vector is zero. The derivative of the sum $\mathbf{u} + \mathbf{v}$ and the products $f\mathbf{u}$, $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\times\mathbf{v}$ are found by rules of the same form as those given in the Calculus for differentiating a sum and a product; in the case of $\mathbf{u}\times\mathbf{v}$, however, the order of the factors must be preserved. Moreover the rule for differentiating a function of a function is like that in the Calculus.

If $\mathbf{r}$ is the position vector at a variable point P of a curve,

$$\frac{d\mathbf{r}}{dt} = \mathbf{T}, \quad \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

where $\mathbf{T}$ and $\mathbf{N}$ are unit vectors along the positive tangent and principal normal at P. The unit vector $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ defines the positive direction along the binormal. The positive number κ is the *curvature* at P; its reciprocal $1/\kappa = \rho$, the *radius of curvature.* If $\mathbf{T}$ for a plane curve makes an angle ψ with a fixed direction in the plane,

$$\kappa = \left|\frac{d\psi}{ds}\right|.$$

If $\mathbf{R}$ and $\mathbf{P}$ are unit vectors in a plane inclined θ and $\theta + \frac{1}{2}\pi$ radians to a fixed direction,

$$\frac{d\mathbf{R}}{d\theta} = \mathbf{P}, \quad \frac{d\mathbf{P}}{d\theta} = -\mathbf{R}.$$

The indefinite integral of $\mathbf{u}(t)$ is any function $\mathbf{U}(t)$ such that $d\mathbf{U}/dt = \mathbf{u}$. The definite integral $\int_a^b \mathbf{u}(t)\,dt$ is defined as the limit of a sum just as in the scalar Calculus; it equals $\mathbf{U}(b) - \mathbf{U}(a)$.

CHAPTER VII

FLEXIBLE CABLES

91. Principle E: Rigidification. We have seen that the statics of particles and rigid bodies may be based on four principles (§ 24). To deal with the equilibrium of deformable bodies, however, we need a fifth principle:

Principle E (Rigidification). *In order that a deformable body be in static equilibrium it is necessary and sufficient that any portion of it, when regarded as a rigid body, be in equilibrium under the external forces acting upon it.*

The "portion" of the deformable body referred to in this principle may be any part "cut off" by an ideal surface or any part in its interior bounded by an ideal closed surface. If the body is divided into two parts I and II by an ideal surface, the external forces acting on I consist of all the forces exerted on its particles by matter not included within its boundaries. They include, in particular, the forces of cohesion exerted by the particles on II on those of I in the immediate neighborhood of the ideal surface of separation. The latter forces are the *surface stresses* exerted by II on I.

In view of the Theorem of § 72 we may state at once the following criterion for the equilibrium of a deformable body.

Theorem E. *A deformable body is in equilibrium when, and only when, the force-sum and moment-sum of the external forces acting on any portion of it vanish.*

92. Flexible Cables. We shall treat a chain or a cable as if its matter were concentrated along a geometrical curve. If a cable is divided into two parts by an ideal section, the cable is said to be *flexible* when the action of either part on the other may be represented by a single force — not a force *and a couple*. When the cable is "cut" at any point P, the force $\mathbf{F}$ shall denote the action of the part to the right on that to the left; the reaction of the part to the left on that to the right is then $-\mathbf{F}$. The numerical value F of these forces is called the *tension* of the cable at P.

Consider, now, a cable acted on by a distributed force $\mathbf{Q}$ per unit of length. In general $\mathbf{Q}$ will vary from point to point. Let

us measure the arc s along the cable from a fixed point A and reckon s positive to the right. A small element PP' of the cable, of length Δs, is then acted on by the forces $-\mathbf{F}$ at P, $\mathbf{F}'$ at P', and the resultant of the distributed load $\mathbf{Q}\Delta s$ acting at same point $\bar{P}$ between P and P' (Fig. 92). In order that this element shall be in equilibrium the force-sum and moment-sum of those forces must vanish (Theorem E); hence

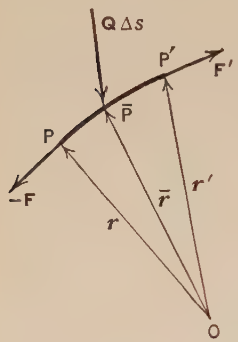


Fig. 92.

$$\mathbf{F}' - \mathbf{F} + \mathbf{Q}\Delta s = 0, \quad \mathbf{r}' \times \mathbf{F}' - \mathbf{r} \times \mathbf{F} + \bar{\mathbf{r}} \times \mathbf{Q}\Delta s = 0$$

where $\mathbf{r}$, $\mathbf{r}'$, $\bar{\mathbf{r}}$ are the position vectors of P, P', $\bar{P}$ relative to any origin O. Now $\mathbf{F}' - \mathbf{F}$ and $\mathbf{r}' \times \mathbf{F}' - \mathbf{r} \times \mathbf{F}$ are the increments of $\mathbf{F}$ and $\mathbf{r} \times \mathbf{F}$ respectively in passing from P to P'. The above equations may therefore be written

$$\frac{\Delta \mathbf{F}}{\Delta s} + \mathbf{Q} = 0, \quad \frac{\Delta(\mathbf{r} \times \mathbf{F})}{\Delta s} + \bar{\mathbf{r}} \times \mathbf{Q} = 0;$$

and on passing to the limit $\Delta s \to 0$,

(1), (2) $$\frac{d\mathbf{F}}{ds} + \mathbf{Q} = 0, \quad \frac{d(\mathbf{r} \times \mathbf{F})}{ds} + \mathbf{r} \times \mathbf{Q} = 0.$$

On expanding the derivative in (2) and reducing by means of (1) we obtain

$$\frac{d\mathbf{r}}{ds} \times \mathbf{F} + \mathbf{r} \times \left(\frac{d\mathbf{F}}{ds} + \mathbf{Q}\right) = \mathbf{T} \times \mathbf{F} = 0,$$

where $\mathbf{T}$ is the unit tangent vector at P. In other words, *the tensile stress at any point of the cable is tangential to the cable.* This result, which is almost intuitive, is thus seen to be a consequence of the moment equation (2).

Since equations (1), (2) ensure that every element of the cable is in equilibrium it is obvious that the same is true of any finite portion of the cable. In fact if we integrate (1) and (2) between the points P_1 and P_2 of the cable we get

$$\mathbf{F}_2 - \mathbf{F}_1 + \int_{s_1}^{s_2} \mathbf{Q}\, ds = 0, \quad \mathbf{r}_2 \times \mathbf{F}_2 - \mathbf{r}_1 \times \mathbf{F}_1 + \int_{s_1}^{s_2} \mathbf{r} \times \mathbf{Q}\, ds = 0;$$

that is, the force-sum and moment-sum of the external forces acting on P_1P_2 both vanish. We conclude, therefore, from Theo-

rem E, that equations (1), (2) give necessary and sufficient conditions for the equilibrium of the cable.

93. Scalar Equations of Equilibrium. Since the moment equation (§ 92, 2) implies that **F** has the direction of **T**, we may write the equations of equilibrium in the form

$$\text{(1), (2)} \qquad \frac{d\mathbf{F}}{ds} + \mathbf{Q} = 0, \quad \mathbf{F} = F\mathbf{T},$$

where F is the tension. If we substitute (2) in (1) and remember that $d\mathbf{T}/ds = \kappa\mathbf{N}$ (§ 86, 1) we obtain

$$\text{(3)} \qquad \frac{dF}{ds}\mathbf{T} + \kappa F\mathbf{N} + \mathbf{Q} = 0.$$

This equation of equilibrium is equivalent to the three scalar equations obtained by taking components along **T**, **N**, **B**:

$$\text{(4)} \qquad \frac{dF}{ds} + \mathbf{Q}\cdot\mathbf{T} = 0,$$

$$\text{(5)} \qquad \kappa F + \mathbf{Q}\cdot\mathbf{N} = 0,$$

$$\text{(6)} \qquad \mathbf{Q}\cdot\mathbf{B} = 0.$$

Example 1. If the distributed load on the cable has a constant direction, as in the case of a cable subject only to the action of gravity, the cable will lie in a plane. For if $\mathbf{Q} = Q\mathbf{e}$, where **e** is a constant unit vector, we have from (1)

$$\mathbf{e}\times\frac{d\mathbf{F}}{ds} = 0; \qquad \text{hence} \qquad \mathbf{e}\times\mathbf{F} = \mathbf{c},$$

a constant vector On multiplying this equation $\mathbf{F}\cdot$ we obtain $\mathbf{F}\cdot\mathbf{c} = 0$. Since **F** is tangential to the cable, **T** is always normal to the constant vector **c**.

Example 2. *Uniform Chain on a Smooth Cylinder.* Consider a uniform chain in contact with a smooth horizontal cylinder along a normal section (Fig. 93*a*). If the y-axis is taken vertically upward and w and R denote the weight and normal pressure on the chain, each per unit of length, then

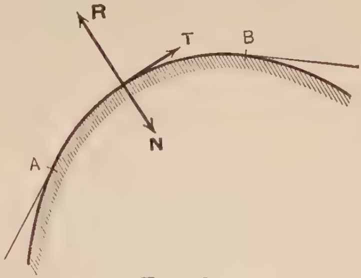


Fig. 93*a*.

$$\mathbf{Q} = -R\mathbf{N} - w\mathbf{j}.$$

Hence

$$\mathbf{Q}\cdot\mathbf{T} = -w\mathbf{j}\cdot\mathbf{T} = -w\frac{dy}{ds} \quad (\S\ 85,\ 5), \quad \mathbf{Q}\cdot\mathbf{N} = -R - w\mathbf{j}\cdot\mathbf{N}.$$

From (4)

$$\frac{dF}{ds} = w\frac{dy}{ds}; \qquad \text{hence} \qquad F = wy + \text{constant}.$$

If $F = F_0$ when $y = y_0$ this gives

(i) $$F - F_0 = w(y - y_0);$$

the difference in the tensions at two points is thus proportional to their difference in level.

From (5)

(ii) $$R = \kappa F - w\,\mathbf{j}\cdot\mathbf{N}.$$

These equations also apply to a chain hanging freely between two points if we put $R = 0$.

PROBLEMS

1. A uniform chain, with its ends hanging freely, makes a turn and a half about a smooth horizontal circular cylinder of radius r (Fig. 93*b*). Show that its ends A, B must be on the same level and that the tension at any point is wy, where y is the vertical distance above A. Prove that the lengths l of free chain must exceed the diameter of the cylinder in order that chain and cylinder shall be in contact at E.

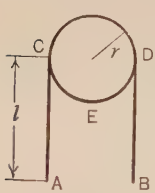


FIG. 93*b*.

2. A uniform cable hangs between two supports. If ρ_0 is the radius of curvature at its lowest point, show that $F_0 = w\rho_0$. If we choose the origin a distance $y_0 = \rho_0$ below the lowest point show that the tension at any point is $F = wy$.

94. String Stretched over a Smooth Surface. Consider a light flexible string stretched over a smooth surface between the points A and B. If we neglect its weight, the distributed load $\mathbf{Q}$ is the normal reaction $\mathbf{R}$ of the surface per unit of length (Fig. 93*a*). Then since $\mathbf{R}\cdot\mathbf{T} = 0$ and $\mathbf{R}\cdot\mathbf{B} = 0$ (§ 93, 6), $\mathbf{R}$ must be parallel to $\mathbf{N}$. In other words, the principal normal of the string is always normal to the surface. A curve of the surface having this property is called a *geodesic*. It may be shown that the shortest distance along a surface between two of its points is along a geodesic; thus the geodesics on a plane are straight lines; on a sphere, great circles.

On a convex surface $\mathbf{Q} = -R\mathbf{N}$. Hence from (§ 93, 4, 5)

$$\frac{dF}{ds} = 0, \quad \kappa F - R = 0;$$

hence
$$F = \text{constant}, \quad R = \frac{F}{\rho}.$$

A flexible weightless string stretched over a smooth surface lies along a geodesic of the surface; its tension is constant and reaction of the surface per unit length of string varies inversely as its radius of curvature.

95. Rope or Belt Friction. We shall now find the relation between the tensions of a rope or belt on the two sides of a rough cylinder (of arbitrary normal section) over which it passes (Fig. 95*a*). If we neglect the weight of the rope, the distributed load $\mathbf{Q}$ is the reaction $\mathbf{R}$ of the cylinder per unit length of rope. In the figure the greater tension F_2 is to the left. Taking $\mathbf{T}$ in this direction,

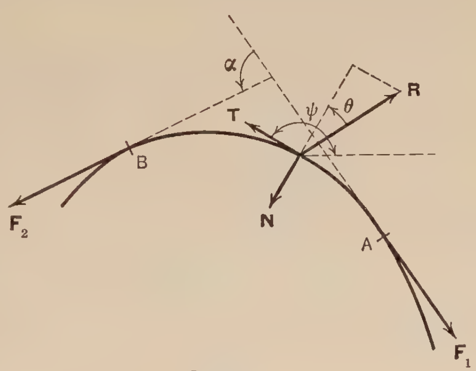


FIG. 95*a*.

$$\mathbf{Q}\cdot\mathbf{T} = -R \sin\theta, \quad \mathbf{Q}\cdot\mathbf{N} = -R\cos\theta;$$

hence from (§ 93, 4, 5)

$$\frac{dF}{ds} = R\sin\theta, \quad \kappa F = R\cos\theta.$$

Putting $\kappa = d\psi/ds$ (§ 87, 3), we have on dividing the first of these equations by the second

$$\frac{1}{F}\frac{dF}{d\psi} = \tan\theta.$$

When the rope is on the point of slipping $\tan\theta = \mu$, the coefficient of friction between rope and cylinder (§ 32); then

$$\frac{dF}{F} = \mu\, d\psi.$$

If F_1 and F_2 denote the tensions on the right and left sides of the rope and $\alpha = \psi_2 - \psi_1$ the angle of wrap expressed in *radians*, we have on integrating the last equation between A and B

$$\ln \frac{F_2}{F_1} = \mu(\psi_2 - \psi_1) = \mu\alpha,^*$$

whence

(1) $$F_2 = F_1 \, e^{\mu\alpha}.$$

For a given value of F_1 this gives the greatest value of F_2 consistent with no slippage.

If the rope is regarded as a band-brake on a wheel of radius r to which a clockwise turning moment is applied, the brake drum will not slip until the turning moment exceeds the frictional moment $(F_2 - F_1)r$.

On a circular cylinder the angle of wrap is equal to the angle AOB subtended at the center O.

Unless a table of values of e^x is at hand, (1) is better adapted for computation after taking the common logarithm of both members. Thus if the rope or belt makes n turns about a cylinder, $\alpha = 2\pi n$ radians, and

(2) $$\log F_2 = \log F_1 + 2\pi\mu n \log e = \log F_1 + 2.7288\,\mu n.$$

Example 1. If a rope makes two complete turns about a cylindrical post and $\mu = \frac{1}{4}$, the limiting ratio of the tensions is given by (2) with $\mu n = \frac{1}{2}$:

$$\log \frac{F_2}{F_1} = 2.7288 \times \tfrac{1}{2} = 1.3644, \quad \frac{F_2}{F_1} = 23.14.$$

In other words, a pull of 1 pound at one end of the rope will balance as much as 23 pounds at the other before the rope begins to slip.

Example 2. In the band-brake shown in Fig. 95*b*, find the least force P that will keep the brake-drum from slipping under the action of counterclockwise turning moment M.

The tension F_2 must be greater than F_1 in order that frictional moment $(F_2 - F_1)r$ may resist M. When slipping is impending

$$(F_2 - F_1)r = F_1 r(e^{\mu\alpha} - 1) = M, \quad F_1 = \frac{1}{e^{\mu\alpha} - 1}\,\frac{M}{r}.$$

Since the bell-crank is in equilibrium, we have on taking moments about O,

$$Pa - F_1 b = 0, \quad \text{whence} \quad P = \frac{b}{a}\,\frac{1}{e^{\mu\alpha} - 1}\,\frac{M}{r}.$$

* Ln x denotes the natural logarithm of x (base e), log x the common logarithm (base 10).

Let us take, for example, $b = a/10$. If $\mu = 0.3$, and the brake-band wraps over 3/4 of the drum ($\alpha = 3\pi/2$), $e^{\mu\alpha} = e^{1.4}$ is about 4. The formula above then gives $P = 0.033\ M/r$.

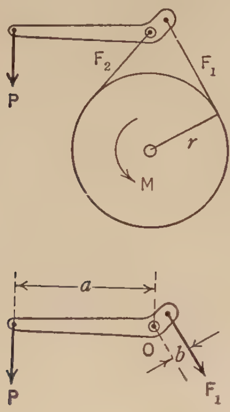


FIG. 95*b*.

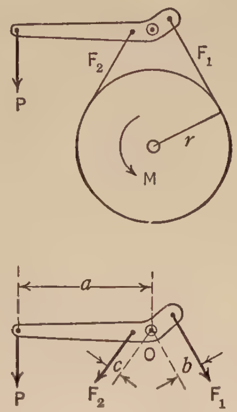


FIG. 95*c*.

Example 3. Let us solve the same problem as in Example 2 for the differential band-brake of Fig. 95*c*. When slipping is impending, F_1 has the value given in Example 2. From the equilibrium of the bell-crank we have, on taking moments about O,

$$Pa + F_2c - F_1b = 0 \quad \text{or} \quad Pa + F_1(ce^{\mu\alpha} - b) = 0.$$

On substituting the value of F_1 in the last equation we find

$$P = \frac{b - ce^{\mu\alpha}}{a(e^{\mu\alpha} - 1)} \frac{M}{r}.$$

In practice b must be greater than $ce^{\mu\alpha}$ in order to avoid rupturing some part of the brake.

In order to compare this brake with the former one, assume the same values as before: thus $b = a/10$, $e^{\mu\alpha} = 4$; also take $c = a/60$. The formula above then gives $P = 0.011\ M/r$.

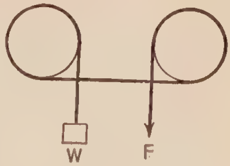


FIG. 95*d*.

PROBLEMS

1. A load of 5000 lb. is supported by a rope making $2\frac{1}{4}$ turns about the drum of a windlass. If $\mu = 0.3$, what force applied to the free end of the rope will just hold the load?

2. In Fig. 95*d* what force F will just hold the load $W = 1000$ lb. if $\mu = \frac{1}{4}$ and the fixed cylindrical drums are of equal radii?

3. A boat exerts a pull of 2000 lb. on its hawser, which is wound

about a mooring-post on the dock. If $\mu = 0.3$ find the number of turns of the rope in order that the boat may be held by a force of 50 lb. at the free end of the hawser.

4. If the turning moment M in Example 2 is clockwise, show that

$$P = \frac{b}{a}\frac{e^{\mu\alpha}}{e^{\mu\alpha} - 1}\frac{M}{r}.$$

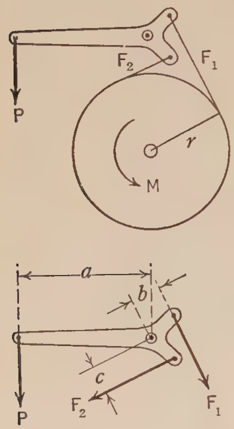


Fig. 95e.

With the numerical data of Example 2 show that $P = 0.133\ M/r$.

5. In the band-brake of Fig. 95e, show that the force

$$P = \frac{M}{r}\frac{b + ce^{\mu\alpha}}{a(e^{\mu\alpha} - 1)} \quad \text{or} \quad \frac{M}{r}\frac{be^{\mu\alpha} + c}{a(e^{\mu\alpha} - 1)}$$

will just brake a counterclockwise or clockwise moment M. Compute P when $M/r = 300$ lb., $b = c = a/10$, $\alpha = 3\pi/2$ and $\mu = 0.3$.

96. Parabolic Cable. We proceed to show that a cable whose load has a uniform horizontal distribution will hang in a parabola with a vertical axis. The cables of a suspension bridge are approximately loaded in this manner; for the roadway they support is a uniform horizontal load which greatly preponderates over the weight of the cables and hangers (Fig. 96a). Although we might use the general equations of § 93 to deal with this case, it is simpler to go back to the first principles of statics.

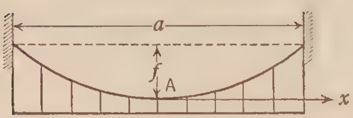


Fig. 96a.

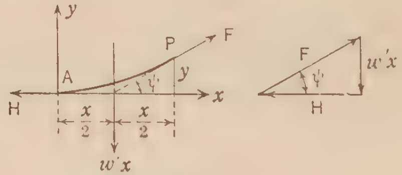


Fig. 96b.

Let us consider a suspension bridge cable with supports on the same level; the span is a, the sag f, the load w' per horizontal foot. Take the origin at the lowest point A of the cable and consider the equilibrium of the part AP (Fig. 96b). This part is subjected to the horizontal tension H at A, the tension F at P, and the vertical load $w'x$ whose resultant bisects the horizontal distance x between A and P. Since these three forces are in equilibrium they must

form a closed triangle and their lines of action must be concurrent (§ 53). Hence

(1), (2) $$F \sin \psi = w'x, \quad F \cos \psi = H,$$

so that

$$\tan \psi = \frac{w'x}{H}. \quad \text{Also} \quad \tan \psi = \frac{y}{\frac{1}{2}x} = \frac{2y}{x}$$

since the forces meet in a point. Equating these values of $\tan \psi$ gives the equation

(3) $$y = \frac{w'}{2H}x^2$$

which represents a parabola with vertex at A and axis vertical. Since $y = f$ when $x = \frac{1}{2}a$,

(4) $$f = \frac{w'a^2}{8H} \quad \text{or} \quad H = \frac{w'a^2}{8f}.$$

By squaring and adding (1) and (2) and then making use of (4) we obtain

(5) $$F^2 = w'^2x^2 + H^2 = w'^2\left(x^2 + \frac{a^4}{64f^2}\right)$$

which gives the tension at any point P. At the points of support ($x = \frac{1}{2}a$) the tension has its greatest value, namely

(6) $$F = \tfrac{1}{2}w'a\sqrt{1 + \frac{a^2}{16f^2}}.$$

To find the length l of the cable having a given span a and sag f, let $n = f/a$ denote the *sag ratio*. Then since

$$\frac{ds}{dx} = \sec \psi = \sqrt{1 + \tan^2 \psi} = \sqrt{1 + \frac{w'^2}{H^2}x^2} = \sqrt{1 + \frac{64f^2}{a^4}x^2},$$

$$\tfrac{1}{2}l = \int_0^{\frac{1}{2}a}\left(1 + 64n^2\frac{x^2}{a^2}\right)^{\frac{1}{2}} dx.$$

Instead of finding the exact value of this integral it is more convenient for purposes of computation to expand the integrand by the binomial theorem and integrate the resulting series term by term. We thus find

$$\tfrac{1}{2}l = \int_0^{\frac{1}{2}a}\left(1 + 32n^2\frac{x^2}{a^2} - 512n^4\frac{x^4}{a^4} + \cdots\right)dx,$$

(7) $$l = a\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4 + \cdots\right).$$

For the small values of n that usually occur in practice this series converges so rapidly that the first two or three terms will give a sufficiently close approximation for the value of l. Thus when $n = 1/8$ the use of three terms gives $l/a = 1.0401$, which is correct to within 0.0001.

If a uniform wire is stretched nearly horizontal, w' is nearly equal to its weight w per unit of length. The wire will therefore hang very nearly in a parabola and the above results are approximately correct with $w' = w$. The tension at the supports is $H + wf$, from (§ 93, i) and, as f is small, differs but slightly from H; thus the tension in the wire is nearly constant.

Example 1. A telegraph wire weighing 0.05 lb./ft. has a span of 200 ft. If its tension is 250 lb., find its sag and its length.

With $w' = 0.05$, $a = 200$, $H = 250$, we find $f = 1$ ft. from (4); then from (7)

$$l = a + \frac{8}{3}\frac{f^2}{a} = 200 + \frac{8}{600} = 200.013 \text{ ft.}$$

The tension varies by only $wf = 0.05$ lb. throughout the wire.

Example 2. The parabolic cable BC (Fig. 96c) has the sag $f = MN$ measured vertically below the mid-point M of the chord BC. If h is the horizontal distance from B to C, prove that

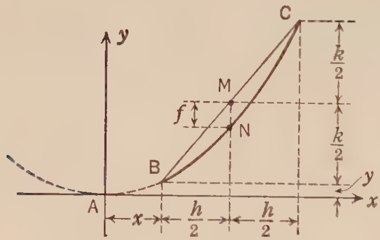


FIG. 96c.

(8) $$f = \frac{w'h^2}{8H}.$$

If the parabola is referred to horizontal and vertical axes through its vertex, its equation will be given by

(3) $$y = cx^2 \quad \text{where} \quad c = w'/2H.$$

The points B, C, N of the curve have the coördinates

$$B(x, y), \quad C(x + h, y + k), \quad N(x + \tfrac{1}{2}h, y + \tfrac{1}{2}k - f)$$

where h and k are given; hence

(i) $$y = cx^2,$$

(ii) $$y + k = c(x + h)^2 = cx^2 + 2\,cxh + ch^2,$$

(iii) $$y + \tfrac{1}{2}k - f = c(x + \tfrac{1}{2}h)^2 = cx^2 + cxh + \tfrac{1}{4}ch^2.$$

Adding (i) and (ii), dividing by 2, and subtracting (iii) from the resulting equation, we obtain

(iv) $$f = \tfrac{1}{4}ch^2 = \frac{w'h^2}{8H}$$

as given in (8). If f and h are given, this equation serves to compute H.

Given h, k and f, we may compute c from (iv), then x and y from (i) and (ii). Then from (7) we can find the length of the arcs AB and AC; their difference gives the length of the cable BC.

Example 3. A standard steel tape weighing 0.0066 lb./ft. reads correctly at 56° F. under a tension of 16 lb. What correction must be made for sag when the tape is used under these conditions for an apparent horizontal measurement of 100 ft.?

If l is the distance read on the tape, a the true distance, and f the sag, we have from (7) and (4)

$$l = a + \frac{8}{3}\frac{f^2}{a}, \quad f = \frac{wa^2}{8H};$$

hence

$$l - a = \frac{8}{3}\frac{w^2a^3}{64H^2} = \frac{1}{24}\frac{w^2l^3}{H^2} \text{ very nearly.}$$

Putting $w = 0.0066$, $l = 100$, $H = 16$ we find

$$l - a = \frac{1}{24}\left(\frac{0.66}{16}\right)^2 \times 100 = 0.0071 \textit{ ft.}$$

as the correction to be deducted.

Example 4. Each cable of a suspension bridge of 500-ft. span supports a uniform load of $w' = 1000$ lb./ft. With a concentrated load of $W = 20{,}000$ lb. at the center, the sag $f = 40$ ft. Compute the tension at the supports and locate the vertex A of the parabolic arc PC (Fig. 96d).

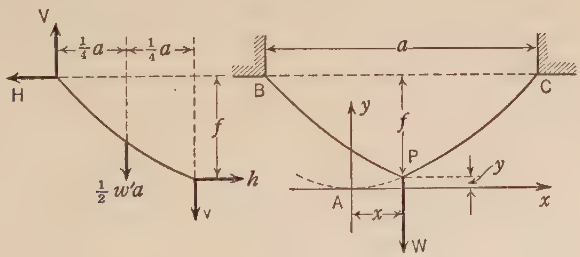


FIG. 96d.

At the supports let $\mathbf{F} = [H, V]$. If we treat the entire cable as a free body, we have

(i) $\qquad 2V - w'a - W = 0, \quad \text{whence} \quad V = \frac{1}{2}(W + w'a).$

Next treat the part BP as a free body, introducing the forces $[H, V]$ at B and $[-h, -v]$ at P. The moment equation about P gives

$$Hf + \tfrac{1}{2}w'a \cdot \tfrac{1}{4}a - V \cdot \tfrac{1}{2}a = 0;$$

hence in view of (i)

(ii) $$H = \frac{a}{4f}(W + \tfrac{1}{2}w'a).$$

Moreover from horizontal and vertical balance

$$h - H = 0, \qquad V - v - \tfrac{1}{2}w'a = 0;$$

hence the stress at P has the components

$$h = H, \quad v = \tfrac{1}{2}W.$$

The last result is also obvious from the equilibrium at the point P.

Now imagine the part CP of the cable prolonged to its vertex A and let (x, y) be the coördinates of P referred to A. Then from (1)

(iii) $$v = w'x, \quad x = \frac{v}{w'} = \frac{W}{2w'}.$$

Also from (3) $y = w'x^2/2H$; hence with the values of x and H above,

(iv) $$y = \frac{W^2 f}{w'a(2W + w'a)}.$$

If we substitute $a = 500$, $w' = 1000$, $W = 20{,}000$, $f = 40$ in the equations above, we find

$$H = 1{,}625{,}000, \quad V = 260{,}000 \text{ lb.}; \quad x = 10, \quad y = 0.059 \text{ ft.}$$

The tension at the supports is

$$F = \sqrt{H^2 + V^2} = 1{,}646{,}000 \text{ lb.}$$

PROBLEMS

1. Each cable of a suspension bridge has a span of 600 ft., a sag of 50 ft. and supports a load of 1000 lb. per foot. Find the length of the cables and their tension at the lowest and highest points.

2. Find the sag of a parabolic cable 510 ft. long which bridges a span of 500 ft.

3. If the sag-ratio n of a parabolic cable is small, say less than 1/8, prove that a small change dl in its length (due, for example, to a change in temperature) will produce the following changes in f, H, and the tension F at the supports:

$$df = \frac{3}{16}\frac{dl}{n}, \quad dH = -\frac{H}{f}df, \quad dF = \frac{H}{F}dH.$$

4. If the coefficient of expansion of the cable in Problem 1 is 0.000007 per degree Fahrenheit, compute the increase in sag due to a temperature rise of 50° F. What change will this make in the tension at the supports?

5. If the element ds of a cable is under a tension F, it will stretch an amount $d\epsilon = F\,ds/AE$, where A is the section-area of the cable

(in.^{2}) and E its modulus of elasticity (lb. per sq. in.) If F denotes the tension of a suspension bridge cable at any point due to the load of the roadway alone, prove that this load will increase its length under its own weight by an amount

$$\epsilon = \frac{2}{AE}\int_0^{\frac{1}{2}l} F\,ds = \frac{2H}{AE}\int_0^{\frac{1}{2}a}\left(1 + 64\frac{n^2}{a^2}x^2\right)dx = \frac{Ha}{AE}\left(1 + \frac{16}{3}n^2\right).$$

6. If the load in Problem 1 is entirely due to the roadway, compute the length of the cable when unloaded, given that $A = 24$ in.^{2} and $E = 30{,}000{,}000$ lb./in.^{2}

7. If a wire of length l is stretched nearly straight between two points A, B at different levels, prove that its sag under the middle point M of the chord AB is approximately $wl^2/8\,F$, where F, the tension, is regarded as constant. [This may be deduced from (8); or consider the part NB of the wire as a free body (Fig. 96c) and take moments about B.]

8. If the line of action of a weight of W lb., applied at the point P of a cable supporting a uniform load of w' lb./ft., divides the span a into the parts a_1, a_2, show that

$$H = \frac{a_1a_2}{af}(W + \tfrac{1}{2}w'a),$$

where f is the sag at P. Find also the vertical component of the tension at each support.

9. A cable BPC of span a supports uniform loads of w' lb./ft. over BP, $2w'$ lb./ft. over PC. If the horizontal projections of BP and PC are $2a/3$ and $a/3$, show that the greatest sag f occurs at a horizontal distance $5a/9$ from B and is given by

$$f = \frac{25}{162}\frac{w'a^2}{H}.$$

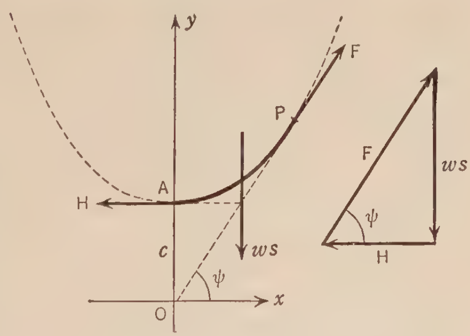


Fig. 97a.

97. The Catenary. If a uniform chain or flexible cable hangs freely between two points, the curve it assumes under the influence of gravity is called a *catenary*. To find its equation consider the equilibrium of the part AP of length s, extending from the lowest point A to any other point P (Fig. 97a). Draw the y-axis vertically upward through A; the position of the horizontal x-axis

is left open for the present. The part AP is subjected to the horizontal tension H at A, the tension F at P, and the vertical load ws whose resultant passes through the point where the lines of action of **H** and **F** meet. From the force triangle

(1), (2) $$F \sin \psi = ws, \quad F \cos \psi = H,$$

so that $\tan \psi = ws/H$, or

(3), (4) $$s = c \tan \psi \quad \text{where} \quad c = \frac{H}{w}.$$

The constant c represents the length of cable whose weight is H; it is called the *parameter* of the catenary. From (3), the *intrinsic equation* of the catenary, we may obtain the coördinates x, y of P as functions of ψ by the following method.* From (§ 85, 4, 5)

$$\frac{dx}{ds} = \cos \psi, \quad \frac{dy}{ds} = \sin \psi; \quad \text{also} \quad \frac{ds}{d\psi} = c \sec^2 \psi$$

from (3); hence

$$\frac{dx}{d\psi} = \frac{dx}{ds}\frac{ds}{d\psi} = \cos \psi \cdot c \sec^2 \psi = c \sec \psi,$$
$$\frac{dy}{d\psi} = \frac{dy}{ds}\frac{ds}{d\psi} = \sin \psi \cdot c \sec^2 \psi = c \tan \psi \sec \psi.$$

On integrating these equations we find

$$x = c \ln (\sec \psi + \tan \psi) + A,$$
$$y = c \sec \psi + B,$$

where A and B are constants of integration. Since $x = 0$ when $\psi = 0$, $A = 0$; and if we choose the x-axis so that $y = c$ when $\psi = 0$, $B = 0$ also. The parametric equations of the catenary with this choice of axes are therefore

(5), (6) $$x = c \ln (\sec \psi + \tan \psi), \quad y = c \sec \psi.$$

We now eliminate ψ between (5) and (6) to obtain the Cartesian equation. From (5)

$$\sec \psi + \tan \psi = e^{\frac{x}{c}},$$

and on taking reciprocals of both members

$$\sec \psi - \tan \psi = e^{-\frac{x}{c}}.$$

* This method applies whenever the intrinsic equation of a curve has the form $s = f(\psi)$.

On adding and subtracting these equations we obtain

$$\sec\psi = \tfrac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) = \cosh\frac{x}{c},$$

$$\tan\psi = \tfrac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) = \sinh\frac{x}{c};$$

hence from (6) and (3)

(7), (8) $$y = c\cosh\frac{x}{c}, \quad s = c\sinh\frac{x}{c}.$$

Here (7) is the Cartesian equation of the catenary, while (8) gives its length from the *vertex* $A(0, c)$ to the point $P(x, y)$. The x-axis, a horizontal line at a distance c below the vertex, is called the *directrix* of the catenary.

From (7) and (8), or from

(6), (3) $$y = c\sec\psi, \quad s = c\tan\psi$$

we obtain the relation

(9) $$y^2 = s^2 + c^2.$$

Again, if we put $\sec\psi = y/c$, $\tan\psi = s/c$ in (5) we get

(10) $$x = c\ln\frac{y+s}{c}.$$

The radius of curvature $\rho = ds/d\psi$ (§ 87, 3); hence on differentiating (3) we find

(11) $$\rho = c\sec^2\psi = y\sec\psi.$$

In particular, at the vertex A, $\psi = 0$ and $\rho = c$; *the parameter of the catenary is equal to the radius of curvature at the vertex.* Hence the flatter the catenary, the larger its parameter.

In Fig. 97*b*, PT and PN are tangent and normal to the catenary at P and QL is drawn perpendicular to PT. Then from (6), (3) and (11),

$$QL = y\cos\psi = c, \quad PL = c\tan\psi = s, \quad PN = y\sec\psi = \rho.$$

Equation (9) is the Pythagorean Theorem applied to the right triangle PQL. Moreover the radius of curvature is represented by the segment of the normal included between the curve and its directrix. Those geometrical relations are true only when the same unit of length is used on both coördinate axes.

Consider now the tension F in the cable at P. From (1) and (2) we see that its vertical component $V = ws$ and that its horizontal component H is constant. The parameter c is defined in

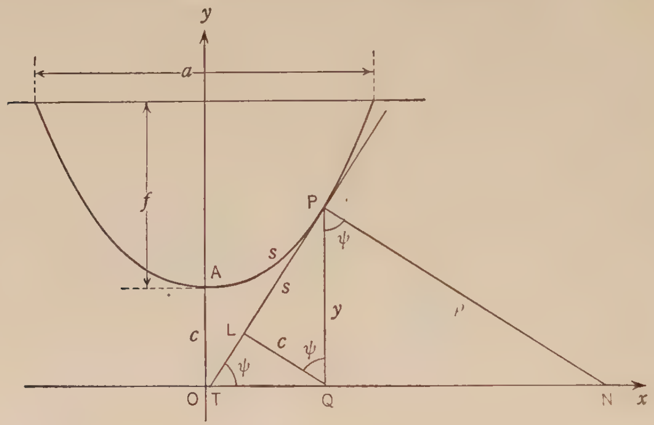


FIG. 97*b*.

(4) so that $H = wc$. Finally, from (2) and (6),

$$F = H \sec \psi = wc\frac{y}{c} = wy;$$

the tension therefore varies as the height above the directrix. In brief:

(12) $$H = wc, \quad V = ws, \quad F = wy.$$

If we choose the parameter c as the unit of length on both axes, the new coördinates are $X = x/c$, $Y = y/c$, and equation (7) of the catenary becomes $Y = \cosh X$. This means that all catenaries are geometrically *similar*.

Example 1. A uniform cable 80 ft. long and weighing 1 lb./ft. hangs between two supports on the same level. If its central sag is 10 ft., find the horizontal span a and the tension F at the supports.

At the right support $s = 40$, $y = c + 10$; hence from (9)

$$(c + 10)^2 = 1600 + c^2, \quad \text{whence} \quad c = 75 \text{ ft.,} \quad y = 85 \text{ ft.}$$

Now from (10)

$$\tfrac{1}{2} a = 75 \ln \frac{85 + 40}{75} = 75 \ln \frac{5}{3} = 75 (\ln 5 - \ln 3)$$
$$= 75 \times 0.51083 = 38.31 \text{ ft.}$$

The span is therefore 76.62 ft.

Since $w = 1$ we have from (12)

$$H = 75 \text{ lb.}, \quad V = 40 \text{ lb.}, \quad F = 85 \text{ lb.}$$

at the supports. This force makes with the horizontal an angle of

$$\psi = \tan^{-1}\frac{V}{H} = \tan^{-1}\frac{40}{75} = \tan^{-1} 0.5333 = 28° 4'.$$

Example 2. A uniform cable of length $l = 40$ ft. and weighing 2 lb./ft. hangs between supports on the same level. If a load of $W = 1000$ lb., attached at its middle point P, produces the sag $f = 10$ ft., find the tension F at the supports (Fig. 97*c*).

Equilibrium at P requires that the vertical components V of the (equal) tensions of two parts of the cable at P shall just balance W; that is, $2V - W = 0$. Let A be the vertex of the catenary PC, c its parameter, s the arc AP and y the ordinate of P. Then from (12),

$$V = ws, \quad \text{and hence} \quad ws = \tfrac{1}{2} W.$$

Since $F = w(y + f)$ at the supports, F may be computed when y is known; we therefore seek the value of y. From (9)

$$y^2 = s^2 + c^2 \quad \text{and} \quad (y + f)^2 = (s + \tfrac{1}{2} l)^2 + c^2;$$

hence on subtracting to eliminate c we get

$$2yf + f^2 = sl + \tfrac{1}{4} l^2 \quad \text{or} \quad y = \frac{sl}{2f} + \frac{l^2}{8f} - \frac{f}{2}.$$

Therefore

$$F = w(y + f) = w\left(\frac{sl}{2f} + \frac{l^2}{8f} + \frac{f}{2}\right) = \frac{l}{4f} W + w\left(\frac{l^2}{8f} + \frac{f}{2}\right).$$

Substituting $l = 40$, $f = 10$, $w = 2$, $W = 1000$ in this formula we obtain

$$F = \frac{40}{40} 1000 + 2\left(\frac{1600}{80} + 5\right) = 1050 \text{ lb.}$$

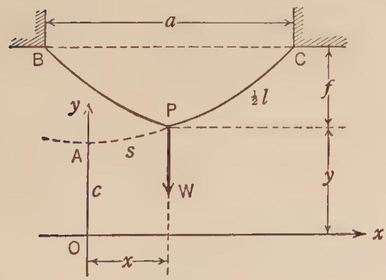


FIG. 97*c*.

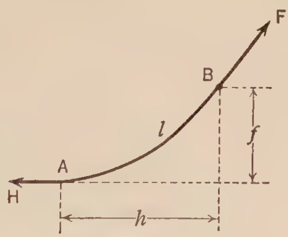


FIG. 97*d*.

Example 3. A chain AB, 20 ft. long and weighing 2 lb./ft., hangs from a fixed point B and has a horizontal force of 30 lb. applied at its lower end A (Fig. 97*d*). Find the position of A relative to B when the chain is in equilibrium and compute the tension at B.

The equilibrium of the cable requires that at B the force

$$\mathbf{F} = [H, ws] = [30, 40]; \quad \text{hence} \quad F = 50 \text{ lb.}$$

From (12)

$$H = 30 = 2c, \quad F = 50 = 2y; \quad \text{hence} \quad c = 15, \quad y = 25 \text{ ft.}$$

Now

$$f = y - c = 10 \text{ ft.};$$

and from (10)

$$h = 15 \ln \frac{25 + 20}{15} = 15 \ln 3 = 16.48 \text{ ft.}$$

Example 4. A uniform chain, hanging between the points B, C, supports the loads W, W' at the points P, P'. Prove that the arcs BP, PP', $P'C$ are portions of catenaries having the same parameter. (Draw the figure.)

Denote the parameters of these arcs by c_1, c_2, c_3 respectively. At P the load W and the tensions in the portions PB, PP' of the cable are in equilibrium. Since the sum of the horizontal components of these forces equals zero, $wc_1 - wc_2 = 0$ and $c_1 = c_2$. Similarly from the equilibrium at P' we have $c_2 = c_3$.

98. Cable with Supports on the Same Level. Denote the length, span and sag of the cable by l, a and f respectively (Fig. 97*b*). When two of the quantities l, a, f are given, we can compute the third and the parameter c of the catenary from the results of § 97. At the right support $s = \frac{1}{2} l$, $x = \frac{1}{2} a$, $y = c + f$; hence if we write

$$u = \frac{a}{2c} \quad \text{or} \quad c = \frac{a}{2u} \tag{1}$$

the equations (7), (8), (9), (10) of § 97 give respectively

$$\frac{2f}{a} = \frac{\cosh u - 1}{u}, \tag{2}$$

$$\frac{l}{a} = \frac{\sinh u}{u} \tag{3}$$

$$(c + f)^2 = \tfrac{1}{4} l^2 + c^2 \quad \text{or} \quad 2cf + f^2 = \tfrac{1}{4} l^2, \tag{4}$$

$$a = 2c \ln \frac{c + f + \frac{1}{2} l}{c}. \tag{5}$$

The tension at the supports is given by (§ 97, 12):

$$F = w(c + f). \tag{6}$$

PROBLEM I. *Given l and a, to find f.*

Find by trial the (positive) root u of (3) with the help of a table of hyperbolic sines. Then compute f from (2); or first compute c from (1) and then f from (4).

Example. Given $l = 150$ ft., $a = 100$ ft., $w = 2$ lb./ft.; find the sag f and the tension at the supports.

We first find a root of the equation $(\sinh u)/u = 1.5$. From the tables we find that

$$u = 1.62 \quad \text{gives} \quad (\sinh u)/u = 1.4985,$$
$$u = 1.63 \quad \text{gives} \quad (\sinh u)/u = 1.5055.$$

Hence, by interpolation,

$$u = 1.62 + \tfrac{15}{70} \times 0.01 = 1.622 \text{ approximately.}$$

Now from (2)

$$f = 50\,\frac{\cosh 1.622 - 1}{1.622} = 50.26 \text{ ft.}$$

Or we may compute c from (1) and f from (4):

$$c = \frac{50}{1.622} = 30.83 \text{ ft.},$$

$$f = \sqrt{\tfrac{1}{4}\,l^2 + c^2} - c = \sqrt{6575} - 30.83 = 50.26 \text{ ft.}$$

The tension at the supports is

$$F = 2\,(30.83 + 50.26) = 162.2 \text{ lb.}$$

PROBLEM II. *Given a and f, to find l.*

Find by trial the root u of (2) with the help of a table of hyperbolic cosines. Then compute l from (3); or first compute c from (1) and then l from (4).

PROBLEM III. *Given l and f, to find a.*

Compute c from (4) and then a from (5). (See § 97, Example 1.)

PROBLEMS

1. A chain BCD, weighing 1 lb./ft., is fixed at B, passes over a smooth pin C and then hangs vertically. If the parts BC, CD are 40 ft. and 30 ft. long respectively and C is 20 ft. below the horizontal through B, show that C is at the vertex of the catenary BC. Also find H, V and F at B.

2. If both parts of the chain in Problem 1 are 40 ft. long, find H, V and F at B. [C is no longer at the vertex of the catenary.]

3. A cable 100 ft. long and weighing 2 lb./ft. hangs between two points on the same level. If the sag is 10 ft., find H, V and F at the supports.

4. Find the length of a wire having a horizontal span of 100 ft. and a sag of 25 ft.

5. Find the sag of a cable 1200 ft. long over a horizontal span of 1000 ft.

6. A wire of length l hangs between two points on the same level. If ψ is its inclination to the horizontal at the supports, show that the sag is $f = \frac{1}{2} l \tan \frac{1}{2} \psi$.

7. Show that a wire of length l and sag f has a horizontal span of

$$a = \left(\frac{l^2}{4f} - f\right) \ln \frac{l + 2f}{l - 2f}.$$

8. A uniform chain of length l hangs symmetrically over two smooth pins on the same level and at a distance a apart. Show that the parameter c of the catenary formed by the central portion must satisfy the equation

$$l = 2\,ce^{\frac{a}{2c}}$$

Thus if $u = a/2\,c$, u must be a root of the equation $e^u/u = l/a$. Prove that this equation has no roots, one root or two roots according as $l < ae$, $l = ae$, $l > ae$.

99. Cable with Supports on Different Levels. Consider now a uniform cable supported at two points B, C on different levels. When the relative positions of B and C, and the length l of the cable, are known, the catenary is completely determined. Our problem, then, is to compute its parameter c and to locate its vertex A.

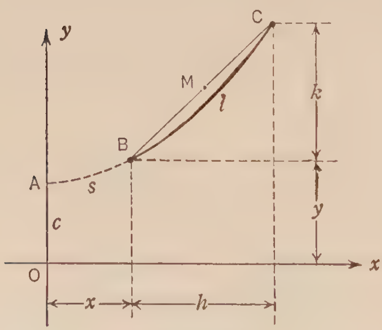


FIG. 99.

As before we choose the directrix (as yet undetermined) and the vertical line through the vertex A as coördinate axes (Fig. 99). Let the left and right points of support be

$$B(x, y), \quad C(x + h, y + k),$$

where h and k are given; and let s = arc AB. We shall then compute c, x and y; then y locates the directrix, x and c the vertex.

From (§ 97, 7, 8) we have

$$k = (y + k) - y = c \cosh \frac{x + h}{c} - c \cosh \frac{x}{c},$$

$$l = (l + s) - s = c \sinh \frac{x + h}{c} - c \sinh \frac{x}{c};$$

on making use of the formulas

$$\cosh u - \cosh v = 2 \sinh \frac{u + v}{2} \sinh \frac{u - v}{2},$$

$$\sinh u - \sinh v = 2 \cosh \frac{u + v}{2} \sinh \frac{u - v}{2},^*$$

these become

(1) $$k = 2\, c \sinh \frac{2\, x + h}{2\, c} \sinh \frac{h}{2\, c},$$

(2) $$l = 2\, c \cosh \frac{2\, x + h}{2\, c} \sinh \frac{h}{2\, c}.$$

From (1) and (2) we have

(3), (4) $$\sqrt{l^2 - k^2} = 2\, c \sinh \frac{h}{2\, c}, \quad \frac{k}{l} = \tanh \frac{2\, x + h}{2\, c}.$$

Equation (3) may be written

(5), (6) $$\frac{\sinh u}{u} = \frac{\sqrt{l^2 - k^2}}{h} \quad \text{where} \quad u = \frac{h}{2\, c}.$$

Since $c > 0$ and $h > 0$ (B is to the *left* of C), $u > 0$. Now (5) has always a single positive root when $\sqrt{l^2 - k^2}/h > 1$†; this condition is always fulfilled since $l > BC$, that is, $l^2 > h^2 + k^2$. Let u denote the positive root of (5). After finding u by trial,

* The identities connecting the trigonometric functions may be changed into their hyperbolic analogues by replacing

$$\sin u, \quad \cos u, \quad \tan u \qquad \text{by} \qquad i \sinh u, \quad \cosh u, \quad i \tanh u$$

where $i = \sqrt{-1}$. Thus, since $i^2 = -1$,

$$\sin^2 u + \cos^2 u = 1 \quad \text{gives} \quad \cosh^2 u - \sinh^2 u = 1;$$

$$\cos u - \cos v = -2 \sin \frac{u + v}{2} \sin \frac{u - v}{2}$$

gives the first of the formulas above.

† This root is the abscissa of the right-hand point of intersection of the curves

$$y = \sinh u, \quad y = \frac{\sqrt{l^2 - k^2}}{h}\, u$$

plotted with u as abscissa and y as ordinate.

with the help of table of hyperbolic sines (see § 98, Example) we may compute $c = h/2\,u$ from (6).

Knowing c, we may now compute x from (4):

$$x + \frac{h}{2} = c \tanh^{-1}\frac{k}{l} = \frac{c}{2}\ln\frac{l+k}{l-k} \,* \tag{7}$$

and tnen y from (§ 97, 7).

We may also compute y without first finding x. Since

$$y + (y + k) = c\cosh\frac{x}{c} + c\cosh\frac{x+h}{c},$$

we have, on making use of the formula

$$\cosh u + \cosh v = 2\cosh\frac{u+v}{2}\cosh\frac{u-v}{2},$$

$$y + \frac{k}{2} = c\cosh\frac{2\,x+h}{2\,c}\cosh\frac{h}{2\,c}.$$

On dividing the members of this equation by those of (2) we obtain

$$y + \frac{k}{2} = \frac{l}{2\tanh u} \tag{8}$$

where u is the root of (5) found above. Note that the left members of (7) and (8) give the coördinates of the mid-point M of the segment BC; these equations, therefore, locate the origin O with reference to M.

The tensions at B and C are wy and $w(y + k)$ respectively.

PROBLFMS

1. If $l = 104$ ft., $h = 64$ ft., $k = 40$ ft., and $w = 2$ lb./ft., find the tensions at the supports.

2. If $l = 130$ ft., $h = 40$ ft., $k = 50$ ft., locate the lowest point of the cable with reference to M, the middle point of the chord joining the supports.

100. Concentrated Load on Cable. We shall only consider the case of a uniform cable with supports on the same level and having a load W attached at its middle point (Fig. 97c).

Suppose that the length l and span a are given; and let the load $W = nw$, where w is the weight of the cable per unit of length. We propose to compute the sag f and the tension F at the supports.

* From the identity $\tanh^{-1} v = \frac{1}{2}\ln\frac{1+v}{1-v}$.

Let A denote the vertex of the catenary CP, c its parameter, s the arc AP, and (x, y) the coördinates of P referred to the directrix and the vertical through A. As in § 97, Example 2, equilibrium at P requires that

$$2\,ws = W = nw, \quad \text{whence} \quad s = \tfrac{1}{2}\,n.$$

Now from (§ 97, 8) we have the two equations

$$(1),\ (2) \qquad \tfrac{1}{2}\,n = c \sinh\frac{x}{c}, \quad \tfrac{1}{2}\,(n + l) = c \sinh\frac{x + \frac{1}{2}\,a}{c}$$

to determine x and c. But since

$$c \sinh\frac{x + \frac{1}{2}\,a}{c} = c \sinh\frac{x}{c}\cosh\frac{a}{2\,c} + c\cosh\frac{x}{c}\sinh\frac{a}{2\,c}$$

$$= \tfrac{1}{2}\,n\cosh\frac{a}{2\,c} + c\cosh\frac{x}{c}\sinh\frac{a}{2\,c},$$

(2) may be written

$$(2)' \qquad (n + l) - n\cosh\frac{a}{2\,c} = 2\,c\cosh\frac{x}{c}\sinh\frac{a}{2\,c}.$$

On multiplying (1) by 2 sinh $(a/2\,c)$, we get

$$(1)' \qquad n\sinh\frac{a}{2\,c} = 2\,c\sinh\frac{x}{c}\sinh\frac{a}{2\,c}.$$

To eliminate x from (1)′ and (2)′ we square the equations and subtract; we thus obtain

$$(n + l)^2 - 2\,n(n + l)\cosh\frac{a}{2\,c} + n^2 = 4\,c^2\sinh^2\frac{a}{2\,c},$$

or on writing $u = a/2\,c$,

$$(3) \qquad \frac{(n + l)^2 + n^2}{a^2} - \frac{2\,n(n + l)}{a^2}\cosh u = \frac{\sinh^2 u}{u^2}.$$

This transcendental equation always has just one positive root. For as u increases from 0 to ∞, the left member steadily decreases from l^2/a^2 (>1) to $-\infty$ while the right member steadily increases from 1 to ∞. Therefore there is just one positive value of u, for which both members have the same value. Having found this positive root u by trial, we may compute $c = a/2\,u$.

From (§ 97, 9)

$$y = \sqrt{s^2 + c^2}, \quad y + f = \sqrt{(s + \tfrac{1}{2}\,l)^2 + c^2};$$

knowing $s = \frac{1}{2}n$ and c, these equations serve to compute y and f. The tension at the supports is $F = w(y + f)$.

101. Summary, Chapter VII. In order that a flexible cable may be in equilibrium it is necessary and sufficient that the force-sum and moment-sum of the external forces acting on any portion of it shall vanish. The moment equation requires that the tensile stress at any point of the cable shall be tangential to the cable. The force equation gives

$$\frac{d\mathbf{F}}{ds} + \mathbf{Q} = 0,$$

where $\mathbf{Q}$ is the distributed load per unit length of cable.

A light string stretched over a smooth surface will trace a geodesic of the surface; its tension is constant.

The tensions F_2 and F_1 ($F_2 > F_1$) on the free sides of a rope or belt, wrapped over a rough circular cylinder through α radians, are related by the equation

$$F_2 = F_1 e^{\mu\alpha}$$

when slipping is impending.

When the load has a uniform horizontal distribution (w' lb. per foot) the cable forms a parabola

$$y = \frac{w'x^2}{2H}; \qquad \text{hence} \qquad H = \frac{w'a^2}{8f}$$

where a is the horizontal span, f the sag. At the supports (on the same level), $V = \frac{1}{2}w'a$.

When the load is due to the uniform weight of the cable (w lb. per foot) the cable forms a catenary

$$y = c \cosh\frac{x}{c}; \quad \text{and} \quad s = c \sinh\frac{x}{c}$$

gives the length of arc measured from the vertex to a point whose abscissa is x; hence $y^2 = s^2 + c^2$. When the supports are on the same level, any two of the quantities l, a, f (length, span, sag) suffice to determine the third and the parameter c. The tension at any point at a distance y above the directrix is $F = wy$.

CHAPTER VIII

KINEMATICS OF A PARTICLE

102. Speed. To define the position P of a particle moving along a curve, choose a point A of the curve from which to measure distances (arcs of the curve) and take a definite direction along the curve as *positive*. Then if the arc $s = AP$ is given as a function of the time,

$$s = f(t),$$

FIG. 102*a*.

the motion of the particle is determined, in the sense that its position is given at every instant.

Now let the particle pass the points P and P' at the instants t and $t' = t + \Delta t$; and let

$$\text{arc } AP = s, \qquad \text{arc } AP' = s' = s + \Delta s.$$

Then the quotient

$$\frac{s' - s}{t' - t} = \frac{\Delta s}{\Delta t}$$

represents the average rate at which the particle is describing its path during the interval Δt.

If this quotient is the same for all intervals Δt, the particle is said to have a *constant speed* of $v = \Delta s/\Delta t$. The particle then describes v units of length during every unit of time. If the time is measured from the instant when the particle is at P_0 ($s = s_0$), we have

$$v = \frac{s - s_0}{t - 0} = \frac{s - s_0}{t},$$

$$s = s_0 + vt \qquad (v \text{ constant}).$$

When $\Delta s/\Delta t$ varies for different time intervals Δt, it is called the *average speed* during the interval Δt; for a particle moving with a constant speed of $\Delta s/\Delta t$ would describe the arc Δs in the time Δt. Now imagine t' to be chosen nearer and nearer

to t. Evidently the smaller the interval Δt between t and t', the more nearly will $\Delta s/\Delta t$ measure the rate at which the particle is describing space at the precise instant t. We therefore define the limiting value of $\Delta s/\Delta t$, as Δt approaches zero, as the *speed at the instant t.* Denoting the speed by v, we have in the notation of the Calculus:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \tag{1}$$

The speed, thus defined, will be positive or negative according as s is increasing or decreasing at the instant considered; that is, according as the motion is in the positive or negative direction of s.

When the units of length and time are chosen, the unit of speed is determined. A particle moving uniformly with unit speed describes a unit of length in a unit of time. In the British system of units, common units of speed are the foot per second (ft./sec.) and the mile per hour (mi./hr.). It is often convenient to remember that

$$60 \text{ mi./hr.} = 88 \text{ ft./sec.}$$

Example 1. The approximate space-time relation for a body falling from rest is

$$s = 16\,t^2;$$

here s is measured in feet from the starting point, t is in seconds, and the $+s$ direction is downwards. To compute the speed when $t = 3$, we note that $s = 144$ at this instant. At a later instant $t = 3 + \Delta t$, $s = 144 + \Delta s$; hence

$$144 + \Delta s = 16\,(3 + \Delta t)^2 = 144 + 96\,\Delta t + 16\,\Delta t^2,$$

$$\Delta s = 96\,\Delta t + 16\,\Delta t^2,$$

$$\frac{\Delta s}{\Delta t} = 96 + 16\,\Delta t.$$

The average speed is seen to be variable and dependent upon the length of the interval Δt; but as $\Delta t \to 0$, the average speed approaches the value 96 as a limit. Thus the speed at the instant $t = 3$ is 96 ft./sec.

Example 2. A man 6 ft. high walks directly away from a light upon the ground towards a house 100 ft. distant. If the speed of the man is 6 ft./sec., find how fast his shadow on the house is moving when he is 30, 60, and 90 ft. from the light.

Let x denote the distance of the man from the light, and s the length of his shadow. The $+x$ direction is towards the house, the $+s$ direction

upward. From Fig. 102*b* we have

$$\frac{s}{l} = \frac{h}{x}, \qquad s = \frac{lh}{x}; \quad \text{hence}$$

$$\frac{ds}{dt} = -\frac{lh}{x^2} \cdot \frac{dx}{dt} = -\frac{3600}{x^2},$$

since $l = 100$ ft., $h = 6$ ft., $dx/dt = 6$ ft./sec. Putting $x = 30, 60, 90$ successively, we obtain the corresponding speeds: $ds/dt = -4, -1, -\frac{4}{9}$ ft./sec. The negative signs show that the shadow is decreasing.

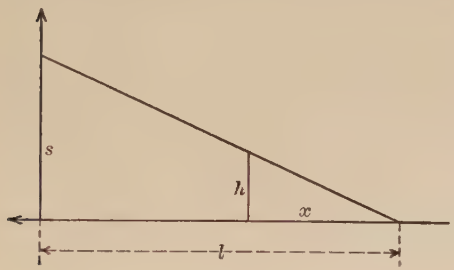


FIG. 102*b*.

PROBLEMS

1. Solve Example 2, when the light is 8 ft. above the ground.

2. A kite is 200 ft. high, with 250 ft. of cord out. If the kite moves horizontally 4 mi./hr. directly away from the boy flying it, how fast is the cord being paid out?

3. A car is drawn along a straight horizontal track by means of a rope wound in on a windlass at the rate of 3 ft./sec. If the rope passes over the windlass 5 ft. above its point of attachment to the car, at what rate is the car moving when its horizontal distance from the windlass is 12 ft.?

4. If $s = t^2 - 10t + 16$, find the speed. Describe the motion.

5. A particle describes a semicircle 2 ft. in diameter in such a manner that its projection upon the diameter moves at the constant speed of 4 ft./sec. What is its speed when $\frac{1}{2}$ ft. above the diameter?

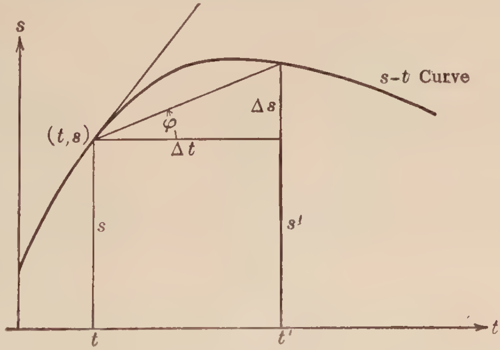


FIG. 103.

103. Space-Time Curve. Any space-time relation, $s = f(t)$, may be plotted as a curve by taking t, s as rectangular coördinates. The curve plotted with t as abscissa and s as ordinate is called the *space-time curve* (Fig. 103).

The average speed $\Delta s/\Delta t$ during the interval $\Delta t = t' - t$ is represented by the slope of the secant joining the points (t, s), (t', s') of this curve:

$$\frac{\Delta s}{\Delta t} = \frac{s' - s}{t' - t} = \tan \phi.$$

As Δt approaches zero, the secant approaches the tangent to the curve at (t, s) as its limiting position, and its slope approaches the slope of this tangent. *Hence the speed at the instant t is represented by the slope of the tangent to the space-time curve at the point* (t, s). In obtaining this slope from the graph, vertical and horizontal distances must be measured in the units of length and time, respectively, employed in plotting the curve.

When the speed is constant, $s = s_0 + vt$ (§ 102); the space-time curve is then a straight line of slope v, intercepting the distance s_0 on the s-axis.

PROBLEMS

1. Plot the space-time curve when $s = 16\,t^2$ (§ 102, Example 1). Obtain graphically the average speed in the interval from $t = 3$ to $t = 4$, and also the speed at the instant $t = 3$.

2. Plot the space-time curve when $s = 600/x$ (§ 102, Example 2). Obtain graphically the speeds when $x = 30, 60, 90$.

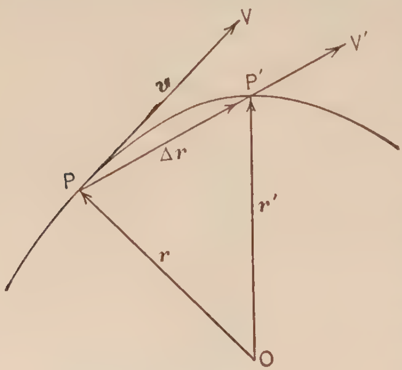


FIG. 104.

104. Velocity. We have seen that the *speed*, $v = ds/dt$, measures the instantaneous rate at which a particle is moving along its path. As the speed gives no information about the instantaneous direction of the motion* we introduce a vector quantity, the *velocity*, which gives the rate at which the particle is changing its position in both magnitude and direction.

Let O be a definite point of some fixed system of reference;

* In the case of rectilinear motion, however, the sign of the speed gives the direction of motion along the line.

for example, O may be thought of as the origin of a system of rectangular axes having an invariable position in space. Then the position P of the moving particle at the instant t is given when its *position* vector, $\mathbf{r} = \overrightarrow{OP}$, is known. At a later instant $t' = t + \Delta t$, let the position vector be $\mathbf{r}' = \overrightarrow{OP'}$. Then the vector

$$\frac{\mathbf{r}' - \mathbf{r}}{t' - t} = \frac{\overrightarrow{PP'}}{\Delta t} = \frac{\Delta \mathbf{r}}{\Delta t},$$

represents the average rate at which the particle is changing its position during the interval Δt, *in direction as well as in magnitude.*

If $\Delta \mathbf{r}/\Delta t$ is invariable in length and direction for all intervals Δt, the particle is said to have a *constant velocity* of $\mathbf{v} = \Delta \mathbf{r}/\Delta t$. If $\mathbf{r} = \mathbf{r}_0$ when $t = 0$, we have, in this case,

$$\mathbf{v} = \frac{\mathbf{r} - \mathbf{r}_0}{t - 0} = \frac{\mathbf{r} - \mathbf{r}_0}{t},$$

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t \qquad (\mathbf{v} \text{ constant}).$$

The end of the vector $\mathbf{r}$ describes a straight line through the end of $\mathbf{r}_0$ and parallel to $\mathbf{v}$. The particle is displaced an amount $\mathbf{v}$ in every unit of time.

When the vector $\Delta \mathbf{r}/\Delta t$ varies for different intervals, it is called the *average velocity* for the interval Δt; for a particle moving with a constant velocity of $\Delta \mathbf{r}/\Delta t$ would undergo the displacement $\Delta \mathbf{r}$ in the time Δt. The average velocity is represented by a vector $\overrightarrow{PV'}$ of length $PP'/\Delta t$, laid off in the direction $\overrightarrow{PP'}$. As the interval Δt varies, $\overrightarrow{PV'}$ will vary in length, in direction, or in both. But as Δt approaches zero, $\overrightarrow{PV'}$ approaches a definite limiting vector $\overrightarrow{PV}$, tangent to the path at P. This limiting vector is defined as the *velocity of the particle at the instant t.* Denoting it by $\mathbf{v}$, we have

$$(1) \qquad \mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}.$$

The velocity of a particle is equal to the time derivative of its position vector from a fixed origin.

It remains to show the connection between the velocity $\mathbf{v}$ and the speed v of the particle. Choose an origin of arcs s, and a

positive direction along the path. From (§ 83, 2) we have

$$\mathbf{v} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = v\,\mathbf{T}$$

where $\mathbf{T}$ is a unit vector tangent to the path at P in the direction of $+s$ (§ 85, 1); hence

(2) $$\mathbf{v} = v\mathbf{T} \quad \text{and} \quad |\mathbf{v}| = |v|.$$

The velocity at P is represented by a vector tangent to the path at P in the direction of instantaneous motion and of length numerically equal to the speed. The velocity vector is thus localized at the moving particle.

105. Acceleration. The velocity of a moving particle at different instants can be graphically represented by drawing the corresponding velocity vectors tangent to the path (Fig. 105*a*).

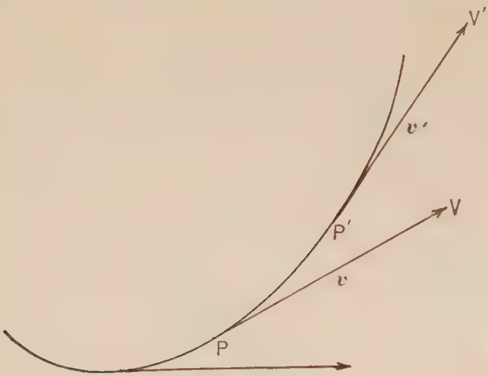


Fig. 105*a*.

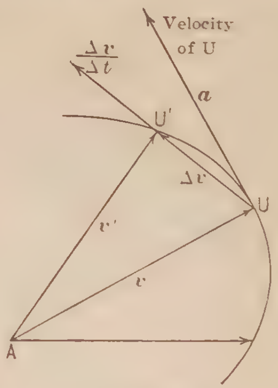


Fig. 105*b*.

To bring into evidence the variation of the velocity, the velocity vectors may be drawn from the same initial point; thus, in Fig. 105*b*, $\overrightarrow{AU} = \overrightarrow{PV}$, $\overrightarrow{AU'} = \overrightarrow{PV'}$. As the particle moves along its path we imagine the velocity vector $\overrightarrow{AU}$ to revolve about A; its end point U will then describe a certain curve, which is called the *hodograph* of the motion. Now the hodograph is related in the same way to the velocity vector as the path of the particle is to its position vector. And just as the velocity has been defined as the rate of change of the position vector, we now define the *acceleration* as the rate of change of the velocity vector. Thus

if the velocity at the instants t and $t' = t + \Delta t$ is $\mathbf{v} = \overrightarrow{PV}$, $\mathbf{v}' = \overrightarrow{PV'}$, the vectorial change in the velocity during the interval Δt is

$$\mathbf{v}' - \mathbf{v} = \overrightarrow{PV'} - \overrightarrow{PV} = \overrightarrow{AU'} - \overrightarrow{AU} = \overrightarrow{UU'} = \Delta\mathbf{v}.$$

The vector $\Delta\mathbf{v}/\Delta t$ represents the average rate at which the velocity is changing during this interval.

If $\Delta\mathbf{v}/\Delta t$ is constant for all intervals Δt, the particle is said to have a *constant acceleration* of $\mathbf{a} = \Delta\mathbf{v}/\Delta t$. If $\mathbf{v} = \mathbf{v}_0$ when $t = 0$, we have in this case

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{v}_0}{t - 0} = \frac{\mathbf{v} - \mathbf{v}_0}{t},$$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t \qquad (\mathbf{a} \text{ constant}).$$

The hodograph is then a straight line through the end of $\mathbf{v}_0$ and parallel to $\mathbf{a}$; and the velocity receives the increment $\mathbf{a}$ in every unit of time.

When the vector $\Delta\mathbf{v}/\Delta t$ varies for different intervals, it is called the *average acceleration* for the interval Δt. The *acceleration at the instant* t is then defined as the limiting vector

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \tag{1}$$

localized at the moving particle. *The acceleration is equal to the time derivative of the velocity.* In the figures, the acceleration at P is equal to the velocity of U on the hodograph; its magnitude is numerically equal to the speed of U.

Since $\mathbf{v} = d\mathbf{r}/dt$, we may also write (1) in the form

$$\mathbf{a} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2}. \tag{2}$$

The unit of acceleration is such that a particle having a constant unit acceleration experiences a *vector* change of one unit of velocity in each unit of time. Unless the body is moving in a straight line, a unit acceleration does not imply a unit change of *speed* in a unit of time.

With the foot and second as units, the unit of acceleration is one foot-per-second per second. This is usually abbreviated as ft./sec.^{2}. When we say that the acceleration of a particle is 8 ft./sec.^{2} at a certain instant, we mean that if, from this time on, the acceleration remained constant in magnitude and direction,

the vector change in velocity would amount to 8 ft./sec. in each second.

Example 1. A train running 45 mi./hr. on a straight track is retarded uniformly to 30 mi./hr. in one minute. Find its acceleration during this interval.

Let **i** denote a unit vector in the direction of motion. Then since the speed decreases uniformly from 66 ft./sec. to 44 ft./sec. in 60 sec.

$$\mathbf{a} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{44\,\mathbf{i} - 66\,\mathbf{i}}{60} = -0.367\,\mathbf{i}\text{ ft./sec.}^2$$

Example 2. If a particle P moves in a circle of radius r with constant speed v, the hodograph is evidently a circle of radius v. The particle makes a complete circuit in the time $T = 2\pi r/v$. Since the end-point U of the velocity vector describes the hodograph in the same time, the speed of U is $2\pi v/T = v^2/r$. This is the magnitude of the acceleration. The acceleration vector **a**, drawn from P, has the same direction as the velocity of U in the hodograph; the acceleration is therefore directed towards the center of the path.

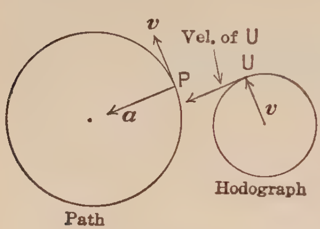


FIG. 105c.

The acceleration may also be found by direct computation. Since $\mathbf{v} = v\mathbf{T}$,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = v\frac{d\mathbf{T}}{dt} = v\frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \frac{v^2}{r}\mathbf{N} \qquad (\S\ 86,\ 1).$$

Example 3. When a wheel of radius b rolls along a straight track, a point P on its rim describes a cycloid (Fig. 87c). The position vector of P is

$$\overrightarrow{OP} = \mathbf{r} = b\theta\mathbf{i} + b\mathbf{j} + b\mathbf{R} \qquad (\S\ 87,\ \text{Problem } 3).$$

We shall compute the velocity and acceleration of P when the wheel is rolling at a uniform rate, that is, when $d\theta/dt$ is constant.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = b\frac{d\theta}{dt}\mathbf{i} + b\frac{d\mathbf{R}}{d\theta}\frac{d\theta}{dt},$$

or, on writing $d\theta/dt = \omega$ and putting $d\mathbf{R}/d\theta = \mathbf{P}$ (§ 85, 2)

$$\mathbf{v} = b\omega(\mathbf{i} + \mathbf{P}).$$

From Fig. 87c we see that the vector $\mathbf{i} + \mathbf{P}$ is perpendicular to IP; and since **v** is tangential to the cycloid at P, IP is a normal to the curve.

Remembering that ω is constant,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = b\omega \frac{d\mathbf{P}}{d\theta}\frac{d\theta}{dt} = -b\omega^2\mathbf{R} \qquad (\S\ 85, 3).$$

Therefore the acceleration of P is always directed toward the center of the wheel.

PROBLEMS

1. If the speed of a particle on any path is constant, show that $\mathbf{a} = (v^2/\rho)\mathbf{N}$. What is the hodograph?

2. A flywheel 6 ft. in diameter is making 75 r.p.m. Find the acceleration of a point on its rim.

3. An automobile runs on a one-mile circular track at 60 mi./hr. Find its acceleration in mi./hr.^{2} and ft./sec.^{2}

106. Rectangular Components of Velocity and Acceleration. If the rectangular coördinates of a moving particle P are given as functions of the time:

$$x = f_1(t), \qquad y = f_2(t), \qquad z = f_3(t),$$

the Cartesian equations of the path are obtained by eliminating t. If this elimination is impossible, the path is still determined by these equations.

Since the position vector of P is

$$\overrightarrow{OP} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

we have for its velocity and acceleration

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k},$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}.$$

The components of velocity and acceleration are therefore

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}, \qquad v_z = \frac{dz}{dt};$$

$$a_x = \frac{d^2x}{dt^2}, \qquad a_y = \frac{d^2y}{dt^2}, \qquad a_z = \frac{d^2z}{dt^2}.$$

If P' is the projection of P on the x-axis (Fig. 11), $\overrightarrow{OP'} = x\mathbf{i}$ and the velocity and acceleration of P' are

$$\frac{dx}{dt}\mathbf{i} = \text{proj}_x\,\mathbf{v}, \quad \frac{d^2x}{dt^2}\mathbf{i} = \text{proj}_x\,\mathbf{a}.$$

Since any axis may be chosen as the x-axis we see that the velocity and acceleration of the projection of a particle on an axis are equal to the projections of its velocity and acceleration on that axis.

Since the coördinates of a point on the hodograph are equal to (v_x, v_y, v_z), its parametric equations are

$$x = f_1'(t), \qquad y = f_2'(t), \qquad z = f_3'(t),$$

where the primes denote differentiation. Its Cartesian equations are obtained from these by eliminating t.

Example. Let the equations of motion be

$$x = A \cos nt, \qquad y = B \sin nt.$$

By eliminating t we obtain

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

as the Cartesian equation of the path; this is an ellipse of semi-axes, A, B.

The velocity components are

$$v_x = -nA \sin nt, \qquad v_y = nB \cos nt.$$

Eliminating t from these equations, and replacing v_x, v_y by x, y we obtain

$$\frac{x^2}{n^2A^2} + \frac{y^2}{n^2B^2} = 1$$

as the equation of the hodograph. This is an ellipse of semi-axes, nA, nB, and is therefore similar to the ellipse forming the orbit.

The acceleration components are

$$a_x = -n^2A \cos nt = -n^2x, \qquad a_y = -n^2B \sin nt = -n^2y;$$

whence

$$\mathbf{a} = -n^2 [x, y] = -n^2\mathbf{r}.$$

The acceleration is therefore always directed towards the center of the ellipse, and its magnitude is proportional to the distance of the particle from the center.

PROBLEMS

Find the components and magnitude of the velocity and acceleration; and obtain the Cartesian equations of the path and hodograph.

1. $x = 80\,t, \qquad y = 80\,t - 16\,t^2.$

2. $x = A \sin nt, \qquad y = B \sin nt.$

3. $x = \sin t, \qquad y = \cos 2t.$

4. $x = \cos t, \qquad y = \frac{1}{2} \sin 2t.$

5. $x = \sin t, \qquad y = \cos t, \qquad z = \sin t.$

107. Tangential and Normal Components of Acceleration. The acceleration of a particle moving along a curve with the velocity $\mathbf{v} = v\mathbf{T}$ may be written

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{dt};$$

or since

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \kappa\mathbf{N}v = \frac{v}{\rho}\mathbf{N}, \qquad (\S\ 86,\ 1),$$

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \frac{v^2}{\rho}\mathbf{N}.$$

Hence the acceleration vector always lies in the plane of tangent and principal normal; and its components in the positive directions of the tangent, principal normal, and binormal, are

(1), (2), (3) $$a_t = \frac{dv}{dt}, \quad a_n = \frac{v^2}{\rho}, \quad a_b = 0.$$

These results are of the first importance and may be stated as follows:

The acceleration of a particle has a tangential component equal to the instantaneous time rate of change of its speed, and a normal component equal to the square of its speed divided by the radius of curvature of the path.

If we regard v as a function of s,

(4) $$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}, \quad \text{or} \quad a_t = v\frac{dv}{ds}.$$

From (1) we see that a_t is positive or negative according as v is increasing or decreasing (algebraically) at the instant considered. If v changes at a uniform rate, $a_t = \Delta v/\Delta t$; that is, a_t equals the algebraic change in speed during any time interval, divided by that interval. Thus if the speed changes uniformly from 7 ft./sec. to 1 ft./sec. in 3 sec., $a_t = (1 - 7)/3 = -2$ ft./sec.^{2}

Since $a_n = \kappa v^2$ and $\kappa \geqq 0$ (§ 86) we see that a_n is positive or zero. Hence the normal projection of the acceleration, when not zero, has the same direction as $\mathbf{N}$ and therefore points toward the center of curvature of the path.

In the case of motion along a straight line, $\rho = \infty$ and $a_n = 0$. Hence

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} = v\frac{dv}{ds}\mathbf{T} \qquad (\rho = \infty),$$

where $\mathbf{T}$ is now a constant unit vector.

If a particle moves along its path with constant speed v, $dv/dt = 0$, and $a_t = 0$. Then

$$\mathbf{a} = \frac{v^2}{\rho}\mathbf{N} \qquad (v \text{ constant}),$$

and the acceleration is always normal to the path.

Example 1. An automobile running 45 mi./hr. on a one-mile circular track is uniformly retarded by the brakes so that it comes to rest in 3 seconds. Find the tangential and normal components of its acceleration one second after the brakes are applied. How far does it travel in coming to rest?

The speed changes uniformly from 45 mi./hr. or 66 ft./sec. to zero in 3 seconds; hence

$$a_t = \frac{\Delta v}{\Delta t} = \frac{0 - 66}{3} = -22 \text{ ft./sec.}^2$$

throughout the retardation.

Since the speed decreases 22 ft./sec. in each second, the speed t seconds after the brakes are applied is $v = 66 - 22\,t$ ft./sec. The radius of the track is $5280/2\,\pi = 840$ ft.; hence after one second

$$a_n = \frac{v^2}{\rho} = \frac{44^2}{840} = 2.3 \text{ ft./sec.}^2$$

To find the distance traveled in coming to rest we have

$$s = \int_0^3 v\,dt = \int_0^3 (66 - 22\,t)\,dt = 66\,t - 11\,t^2 \Big|_0^3 = 99 \text{ ft.}$$

Example 2. On applying four wheel brakes, an automobile traveling at 30 mi./hr. comes to rest in 50 ft. Assuming a uniform change in speed, compute the tangential component of its acceleration.

From (4) we have $a_t\,ds = v\,dv$; and since s varies from zero to 50 ft. while v varies from 44 ft./sec. to zero,

$$a_t\int_0^{50} ds = \int_{44}^{0} v\,dv \quad \text{or} \quad 50\,a_t = -\frac{44^2}{2};$$

hence $$a_t = -19.36 \text{ ft./sec.}^2.$$

Example 3. When the velocity and acceleration are known, the tangential and normal components of the acceleration may be computed as follows. On multiplying

$$\mathbf{v} = v\mathbf{T} \quad \text{and} \quad \mathbf{a} = a_t\mathbf{T} + a_n\mathbf{N}$$

member for member, first with the dot and then with the cross, we obtain

$$\mathbf{v}\cdot\mathbf{a} = va_t \quad \text{and} \quad \mathbf{v}\times\mathbf{a} = va_n\mathbf{B}$$

since $\mathbf{T}\cdot\mathbf{N} = 0$, $\mathbf{T}\times\mathbf{N} = \mathbf{B}$ (§ 86). Hence

$$a_t = \frac{\mathbf{v}\cdot\mathbf{a}}{v}, \quad a_n = \left|\frac{\mathbf{v}\times\mathbf{a}}{v}\right|.$$

If we equate this value of a_n to v^2/ρ we obtain

$$\rho = \frac{|v^3|}{|\mathbf{v}\times\mathbf{a}|}$$

for the radius of curvature of the path.

In plane motion we may write $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$, and $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$. The results above then give

$$a_t = \frac{v_xa_x + v_ya_y}{v}, \quad a_n = \left|\frac{v_xa_y - v_ya_x}{v}\right|; \quad \rho = \left|\frac{v^3}{v_xa_y - v_ya_x}\right|.$$

PROBLEMS

1. A flywheel 6 ft. in diameter and making 200 r.p.m. is brought uniformly to rest in 1 minute by a braking resistance. Compute

(*a*) the angular acceleration in rad./sec.²;

(*b*) the angular speed in rad./sec. 30 sec. after the brake is applied;

(*c*) a_t and a_n at this instant for a particle on the rim;

(*d*) the number of revolutions made before stopping.

2. The speed of a train changes uniformly from 15 mi./hr. to 30 mi./hr. in 1 minute; compute a_t in ft./sec.² How far does the train travel in this time?

If the above change in speed occurs in $\frac{1}{2}$ mile, compute a_t in ft./sec.². What time elapses during the change?

3. Using the results of Example 3, § 105, show that

$$a_t = b\omega^2 \cos \tfrac{1}{2}\theta, \quad a_n = b\omega^2 \sin \tfrac{1}{2}\theta.$$

4. A projectile describes the parabola

$$x = 80\,t, \qquad y = 160\,t - 16\,t^2.$$

Find a_t and a_n at the highest point of the path, and where the projectile strikes the ground ($y = 0$). Compute the total acceleration and ρ at these points.

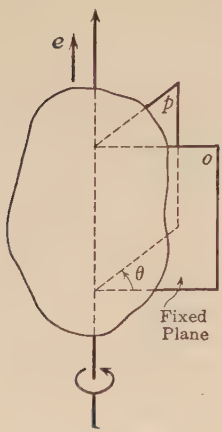


FIG. 108.

108. Rotation about a Fixed Axis. When a body revolves about a fixed axis, the amount of rotation is measured by the angle θ between an axial plane o fixed in space and an axial plane p fixed in the body (Fig. 108). By choosing a positive direction on the axis (unit vector $\mathbf{e}$) we fix

the positive sense of θ by the rule of the right-hand screw. Then the *angular velocity* and the *angular acceleration* of the body are defined as

$$\boldsymbol{\omega} = \frac{d\theta}{dt}\,\mathbf{e} \qquad \text{and} \qquad \boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}.$$

If we write

(1), (2) $$\omega = \frac{d\theta}{dt} \qquad \text{and} \qquad \alpha = \frac{d\omega}{dt},$$

we have

(3), (4) $$\boldsymbol{\omega} = \omega\mathbf{e} \qquad \text{and} \qquad \boldsymbol{\alpha} = \alpha\mathbf{e}.$$

Thus $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are vectors parallel to the axis of rotation.

When $\mathbf{e}$ and the scalars ω and α are given, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are determined. For this reason it is customary to call the scalars ω and α the angular velocity and acceleration. We shall follow this usage in dealing with rotation about a *fixed* axis. The notation will show whether the quantity in question is a scalar or vector.

When ω changes uniformly with the time, α is constant and equal to $\Delta\omega/\Delta t$. Thus if ω decreases from 5 to 2 rad./sec. in 10 sec.,

$$\alpha = \frac{2-5}{10} = -0.3 \text{ rad./sec.}^2$$

If we regard ω as a function of θ,

(5) $$\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} \qquad \text{or} \qquad \alpha = \omega\frac{d\omega}{d\theta}.$$

When θ is measured in radians and t in seconds the units of angular velocity and acceleration are the radian per second (rad./sec.) and the radian-per-second per second (rad./sec.^{2}). These units should always be understood when others are not explicitly mentioned. In engineering practice the angle is often measured in revolutions and the time in minutes; the units of ω and α are then the revolution per minute (rev./min. or r.p.m.) and the revolution-per-minute per minute (rev./min.^{2}). Since 2π radians are comprised in a complete revolution,

$$1 \text{ rev./min.} = \frac{2\pi}{60}\text{ rad./sec.}, \quad 1 \text{ rev./min.}^2 = \frac{2\pi}{60^2}\text{ rad./sec.}^2$$

Example. A braking resistance is applied to a flywheel making 180 r.p.m. If it makes 30 revolutions before coming to rest, find its angular

acceleration (assumed constant) and the time during which it was retarded.

As ω changes from 180 to 0 r.p.m., θ changes from 0 to 30 revolutions. On integrating (5), written $\alpha\, d\theta = \omega\, d\omega$, between these limits, we have

$$\alpha \int_0^{30} d\theta = \int_{180}^{0} \omega\, d\omega \quad \text{or} \quad 30\,\alpha = \tfrac{1}{2}(0 - 180^2);$$

hence $\alpha = -540$ rev./min.^{2} If we wish to express α in rad./sec.^{2},

$$\alpha = -540 \frac{2\,\pi}{60^2} = -\frac{3\,\pi}{10} = -0.942 \text{ rad./sec.}^2.$$

If Δt denotes the braking interval, we have since α is constant

$$\alpha = \frac{\Delta\omega}{\Delta t} \quad \text{or} \quad -540 = \frac{-180}{\Delta t}; \quad \text{hence} \quad \Delta t = \tfrac{1}{3} \text{ min.}$$

109. Circular Motion. The particles of a body revolving about a fixed axis describe circles with centers on the axis. Consider a particle P revolving in a circle of radius r about the point O (Fig. 109a). Then if s = arc AP and $\theta = \angle AOP$ are measured in the same sense, $s = r\theta$ when θ is expressed in radians. The speed of P is therefore

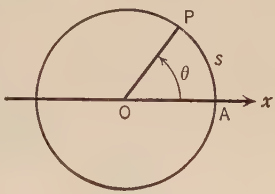


FIG. 109a.

(1) $$v = \frac{ds}{dt} = r\frac{d\theta}{dt} \quad \text{or} \quad v = r\omega.$$

Thus if the body is revolving at the rate of 4 rad./sec., a particle 2 ft. from the axis will have a speed of 8 ft./sec.

The acceleration of P has the tangential and normal components

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}, \quad a_n = \frac{v^2}{r} = \frac{r^2\omega^2}{r} \quad (\S\ 107);$$

hence, for circular motion,

(2), (3) $$a_t = r\alpha, \quad a_n = r\omega^2.$$

In these important formulas *the angle must be measured in radians.* If r is given in ft., ω in rad./sec., α in rad./sec.^{2}, a_t and a_n will be in ft./sec.^{2}.

Example. A braking resistance uniformly retards a flywheel from 120 to 60 r.p.m. while it makes 12 revolutions. Find v, a_t, a_n for a particle 3 ft. from the axis when ω = 90 r.p.m.

Since α is constant, we have on integrating $\alpha\, d\theta = \omega\, d\omega$ (§ 108, 5) over the braking interval

$$\alpha \int_0^{12} d\theta = \int_{120}^{60} \omega\, d\omega \quad \text{or} \quad 12\,\alpha = \tfrac{1}{2}\,(60^2 - 120^2);$$

hence $$\alpha = -450 \text{ rev./sec.}^2.$$

On changing units to radians and seconds, we have at the instant in question

$$\omega = \frac{90 \times 2\,\pi}{60} = 3\,\pi \text{ rad./sec.}, \quad \alpha = -\frac{450 \times 2\,\pi}{60 \times 60} = -\frac{\pi}{4} \text{ rad./sec.}^2$$

Hence from (1), (2) and (3),

$$v = 3 \times 3\,\pi = 28.27 \text{ ft./sec.};$$

$$a_t = -\frac{3\,\pi}{4} = -2.356, \quad a_n = 3 \times (3\,\pi)^2 = 266.5 \text{ ft./sec.}^2$$

The velocity of any particle of a body revolving about a fixed axis is equal to the vector product of the angular velocity and the position vector of the particle referred to an origin on the axis:

(4) $$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}.$$

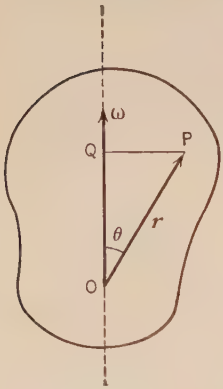


FIG. 109*b*.

For $\boldsymbol{\omega}\times\mathbf{r}$ is a vector normal to the plane OPQ in the direction of motion (Fig. 109*b*) and

$$|\boldsymbol{\omega}\times\mathbf{r}| = \omega \cdot OP \sin\theta = \omega \cdot QP.$$

From (4), the acceleration of P is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \boldsymbol{\alpha}\times\mathbf{r} + \boldsymbol{\omega}\times\mathbf{v};$$

or on replacing $\mathbf{v}$ by $\boldsymbol{\omega}\times\mathbf{r}$,

(5) $$\mathbf{a} = \boldsymbol{\alpha}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}).$$

Since $\boldsymbol{\alpha}\times\mathbf{r}$ and $\boldsymbol{\omega}\times\mathbf{v}$ are tangent and normal to the path, the projections of $\mathbf{a}$ on the tangent and normal are $\boldsymbol{\alpha}\times\mathbf{r}$ and $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$.

PROBLEMS

1. When ω is negative and $|\omega|$ increasing, what is the sign of α?

2. A pendulum of length l and vibrating through an angle 2β has its angular velocity given by $\omega^2 = \dfrac{2g}{l}(\cos\theta - \cos\beta)$, when it makes an angle θ with the vertical. Find its angular acceleration.

3. A wheel running lightly on a shaft at 300 r.p.m. has its motion uniformly retarded by axle friction. If its angular velocity decreases to

240 r.p.m. while it makes 900 revolutions, how many revolutions will it make before coming to rest? What time elapses while the wheel is coming to rest?

4. A flywheel is making 100 r.p.m. when a braking resistance is applied. If it is brought to rest in $\frac{1}{2}$ minute, what is its angular acceleration (assumed constant) in rad./sec.^{2}? How many revolutions does it make in this time? Find ω after 15 revolutions.

5. In Problem 2 find a_t and a_n for the bob of the pendulum when the string makes an angle θ with the vertical. Find also the extreme values of a_t and a_n and the angles at which they occur.

6. In Problem 4 find a_t and a_n for a point 2 ft. from the center of the wheel 18 sec. after the brake is applied.

110. Relative Motion. Hitherto the motion of a particle P has been referred to a system of reference regarded as having a fixed position in space. Suppose now that the motion of P is referred to a body b which is itself moving with respect to the fixed system. The motion of P referred to the fixed system is called its *absolute motion*, but when referred to the moving system, its *relative motion*, or more precisely, its *motion relative to b*. The latter motion is that which would be assigned to P by an observer at rest in b and unconscious of his own motion. Similarly, the velocity and acceleration of P are called *absolute* or *relative* according as they refer to its absolute or relative motion.

At any instant the particle P coincides with a certain point Q fixed in the moving body b; the velocity and acceleration of Q (due to the motion of b) are called the *body velocity* and the *body acceleration* for P at that instant. We shall now prove the following theorem known as

THE COMPOSITION OF VELOCITIES: *If the motion of a particle is referred to a moving body of reference, its absolute velocity is equal to the vector sum of the body velocity* $\mathbf{v}_b$ *and the relative velocity* $\mathbf{v}_r$:

$$\mathbf{v} = \mathbf{v}_b + \mathbf{v}_r. \tag{1}$$

Proof. Let the particle P coincide with the point Q of the body b at the instant t. At a later instant $t + \Delta t$, let P and Q have the positions P' and Q'. Now

$$\overrightarrow{PP'} = \overrightarrow{PQ'} + \overrightarrow{Q'P'} = \overrightarrow{QQ'} + \overrightarrow{Q'P'};$$

here $\overrightarrow{PP'}$ and $\overrightarrow{Q'P'}$ are the absolute and relative displacements of P while $\overrightarrow{QQ'}$ is the displacement of Q in the interval Δt. If we

divide this equation by Δt and pass to the limit $\Delta t \to 0$,

$$\lim \frac{\overrightarrow{PP'}}{\Delta t} = \mathbf{v}, \quad \lim \frac{\overrightarrow{QQ'}}{\Delta t} = \mathbf{v}_b, \quad \lim \frac{\overrightarrow{Q'P'}}{\Delta t} = \mathbf{v}_r,$$

and we obtain equation (1).

This equation is frequently used to compute the relative velocity: $\mathbf{v}_r = \mathbf{v} - \mathbf{v}_b$. In many problems b has a motion of *translation*, that is, at any instant all of its points have the same velocity $\mathbf{v}_b$.

Example 1. The speed of a boat in still water is 10 mi./hr. In what direction must it be steered in order to make a N. E. course, if a N. W. current of 2 mi./hr. is flowing? With what speed will the boat go N. E.?

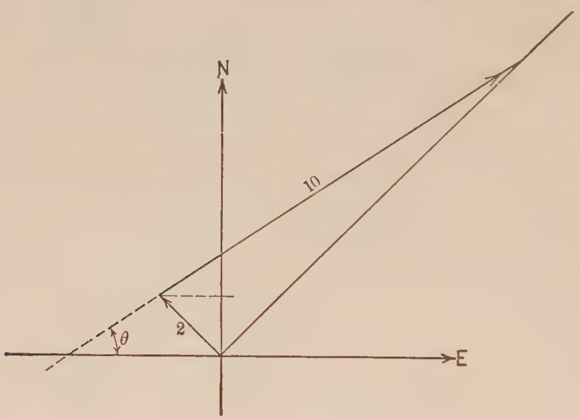


FIG. 110*a*.

Regard the land and the water as the "fixed" and moving bodies of reference respectively. The velocity $\mathbf{v}_b$ of the water relative to the land is 2 mi./hr. N.W. The velocity $\mathbf{v}_r$ of the boat relative to the water, numerically equal to 10 mi./hr., must point in a direction $\theta°$ N. of E. such that $\mathbf{v}_b + \mathbf{v}_r$, the velocity of the boat relative to the land, points to the N.E. From the figure we have

$$\cos(45° + \theta) = 0.2; \qquad 45° + \theta = 78.5°, \qquad \theta = 33.5°.$$

The boat must be steered 33.5° N. of E. The speed on this course is

$$v = \sqrt{100 - 4} = \sqrt{96} = 9.8 \text{ mi./hr.}$$

Example 2. A ship A is 75 miles due west of a ship B. A steams east 9 mi./hr., while B steams north 12 mi./hr. When will the ships be the least distance apart, and what is this least distance?

From Fig. 110*b*, it is clear that the velocity of A relative to B, $\mathbf{v}_A - \mathbf{v}_B$, is

$$\sqrt{9^2 + 12^2} = \sqrt{225} = 15 \text{ mi./hr.}$$

at an angle

$$\theta = \tan^{-1} \tfrac{4}{3} \qquad \text{S. of E.}$$

The course of A relative to B is therefore a straight line drawn in this direction; and the least distance between the ships is given by the length of the perpendicular, BC, dropped from B upon this line:

$$BC = 75 \sin \theta = 75 \cdot \tfrac{4}{5} = 60 \text{ mi.}$$

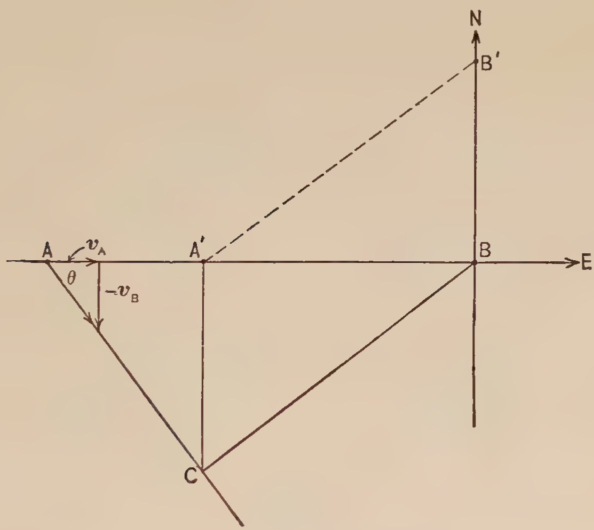


Fig. 110*b*.

Before the ships are nearest together, the portion

$$AC = 75 \cos \theta = 75 \cdot \tfrac{3}{5} = 45 \text{ mi.}$$

of A's course relative to B must be traversed; and the time required is $45/15 = 3$ hours. The actual displacements of A and B during this time are $\overrightarrow{AA'}$ and $\overrightarrow{BB'} = \overrightarrow{CA'}$, of length 27 mi. and 36 mi. respectively.

Example 3. A motorcyclist, sent from the rear of an army column l miles long and marching u mi./hr., delivered a message to an officer at its head and returned immediately to the rear. If the messenger rode at a uniform rate of v mi./hr., what was the time and length of the journey, and how far was the messenger from his starting point at the end?

The velocity of the messenger relative to the column is of magnitude $v - u$ on the forward trip, $v + u$ on the rearward trip. Hence the respec-

tive times of these trips, and the total time, are

$$t_1 = \frac{l}{v - u}, \qquad t_2 = \frac{l}{v + u}, \qquad T = t_1 + t_2 = \frac{2\,vl}{v^2 - u^2} \text{ hr.}$$

The total distance traveled by the messenger is vT; and his net distance forward is the same as the progress of the army in the time T, namely, uT. To check the latter result we note that

$$uT = \frac{2\,uvl}{v^2 - u^2} = vt_1 - vt_2.$$

PROBLEMS

1. At what angle with the shore should a boat be directed in order to reach a point on the other bank directly opposite, if the speed of the boat relative to the water is 5 mi./hr., and that of the stream 2 mi./hr. If the distance is 1 mile, how long will it take to make the crossing?

2. When a steamer is going due east at 10 mi./hr., a vane at the mast-head points N. N. E. When the steamer stops the vane points N. W. Find the speed of the wind.

3. A ship A is steaming due east at 16 mi./hr. while a ship B is steaming 30° east of north at 10 mi./hr. Find the velocity of A relative to B.

4. Two freight trains $\frac{1}{2}$ mile and $\frac{1}{4}$ mile long have speeds of 10 mi./hr. and 15 mi./hr. respectively. How long will it take them to pass each other when traveling in the same direction? In opposite directions?

5. A swimmer's speed is 50 yd./min. in still water and the current is 30 yd./min. How long will it take him to swim 100 yards to a point directly across stream and back? How long will it take him to swim 100 yards upstream and back?

6. Light travels with the speed c from a mirror A to a mirror B, at a distance l from A, and is reflected back to A. The apparatus to which A and B are rigidly attached is moving with the speed v through space. Show that the time of passage over the path ABA is $2cl\,(c^2 - v^2)^{-1}$ or $2l\,(c^2 - v^2)^{-\frac{1}{2}}$ according as the apparatus is moving parallel or perpendicular to the line AB.

7. A steamer travels 60 miles up a river, and then back to its starting point in 16 hours. If the speed of the current is 2 mi./hr., find the speed of the steamer relative to the water.

8. Two runners, A and B, are $\frac{1}{4}$ mile apart on a circular one-mile track. If both start at the same time and run in the same direction, A overtakes B in $7\frac{1}{2}$ minutes; but if they run in opposite directions, A meets B in $2\frac{1}{2}$ minutes. Find the speeds of A and B.

9. A steamer d miles away from a port is approaching it with a speed of

u mi./hr., while another is leaving the port with a speed of v mi./hr. If their courses are straight and include an angle α, the least distance D between the steamers, and the time T of reaching this position, are given by

$$D = \frac{dv \sin \alpha}{\sqrt{u^2 + v^2 + 2uv \cos \alpha}}, \qquad T = \frac{d(u + v \cos \alpha)}{u^2 + v^2 + 2uv \cos \alpha}.$$

Prove this: (*a*) by the method of Example 2; (*b*) by the method of the Calculus for finding extremes. What is the velocity of the first steamer relative to the second?

111. Summary, Chapter VIII. The *speed* v of a moving particle is defined as ds/dt. If the arc s is plotted as ordinate against the time t as abscissa we obtain the space-time curve of the motion; the slope of this curve at any point (t, s) gives the speed at the instant t.

Let $\mathbf{r}$ be the position vector of a moving particle P referred to a fixed origin. The *velocity* and *acceleration* of the particle are defined as the vectors

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2},$$

localized at P.

On forming the first and second time derivatives of $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we obtain the rectangular components of $\mathbf{v}$ and $\mathbf{a}$:

$$\mathbf{v} = \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right], \quad \mathbf{a} = \left[\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right].$$

Velocity and speed are connected by the relation $\mathbf{v} = v\mathbf{T}$. On differentiating this equation with respect to t, we obtain

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \frac{v^2}{\rho}\mathbf{N}; \quad \text{hence} \quad a_t = \frac{dv}{dt}, \quad a_n = \frac{v^2}{\rho}$$

are the components of the acceleration along the tangent and principal normal to the path.

The angular velocity and acceleration of a body revolving about a fixed axis of unit vector $\mathbf{e}$ are defined as

$$\boldsymbol{\omega} = \omega\mathbf{e} \quad \text{where} \quad \omega = \frac{d\theta}{dt},$$

$$\boldsymbol{\alpha} = \alpha\mathbf{e} \quad \text{where} \quad \alpha = \frac{d\omega}{dt};$$

θ denotes the angle of revolution taken positive in the sense determined by $\mathbf{e}$. It is customary to also call the scalars ω and α

the angular velocity and acceleration. A particle P of the body at a distance r from the axis has the speed and acceleration components

$$v = r\omega, \quad a_t = r\alpha, \quad a_n = r\omega^2,$$

provided the angle θ is measured in radians. If O is any point on the axis of rotation and $\mathbf{r} = \overrightarrow{OP}$, the velocity of P is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

If the motion of a particle is referred to a moving body of reference, its absolute velocity is equal to the vector sum of the body velocity and the relative velocity.

CHAPTER IX

KINEMATICS OF RECTILINEAR MOTION

112. Rectilinear Motion. In treating the motion of a particle P along a straight line, we choose an origin, O, of abscissas and a positive direction on the line. Then, if $\mathbf{i}$ denotes a positive unit vector along the line, the position vector of the particle is

$$\mathbf{r} = \overrightarrow{OP} = x\mathbf{i},$$

and its velocity and acceleration are given by

$$(1) \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i}, \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\mathbf{i}.$$

As the direction and magnitude of $\mathbf{v}$ and $\mathbf{a}$ are completely specified by the sign and numerical value of dx/dt, d^2x/dt^2, we shall, for the sake of brevity, call these scalars the velocity and acceleration respectively, and write

$$(2), (3) \qquad v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

If we regard v as a function of x,

$$(4) \qquad \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} \quad \text{or} \quad a = v\frac{dv}{dx}.$$

In rectilinear motion x takes the place of the arc s; v is really the speed and a the tangential component of the acceleration (the normal component being zero). Thus (4) is but a special case of (§ 107, 4).

From (1) we see that v and a are positive when $\mathbf{v}$ and $\mathbf{a}$ are positively directed; this is the case when x and v, respectively, increase with the time. It should be noted that this increase refers to the *algebraic* values of x and v, and not to their numerical values. Thus:

$$a > 0 \text{ when} \begin{cases} v > 0, & |v| \text{ increasing}, \\ v < 0, & |v| \text{ decreasing}; \end{cases}$$

$$a < 0 \text{ when} \begin{cases} v > 0, & |v| \text{ decreasing}, \\ v < 0, & |v| \text{ increasing}. \end{cases}$$

Thus a positive acceleration does not necessarily mean that the motion is becoming more rapid; nor does a negative acceleration always mean that the motion is being retarded.

When the position, x, of the particle is given as a function of the time, the velocity and acceleration at any instant may be computed from (2) and (3).

When the velocity of the particle is given as a function of the position, the acceleration may be obtained from (4) as a function of the position.

Example 1. If the relation between the velocity and the time is

$$v^2 = Ax^2 + Bx + C,$$

we have

$$2\,v\frac{dv}{dx} = 2\,Ax + B = 2\,A\left(x + \frac{B}{2\,A}\right).$$

The acceleration,

$$a = v\frac{dv}{dx} = A\left(x - \frac{-B}{2\,A}\right),$$

is therefore directly proportional to the distance of the particle from the fixed point, $x = -B/2\,A$.

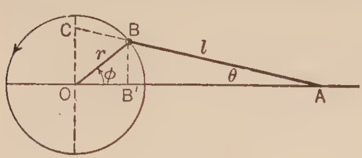


FIG. 112*a*.

Example 2. *Velocity and Acceleration of Piston in a Direct-acting Engine.* In Fig. 112*a*, OB and BA represent the crank and connecting-rod of a direct-acting engine. Then $x = OA$ represents the displacement of the cross-head from the crank-shaft, and dx/dt, d^2x/dt^2 give the velocity and acceleration of the piston. Let $n = l/r$ denote the ratio of the length of the connecting-rod to the length of the crank; then

(5) $$x = OB' + B'A = r\cos\phi + \sqrt{l^2 - r^2\sin^2\phi} = r\,(\cos\phi + \sqrt{n^2 - \sin^2\phi}),$$

(6) $$v = \frac{dx}{dt} = -r\omega\left\{\sin\phi + \frac{\sin 2\,\phi}{2\,(n^2 - \sin^2\phi)^{\frac{1}{2}}}\right\},$$

where $\omega = \dfrac{d\phi}{dt}$ is the angular velocity of the crank in radians per second.

Furthermore, if ω is constant,

(7) $$a = \frac{dv}{dt} = -r\omega^2\left\{\cos\phi + \frac{n^2\cos 2\,\phi + \sin^4\phi}{(n^2 - \sin^2\phi)^{\frac{3}{2}}}\right\}.$$

In actual practice n usually lies between 3 and 8, so that $\sin^2\phi$ is sufficiently small in comparison with n^2 to permit of its being neglected without seriously affecting the accuracy. When this is done we obtain from

(6) and (7) the *approximate* results:

(8) $$v = -r\omega\left(\sin\phi + \frac{\sin 2\phi}{2n}\right),$$

(9) $$a = -r\omega^2\left(\cos\phi + \frac{\cos 2\phi}{n}\right).$$

These, moreover, fulfill the relation $a = dv/dt$, and are accurately true at the dead centers (where $\sin\phi = 0$). At these points we have

$$a\Big|_{\phi=0} = -r\omega^2\left(1 + \frac{1}{n}\right), \qquad a\Big|_{\phi=\pi} = r\omega^2\left(1 - \frac{1}{n}\right).$$

The crank angle ϕ for which $a = 0$ satisfies the equation obtained by putting the expression in braces on the right-hand side of (7) equal to zero. When this equation is rationalized, the cosines expressed in terms of $\sin\phi$, and the common factor $n^2 - 1$ removed, it reduces to

(10) $$\sin^6\phi - n^2\sin^4\phi - n^4\sin^2\phi + n^4 = 0,$$

a cubic in $\sin^2\phi$, which, when $n > 1$, has always a single root between 0 and 1. Of course an approximate value of this angle is readily obtained from (9).

PROBLEMS

1. When $v^2 = Ax + B$, show that the acceleration is constant.

2. When the radius OQ in Fig. 119*a* revolves about O at the constant rate of n rad./sec., the point P, the projection of Q on a diameter, oscillates to and fro along AB. Show that the velocity and acceleration of P are proportional to PQ and OP respectively. [Let $t = 0$ when P and Q coincide with A; then $x = OP = h\cos nt$.]

3. If $t = x^2 - 2x + 3$, show that the acceleration is inversely proportional to the cube of the distance of the particle from the point $x = 1$.

4. If $x = Ae^{nt} + Be^{-nt}$ (n constant), express the acceleration in terms of x.

5. In Example 2 we have, by the cosine law,

$$l^2 = r^2 + x^2 - 2rx\cos\phi.$$

By differentiating this equation with respect to t, prove that

$$\frac{dx}{dt} = -x\tan\theta\cdot\omega = -OC\cdot\omega.$$

6. If, in Example 2, $n = 5$, find from (10) the crank angle for which $a = 0$. Compare this with the approximate value derived from (9).

7. From (9), Example 2, find the values of ϕ for which the acceleration is a maximum or a minimum. Consider the three cases $n < 4$, $n = 4$, $n > 4$.

8. In Fig. 112b, the body A, which can slide over a vertical rod, is attached to a cord passing over a pulley at a distance d from the rod. When the end of the cord is drawn with the constant velocity V, show that A has an upward velocity and acceleration of $V \sec \theta$ and $(V^2/d) \tan^3 \theta$ respectively, where θ is the angle between the cord and the rod.

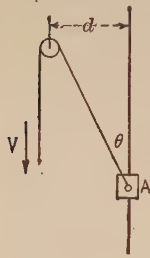


Fig. 112b.

113. Velocity-Time Curve. Any relation between the velocity and the time may be represented graphically by plotting t as abscissa and v as ordinate. The curve so obtained is called a *velocity-time curve.*

Since $a = dv/dt$, the slope of the velocity-time curve at any point (t, v) represents the acceleration at the instant t. In computing this slope from the graph it should be remembered that vertical segments must be measured on the scale of velocity, horizontal segments on the scale of time.

When the velocity is constant, the space traversed during an interval $t_2 - t_1$ is given by $v(t_2 - t_1)$. When the velocity is variable and given as a function of the time, the space traversed in an interval $t_2 - t_1$ may still be approximately computed by dividing $t_2 - t_1$ into a number of smaller intervals Δt, in each of which the velocity is assumed to have a constant value, namely, its actual value at the beginning of the subinterval. Graphically, this amounts to replacing the actual v-t curve by a series of horizontal lines whose left-hand ends are points of the curve. If there are n equal subintervals, $\Delta t = (t_2 - t_1)/n$, and the space traversed with this discontinuous velocity is given by the sum of n terms of the type $v\,\Delta t$. This sum will differ from the actual space by an amount that can be made as small as we please by taking n sufficiently large. With reference to the v-t curve, the separate terms of $\sum v\,\Delta t$ represent the areas of rectangles with equal bases, Δt, on the t-axis; and their sum forms the area under a flight of steps guided by the curve. As n, the number of

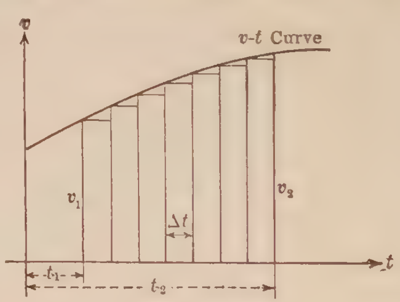


Fig. 113a.

steps, becomes infinite, the area between them and the t-axis approaches the area under the v-t curve as a limit. Thus the space passed over in the interval from t_1 to t_2 is represented by the area included between the v-t curve, the t-axis, and the ordinates at t_1 and t_2. The unit to be used in estimating this area is the rectangle formed by the units of the t and v scales. Thus if 1 in. $= p$ sec. on the t-scale, 1 in. $= q$ ft./sec. on the v-scale, 1 sq. in. of area represents pq ft. passed over.

The above result follows at once upon integrating the equation $dx/dt = v$ with respect to t between the limits t_1 and t_2,

$$(1) \qquad x_2 - x_1 = \int_{t_1}^{t_2} v\,dt,$$

and interpreting this definite integral as an area.

Thus far we have tacitly assumed that the velocity does not change sign in the interval considered. When the velocity changes sign, the area between the v-t curve and the t-axis consists of two or more portions, and if the various portions are all reckoned as positive, their sum will still give the total space passed over. The integral in (1), however, gives the *net change in position* from t_1 to t_2, and not the space traversed. For example, if $v = 10 - 2t$, $t_1 = 0$, $t_2 = 10$,

$$x_2 - x_1 = \int_0^{10} (10 - 2t)\,dt = 0,$$

and the particle has the same position at both instants t_1, t_2. The v-t curve (Fig. 113*b*) shows that the total space passed over is

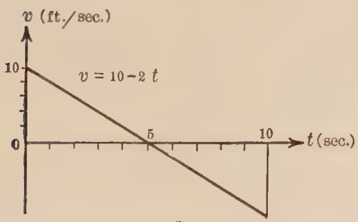


Fig. 113*b*.

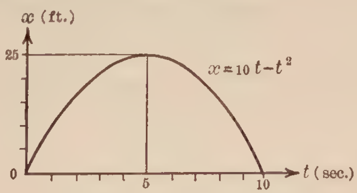


Fig. 113*c*.

represented by the sum of two equal triangles, and is $25 + 25 = 50$ feet. The motion consists of two opposite displacements of 25 feet, each requiring 5 seconds. This is clearly shown by the

space-time curve (Fig. 113*c*), which, if $x = 0$ when $t = 0$, has for its equation

$$x = \int_0^t (10 - 2t)\,dt = 10t - t^2.$$

The x-t curve is a parabola with the line $t = 5$ as axis.

114. Velocity-Space Curve. If a relation between the velocity of a particle and its position is represented graphically by plotting x as abscissa and v as ordinate, the curve so obtained is called a *velocity-space curve.*

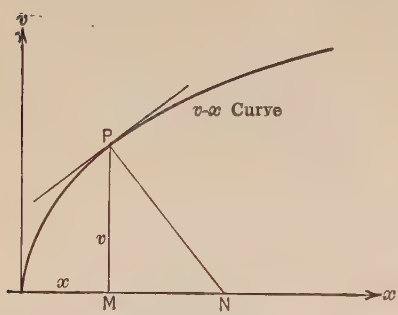


FIG. 114*a*.

By means of the formula (§ 112, 4) we may show that the acceleration at any point $P(x, v)$ of the v-x curve is represented by the length of the subnormal at P. For the slope at P is

$$(1) \qquad \frac{dv}{dx} = \tan MPN = \frac{MN}{v}; \quad \text{hence} \quad MN = v\frac{dv}{dx} = a.$$

Note that the subnormal is a *directed* segment measured from the foot of the ordinate at P to the point where the normal meets the x-axis, and is positive in the $+x$ direction. If 1 in. $= p$ ft. on the x-scale, 1 in. $= q$ ft./sec. on the v-scale, we see from (1) that 1 inch of subnormal represents q^2/p ft./sec.^{2}.

PROBLEMS

1. The v-t curve is a straight line joining $t = 0$, $v = 160$ and $t = 10$, $v = -160$. If t is in sec., v in ft./sec., what is the acceleration? Calculate the space traversed in 3, 5, 8 and 10 seconds.

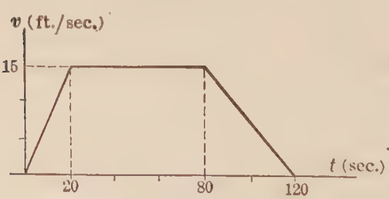


FIG. 114*b*.

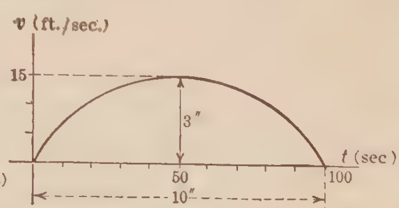


FIG. 114*c*.

2. The v-t curve is shown in Fig. 114*b*. Draw to scale the a-t and x-t curves.

3. The v-t curve is the circular arc shown in Fig. 114*c*. Draw the a-t curve on the same time scale, letting 10 in. = 1 ft./sec.². What is the total space traversed during the motion?

4. The v-x curve is a circle of 5 in. radius about the origin. An inch vertical represents 4 ft./sec., an inch horizontal, 2 ft. Draw the a-t curve. What is the acceleration when $x = -5$ and 5?

5. The v-x curve is a straight line cutting the v-axis at $v = 10$ ft./sec., the x-axis at $x = 32$ ft.; it extends from $x = 0$ to $x = 64$. The units of the v and x scales are $\frac{1}{2}$ and $\frac{1}{8}$ in. respectively. What is the unit of the acceleration scale for measuring subnormals? Draw the a-x curve.

6. Construct curves showing the variation of piston speed and acceleration with the crank angle from the approximate equations (8) and (9) of § 112. Assume $r = 1.5$ ft., $l = 6$ ft., and take the angular velocity of the crank as 60 r.p.m.

[To obtain the v-ϕ curve, first draw the curves for $\sin \phi$ and $\sin 2\,\phi/2\,n$; proceed similarly for the a-ϕ curve.]

7. Show how the v-x and a-x curves for the piston may be obtained from the v-ϕ and a-ϕ curves respectively.

115. Equation of Motion. Suppose that a particle, moving in a straight line, has, at the instant t_0, the position x_0 and the velocity v_0. In brief, the *initial conditions* are

$$(1) \qquad t = t_0, \qquad x = x_0, \qquad v = v_0.$$

Then, if the acceleration is a known function of t, x, and v,

$$(2) \qquad a = f(t, x, v) \qquad \text{or} \qquad \frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right),$$

the motion is, in general, fully determined. The problem of finding the velocity and position of the particle at any instant from the above *equation of motion* may be exceedingly difficult, if soluble at all. In fact, no general method of solution is known. Some simple cases, however, that arise when a is a function of but one of the variables, t, x, v, may be dealt with in an elementary and general manner. These we proceed to consider.

1. When the acceleration is given as a function of the time,

$$\frac{d^2x}{dt^2} = f(t),$$

we may integrate at once with respect to t, obtaining

$$(3) \qquad v = \frac{dx}{dt} = \int f(t)\,dt + C = F(t) + C,$$

where $F(t)$ is any indefinite integral of $f(t)$, and C is the constant of integration. To determine C we put $v = v_0$, $t = t_0$ in (3).

Integrating again, we have

$$x = \int F(t)\,dt + Ct + C', \tag{4}$$

in which C', the new constant of integration, may be found by substituting $x = x_0$, $t = t_0$ in (4).

Equations (3) and (4) give the velocity and the position of the particle as functions of the time. If t is eliminated between these equations, we obtain a relation between the velocity of the particle and its position.

2. When the acceleration is given as a function of the position, we have from (§ 112, 4)

$$v\frac{dv}{dx} = f(x).$$

Multiplying this equation by 2 and integrating with respect to x, we have

$$v^2 = 2\int f(x)\,dx + C = F(x) + C; \tag{5}$$

C is determined from the condition that $v = v_0$ when $x = x_0$.

This equation may be written

$$\frac{dx}{dt} = \pm\sqrt{F(x) + C},$$

where the double sign refers to the two directions in which the particle may pass through the position x. If the motion is unidirectional, the proper sign may be chosen at once; otherwise it is necessary to consider both cases. Equating the reciprocals of the members of the last equation, and integrating with respect to x, we have

$$t = \pm\int \frac{dx}{\sqrt{F(x) + C}} + C'; \tag{6}$$

C' is determined from the condition that $t = t_0$ when $x = x_0$.

Equations (5) and (6) express the velocity and the time as functions of the position. By eliminating x between these equations we obtain a relation connecting the velocity and the time.

3. When the acceleration is given as a function of the velocity,

we may utilize at once both (§ 112, 3) and (§ 112, 4):

$$\frac{dv}{dt} = f(v), \qquad v\frac{dv}{dx} = f(v).$$

If these equations are written

$$\frac{dt}{dv} = \frac{1}{f(v)}, \qquad \frac{dx}{dv} = \frac{v}{f(v)},$$

both may be integrated with respect to v; thus

$$(7), (8) \qquad t = \int \frac{dv}{f(v)} + C, \qquad x = \int \frac{v\,dv}{f(v)} + C'.$$

The initial conditions (1) determine the constants C and C'.

Equations (7) and (8) give the time and position as functions of the velocity. By eliminating v between these equations we obtain a relation connecting the position and the time. Or we may solve (7) for v in terms of t, put $v = dx/dt$, and integrate to obtain x as a function of t.

Example. Suppose that the particle moves in a medium that produces a retardation directly proportional to its velocity. Then if the direction of motion is taken as positive, $a = -kv$, where k is a positive constant. Following the third method above, we have

$$\frac{dv}{dt} = -kv, \qquad v\frac{dv}{dx} = -kv;$$

$$\frac{dt}{dv} = -\frac{1}{kv}, \qquad \frac{dx}{dv} = -\frac{1}{k};$$

$$t = -\frac{1}{k}\ln v + C, \qquad x = -\frac{1}{k}v + C'.$$

If $x = 0$, $v = v_0$, when $t = 0$,

$$C = \frac{1}{k}\ln v_0, \qquad C' = \frac{v_0}{k};$$

whence

$$t = \frac{1}{k}\ln\frac{v_0}{v}, \qquad x = \frac{1}{k}(v_0 - v).$$

By solving the first of these equations for v and substituting its value in the second, we obtain the relation between x and t: thus

$$v = v_0 e^{-kt}, \qquad x = \frac{v_0}{k}(1 - e^{-kt}).$$

As t becomes infinite, v approaches zero and x approaches the value v_0/k. Hence if the equation $a = -kv$ held rigorously throughout the motion, the particle would never come to rest and never quite reach the limiting position v_0/k.

PROBLEMS

Find the relations between v and t, x and t, v and x, under the following conditions.

1. $a = 4;\quad x = 2,\quad v = -6,\quad$ when $t = 1$.
2. $a = 2t;\quad x = 0,\quad v = 1,\quad$ when $t = 0$.
3. $a = -\cos 2t;\quad x = 0,\quad v = 0,\quad$ when $t = 0$.
4. $a = e^{-t};\quad x = 1,\quad v = 0,\quad$ when $t = 0$.
5. $a = -2/x^2;\quad x = 1,\quad v = 2,\quad$ when $t = 0;\quad v > 0$.
6. $a = 32 - 4v;\quad x = 0,\quad v = 4,\quad$ when $t = 0$.
7. $a = -kv^2;\quad x = x_0,\quad v = v_0,\quad$ when $t = 0$.
8. $a = -4x;\quad x = 3,\quad v = 0,\quad$ when $t = 0$.

116. Uniformly Accelerated Motion. Rectilinear motion is said to be *uniformly accelerated* when the acceleration is constant. The equation of motion is

$$\frac{d^2x}{dt^2} = a,$$

where a denotes the constant acceleration. Applying the first method of § 115, we have, upon successive integration,

$$v = \frac{dx}{dt} = at + C,$$
$$x = \tfrac{1}{2}at^2 + Ct + C'.$$

If the instant when $x = x_0$, $v = v_0$, is chosen as the zero for reckoning time, the initial conditions

$$x = x_0, \qquad v = v_0, \qquad \text{when } t = 0,$$

require that $C = v_0$, $C' = x_0$. Hence the relations giving the velocity and position of the particle as functions of the time are

(1) $$v = v_0 + at,$$

(2) $$x = x_0 + v_0t + \tfrac{1}{2}at^2.$$

By eliminating t between these equations, we obtain the relation between the velocity and the position:

(3) $$v^2 - v_0^2 = 2a(x - x_0).$$

If we apply the second method of § 115 the equation of motion is written

$$v\frac{dv}{dx} = a;$$

hence

$$v^2 = 2\,ax + C'', \qquad v_0^2 = 2\,ax_0 + C'',$$

and the relation (3) follows at once.

Equations (1), (2) and (3) form the complete solution of our problem. In applying them, the acceleration, *if known*, must be given its proper sign. The remarks in § 112 relative to the sign of a should be carefully observed in this connection. If the acceleration is unknown, its value derived from the above equations will have the correct sign.

With the increment notation

$$\Delta t = t - 0, \quad \Delta x = x - x_0, \quad \Delta v = v - v_0, \quad \Delta(v^2) = v^2 - v_0^2,$$

and (1) and (3) become

(1) $$\Delta v = a\,\Delta t,$$

(3) $$\Delta(v^2) = 2\,a\,\Delta x.$$

As for (2), we have

$$x - x_0 = \frac{v_0 + (v_0 + at)}{2}\,t = \frac{v_0 + v}{2}\,t,$$

(4) $$\Delta x = \frac{v_0 + v}{2}\,\Delta t.$$

Since the time may be reckoned from any instant as zero, these equations apply to any interval, whether measured from $t = 0$ or not. Thus (4) may be stated as follows: *The distance traveled by a uniformly accelerated body in any time is equal to the mean of the terminal velocities multiplied by the time interval.*

If the values of the velocity are known at two given positions, or at two given instants, the acceleration may be computed from (3) and from (1) respectively. For example, suppose that a train running at 30 miles an hour is uniformly retarded by the brakes so that it comes to rest in 2 minutes. Since 30 mi./hr. = 44 ft./sec.,

$$a = \frac{\Delta v}{\Delta t} = \frac{0 - 44}{120} = -\frac{11}{30} = -0.367 \text{ ft./sec.}^2.$$

The distance traveled in this time may be computed from either (4) or (3):

$$\Delta x = \frac{44 + 0}{2}\,120 = 44 \times 60 = 2640 \text{ ft.};$$

$$\Delta x = \frac{\Delta\,(v^2)}{2\,a} = \frac{0 - 44^2}{-\frac{11}{15}} = 4 \times 44 \times 15 = 2640 \text{ ft.}$$

The train thus comes to rest in a half mile.

All of the results above apply also to curvilinear motion in which the speed ($v = ds/dt$) is changing uniformly. Then $a_t = dv/dt$ is constant and we need only replace x by s and a by a_t. Thus in § 107, Example 1, a car going 66 ft./sec. on a circular track is uniformly brought to rest in 3 seconds; hence from (4) it will travel $\frac{1}{2}\, 66 \times 3 = 99$ ft. in this time.

The velocity-time curve, given by (1), is a straight line of slope a. The trapezoidal area bounded by this line, the t-axis, and the ordinates v_1, v_2 is $\frac{1}{2}\,(v_1 + v_2)\Delta t$; this gives the distance traversed in the time Δt (§ 113), in agreement with (4).

The space-time curve, given by (2), is a parabola with its axis vertical; $v = dx/dt = 0$ at the vertex. The parabola will be concave upward or downward according as $d^2x/dt^2 = a$ is positive or negative.

The velocity-space curve, given by (3), is a parabola with its axis along the x-axis; $v = 0$ at the vertex. Since the acceleration is represented by the subnormal of this parabola (§114), the length of the subnormal must be constant. This is a well-known property of the parabola.

Example 1. A steamer making a landing is uniformly retarded so that it passes over 1000 ft. and 600 ft. respectively in two consecutive minutes. What is its retardation? What was its velocity at the beginning of the first minute? When will it lose its headway?

Let d_1 and d_2 denote the distances traversed in the two equal intervals of t_1 seconds. Then taking the direction of motion as positive and the zero instant at the beginning of the first interval, we have from (2)

$$d_1 = v_0 t_1 + \tfrac{1}{2}\, a t_1^2,$$
$$d_1 + d_2 = 2\, v_0 t_1 + 2\, a t_1^2.$$

Solving these equations for v_0 and a gives

$$v_0 = \frac{3\, d_1 - d_2}{2\, t_1}, \qquad a = \frac{d_2 - d_1}{t_1^2}.$$

From (1) we see that the velocity will vanish after $-v_0/a$ seconds.

Putting $d_1 = 1000$, $d_2 = 600$, $t_1 = 60$, we have

$$v_0 = 20 \text{ ft./sec.}, \qquad a = -\tfrac{1}{9} \text{ ft./sec.}^2,$$

and the steamer will lose its headway in 180 sec., or 3 min., after the initial instant.

Example 2. If the greatest possible acceleration and retardation of a train are 2 ft./sec.^{2} and 1 ft./sec.^{2} respectively, and if its highest speed is 60 mi./hr., find the least time, from rest to rest, between two stations 30 mi. apart. How much time is lost in starting and stopping the train?

Let a_1, a_2 and V denote the greatest acceleration, retardation and speed, respectively. Then V/a_1 is the time spent in getting up speed, and V/a_2

the time in coming to rest from full speed. The distances traveled in these stages are, from (3), $V^2/2\,a_1$ and $V^2/2\,a_2$; hence, if d is the total distance, the distance traveled at full speed is $d - V^2/2\,a_1 - V^2/2\,a_2$. The total time of passage is therefore

$$T = \frac{V}{a_1} + \frac{V}{a_2} + \frac{d - V^2/2\,a_1 - V^2/2\,a_2}{V} = \frac{V}{2}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) + \frac{d}{V}.$$

In this expression d/V is the time necessary to traverse the whole distance at full speed. Hence the time lost in starting and stopping is $\frac{V}{2}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)$.

Using the above data reduced to feet and seconds, we have

$$T = \frac{88}{2}\left(\frac{1}{2} + 1\right) + \frac{30 \times 5280}{88} = 66 + 30 \times 60 \text{ sec.} = 31 \text{ min., } 6 \text{ sec.}$$

The time lost in starting and stopping is 66 sec.

The plan, adopted in the above examples, of solving a problem in general terms and then substituting the numerical data in the formulas deduced, is usually advantageous when the desired quantities cannot be computed at once, for this method puts into evidence the structure of the final results and often facilitates the numerical calculation. Moreover the method allows of a valuable check in the correctness of the reasoning; the nature of this check will be explained later on.

PROBLEMS

1. A "puck," used in hockey, is struck with a velocity of 40 ft./sec. If it comes to rest in 1200 ft., what was its retardation (supposed uniform) caused by friction with the ice? How long was the puck in motion?

2. A train, starting from rest and uniformly accelerated, acquires a velocity of 30 mi./hr. in 5 minutes. What is its acceleration? What distance does it cover in this time?

3. A train, starting from rest, travels a distance d_1 with the acceleration a_1, then comes to rest in a further distance d_2 under the retardation a_2. Given the total distance $d = d_1 + d_2$, find (1) its greatest velocity; (2) the distance d_1; (3) the time of passage. Show that the time is the same as if the distance $2\,d$ had been traversed with the maximum velocity.

Obtain the numerical results when $a_1 = \frac{1}{4}$ ft./sec.^{2}, $a_2 = \frac{1}{8}$ ft./sec.^{2}, $d = 2$ mi.

4. If a uniformly accelerated body passes over the distances d_1, and d_2 in two consecutive intervals T_1 and T_2, show that its acceleration is

$$a = \frac{2\,(d_2T_1 - d_1T_2)}{T_1T_2\,(T_1 + T_2)}.$$

5. Two particles begin moving in the same straight line at the same time with initial velocities of 1 and 3 ft./sec., and accelerations of 2 and 1 ft./sec.^{2}, respectively. If, at the outset, the first is 1.5 ft. ahead of the second, when will they meet?

6. A uniformly accelerated body moves 55 ft. in 2 sec., then 77 ft. in the next 2 sec. Find its initial velocity and acceleration. How far will the body move in the next 4 sec.?

117. Uniformly Accelerated Rotation. The methods of § 116 apply also to the case of uniformly accelerated rotation about a fixed axis. Corresponding to

$$x, \quad v = \frac{dx}{dt}, \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}, \quad a = v\frac{dv}{dx},$$

we now have (§ 108)

$$\theta, \quad \omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}, \quad \alpha = \omega\frac{d\omega}{d\theta}.$$

When α is constant and

$$\theta = \theta_0, \quad \omega = \omega_0 \quad \text{when} \quad t = 0,$$

we may deduce the exact analogues of (1), (2), (3), (4) of § 116 in which θ, ω, α take the place of x, v, a:

(1) $$\omega = \omega_0 + \alpha t \qquad \text{or} \quad \Delta\omega = \alpha\,\Delta t,$$

(2) $$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\,\alpha\,t^2,$$

(3) $$\omega^2 - \omega_0^2 = 2\,\alpha(\theta - \theta_0) \qquad \text{or} \quad \Delta(\omega^2) = 2\,\alpha\,\Delta\theta,$$

(4) $$\Delta\theta = \frac{\omega_0 + \omega}{2}\,\Delta t.$$

Example. A flywheel making 180 r.p.m. is uniformly retarded so that it comes to rest in 30 revolutions. Find α and the time of retardation (§ 108, Example).

With the revolution and minute as units, we have from (3) and (4):

$$\alpha = \frac{0 - 180^2}{2 \times 30} = -540 \text{ rev./min.}^2,$$

$$30 = \frac{180 + 0}{2}\,\Delta t, \quad \Delta t = \tfrac{1}{3} \text{ min}$$

118. The Acceleration of Gravity. Experiment has shown that in any definite locality all bodies fall, in vacuo, with the same acceleration, g. The exact value of g varies, however, with the locality, increasing with the latitude, and decreasing slightly

with the height above sea-level. For most problems it is sufficiently accurate to take

$$g = 32 \text{ ft./sec.}^2 \quad \text{or} \quad 980 \text{ cm./sec.}^2.$$

From the most recent investigations of the U. S. Coast and Geodetic Survey, the best general formula * for the value of g at sea-level and latitude ψ is

(1) $g = 978.039\,(1 + 0.005294 \sin^2\psi + 0.000007 \sin^2 2\psi)$ cm./sec.^{2}.

To obtain g in ft./sec.^{2} the first constant in (1) must be replaced by 32.08783. The *form* of this formula is based upon theory,† while the constants it contains have been derived from numerous determinations of g in different parts of the world.

The variation of the sea-level values of g in the United States, as computed from formula (1), is shown in the following table:

Latitude	g (cm./sec.^{2})	g (ft./sec.^{2})
25° (Key West)	978.960	32.1180
30° (New Orleans)	979.329	32.1302
35° (Chattanooga)	979.737	32.1435
40° (Philadelphia)	980.171	32.1578
45° (Minneapolis)	980.621	32.1735
50° (N.W. Boundary, 49°)	981.071	32.1873

The simplest correction for the effect of elevation is made upon the assumption that the locality is in the air above a sea-level earth. This "free air" correction is

(2) $-0.0003086\, h_m$ cm./sec.^{2}, or $-0.000003086\, h_f$ ft./sec.^{2},

where h_m denotes the height above sea-level in meters, h_f in feet. The correction thus amounts to about $-$ 0.003 ft./sec.^{2} for each 1000 feet.

For extreme accuracy other corrections are applied to the values of g computed as above. The nature of these cannot be considered here; it must suffice to say that they take account of the topography circumjacent to the locality in question, its position relative to elevated land masses or sea depressions, and the

* Special Publication No. 40, Wm. Bowie, p. 134 (1917).

† In this theory the earth is regarded as an ellipsoid of revolution which nearly coincides with the sea-level surface of the earth, and whose density at any point depends only upon the distance from the center.

density of the underlying strata. Even after all rational corrections have been applied there is still a slight discrepancy between the observed and computed values of g. These discrepancies, or *anomalies*, are plotted upon maps in the publications of the U. S. Coast and Geodetic Survey.

Any locality in which the falling acceleration, in vacuo, is

$$(3) \qquad g_0 = 980.665 \text{ cm./sec.}^2 = 32.1740 \quad \text{ft./sec.}^2$$

is called a *standard locality;* and g_0 is called the *standard value of* g. When this value was adopted as a standard of reference by the International Committee of Weights and Measures in 1901, it was regarded as the average value of g at sea-level, latitude 45 degrees. But more recent measurements, as incorporated in formula (1), give the value for 45 degrees listed in the table above. Nevertheless the arbitrary value of g_0 given in (3) will remain the standard until altered by international agreement.

The equations of § 116, with $a = \pm g$, apply to the free vertical motion of bodies, near the surface of the earth, under the influence of gravity, when the resistance of the air is neglected. Note that $a = g$ or $-g$ according as the positive direction is taken downward or upward.

From (§ 116, 3) we see that a body falling from rest through a height h acquires a velocity

$$v = \sqrt{2\,gh}.$$

This is sometimes called the *velocity due to the height* (or *head*) h.

The form of equation (§ 116, 3) shows that a body thrown vertically upward passes a certain position with the same numerical velocity, whether ascending or descending.

Example 1. A stone was thrown vertically upward from the roof of a building 80 ft. high so that it cleared the building and fell to the ground in 5 seconds. What was its initial velocity? When did it pass the point of projection?

Taking the origin at the ground and the positive direction upward, we have from (§ 116, 2)

$$x = 80 + v_0 t - 16\,t^2.$$

Since $x = 0$ when $t = 5$, this gives

$$v_0 = \frac{16 \times 25 - 80}{5} = 64 \text{ ft./sec.}$$

Again, when $x = 80$,

$$64\,t - 16\,t^2 = 0, \qquad t = 0 \text{ or } 4;$$

hence the stone passed the point of projection 4 seconds after the throw.

Example 2. Two particles fall from rest from the same point, the second beginning its motion after the first has fallen a distance δ. How far will they be apart after the first has fallen through a height h?

The first particle falls through the distances δ and h in the times $\sqrt{2\,\delta/g}$, $\sqrt{2\,h/g}$ respectively. Hence when the first has fallen through the height h, the second has traversed the distance

$$h' = \tfrac{1}{2}\,g\left[\sqrt{\frac{2\,h}{g}} - \sqrt{\frac{2\,\delta}{g}}\right]^2 = h - 2\sqrt{h\delta} + \delta,$$

and is separated from the first by an amount

$$h - h' = 2\sqrt{h\delta} - \delta.$$

It is noteworthy that the amount of this separation is entirely independent of the magnitude of the acceleration.

As a numerical example, consider two particles of water, 0.001 inch from each other, passing over the brink of Yosemite Falls. At the base of the upper cataract, which is 1400 feet high, they will be separated by

$$2\sqrt{1400 \times 12 \times .001} - .001 = 8.2 \text{ in.}$$

This effect of gravity in changing the solid sheet of a high waterfall into a fine spray near its base is well known. Actually, the above result would be considerably modified by the retarding effect of the air.

Example 3. A ball, thrown vertically upward, passes a certain window of an office building after t_1 seconds, and repasses after t_2 seconds. How high is the window, and what was the initial velocity of the ball?

Let h be the height of the window above the point of projection. Taking this point as origin and the positive direction upward, we know that t_1 and t_2 are roots of the equation

$$h = v_0t - \tfrac{1}{2}\,gt^2, \quad \text{or} \quad t^2 - \frac{2\,v_0}{g}\,t + \frac{2\,h}{g} = 0.$$

The relations between the roots and coefficients of a quadratic equation require that

$$t_1 + t_2 = \frac{2\,v_0}{g}, \qquad t_1t_2 = \frac{2\,h}{g};$$

hence

$$v_0 = \tfrac{1}{2}\,g\,(t_1 + t_2), \qquad h = \tfrac{1}{2}\,gt_1t_2.$$

PROBLEMS

1. A stone, thrown vertically upward, returned to the ground in 5 seconds. How high did it go, and what was its initial velocity?

2. A stone dropped from a balloon rising at the rate of 24 ft./sec. hits the ground in 10.5 seconds. How high was the balloon when the stone was dropped? What was the striking velocity of the stone?

3. A bullet is shot upward with a muzzle velocity of 1600 ft./sec. How long does it stay in the air? To what height will it rise? With what velocity will it strike the ground?

4. If the hammer of a pile driver drops 10 ft. upon a pile, what is its striking velocity?

5. Neglecting the resistance of the air, with what velocity must a bullet be shot upward to go one mile high?

6. If a body is dropped from rest find the space traversed during the nth second of the motion.

7. A stone dropped over a cliff is heard to strike after 6.5 seconds. How high is the cliff? (Take the velocity of sound as 1120 ft./sec.)

8. A body is dropped to the ground from a height of 80 ft., and at the same time a second body is thrown upward with an initial velocity of 40 ft./sec. When and where will they pass?

9. A stone is thrown upward with a velocity v_0, and T seconds later a second stone is thrown upward with the same initial velocity. At what time after the projection of the second stone will they meet?

10. Two balls are dropped from a tower 100 ft. high, the second $\frac{1}{2}$ sec. after the first. What distance will the first ball gain on the second during its descent?

119. Simple Harmonic Motion. If a particle P moves in a straight line with an acceleration always directed toward a fixed point O of the line and directly proportional to its distance from O, the particle is said to have *simple harmonic motion* (*s.h.m.*).

To put this definition in analytic form we choose O as the origin of abscissas; then

$$\mathbf{a} = k\overrightarrow{PO} = -k\overrightarrow{OP} = -kx\mathbf{i},$$

and

$$a = -kx,$$

where k is a positive constant, namely, the numerical value of the acceleration at unit distance from O. The subsequent notation is somewhat simplified by writing $n = \sqrt{k}$. The equation of motion then becomes

$$a = \frac{d^2x}{dt^2} = -n^2x. \tag{1}$$

This equation is characteristic for simple harmonic motion.

Since the acceleration is a function of x alone we may apply the second method of § 115. We prefer, however, the following treatment in order to avoid separate consideration of the cases $v > 0$, $v < 0$.

Equation (1) states that x must be such a function of t that its second derivative is the same function multiplied by the negative constant $-n^2$. Both $\cos nt$ and $\sin nt$ have precisely this property; the same is true of the more general functions.

$$A \cos nt + B \sin nt, \quad A \cos (nt + B), \quad A \sin (nt + B)$$

involving two arbitrary constants A, B. Any simple harmonic motion may be represented by equating x to one of these functions and then satisfying the initial conditions by a suitable choice of A and B.

Thus let us write

$$x = h \cos (nt + \epsilon) \tag{2}$$

where h and ϵ are now the arbitrary constants; the velocity is then

$$v = \frac{dx}{dt} = -nh \sin (nt + \epsilon). \tag{3}$$

In order that this simple harmonic motion satisfy the initial conditions

$$x = x_0, \quad v = v_0, \quad \text{when } t = 0$$

we must have

$$x_0 = h \cos \epsilon, \quad v_0 = -nh \sin \epsilon. \tag{4}$$

On solving these equations for h and ϵ we find

$$h = \sqrt{x_0^2 + \frac{v_0^2}{n^2}}, \quad \tan \epsilon = -\frac{v_0}{nx_0}.$$

We have chosen the positive value for h; ϵ must now be chosen so that $\cos \epsilon$ and $\sin \epsilon$ have the signs required by (4). With these values of h and ϵ, (2) represents the simple harmonic motion that satisfies the prescribed initial conditions. The extreme values of x are clearly $\pm h$; hence h is called the *amplitude* of the simple harmonic motion. The velocity is given by (3) as a function of t. The relation between v and x is obtained by eliminating t from (2) and (3):

$$v^2 = n^2(h^2 - x^2). \tag{5}$$

A clear idea of the character of simple harmonic motion may be obtained by regarding $OP = x = h \cos (nt + \epsilon)$ as the x-component of a vector $\overrightarrow{OQ}$ of length h revolving about O. As OQ makes

the angle $\theta = nt + \epsilon$ with the x-axis, it revolves with the constant angular velocity $d\theta/dt = n$. While Q is revolving uniformly in a circle, its projection P on the x-axis oscillates to and fro between A and B. These oscillations form a simple harmonic motion that satisfies (1). In brief:

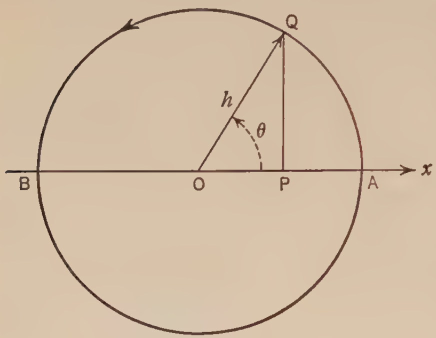


FIG. 119*a*.

The projection of uniform circular motion upon a diameter is simple harmonic.

The simple harmonic motion consists of repeated oscillations about O of constant amplitude h. The time T of a complete oscillation (to and fro) is called the *period* of the motion; since this is the time in which Q makes a complete revolution,

$$T = \frac{2\pi}{n}. \tag{6}$$

At the end of any interval of T seconds both x and v resume the same values they had at the beginning of the interval. The period is the same throughout the motion and moreover is entirely independent of the initial conditions. The *frequency*, or number of complete oscillations per second, is $1/T$.

The angle $\theta = nt + \epsilon$ between OA and OQ is called the *phase* of the motion at the instant t. Since $\theta = \epsilon$ when $t = 0$, ϵ is called the *initial phase*. If Q is at Q_0 when $t = 0$, ϵ is the angle AOQ_0.

Since from (3)

$$v = -nOQ \sin\theta = -nPQ,$$

we see that the velocity is proportional to the ordinate PQ.

Example 1. A particle in simple harmonic motion has an acceleration of -4 ft./sec. when at a distance of 1 ft. from the center. Find the period. If $x = 4$ ft., $v = -6$ ft./sec. when $t = 0$, find the amplitude of the motion and its initial phase.

Since $a = -n^2x$ we have $-4 = -n^2$; hence

$$n = 2 \quad \text{and} \quad T = \frac{2\pi}{n} = \pi \text{ sec.}$$

With $x_0 = 4$, $v_0 = -6$, $n = 2$, equations (4) become

$$4 = h \cos \epsilon, \quad 3 = h \sin \epsilon;$$

on solving these equations for h (positive) and ϵ we find

$$h = 5 \text{ ft.}; \quad \tan \epsilon = 0.75, \quad \epsilon = 36° 52'.$$

The angle ϵ lies in the first quadrant as its sine and cosine are both positive.

Example 2. In Example 2, § 112, let us consider the motion of the piston when the connecting-rod is long in comparison with the crank, that is, for large values of $n = l/r$. It is clear from the figure that the motion of A will be nearly the same as that of B' when this ratio is large. Now when the crank OB revolves uniformly, the motion of B' (the projection of the crank-pin B) is a simple harmonic motion. We conclude therefore that the motion of A, and hence that of the piston, is nearly a simple harmonic motion when n is large and the angular velocity ω of the crank is constant.

Thus conclusion also follows from (§ 112, 7). For when n is large the last term in braces is small and the acceleration of A is approximately

$$a = -r\omega^2 \cos \phi = -\omega^2 \, OB';$$

this is precisely the acceleration of B' in a simple harmonic motion about O.

PROBLEMS

1. In the mechanism shown in Fig. 119*b*, the part *a* is fixed; when the crank *b* revolves, the block *c* moves up and down in the slot of the sliding bar *d*, which is given a reciprocating motion. Show that when the angular velocity ω of the crank is constant, the motion of *d* is simple harmonic.

2. If the crank *b* is 6 in. long and makes 45 r.p.m., what is the maximum velocity and acceleration of the sliding bar *d*?

3. A particle in simple harmonic motion makes 60 complete oscillations per minute. If the amplitude is $\frac{1}{2}$ ft., find v and a when $x = \frac{1}{4}$ ft.

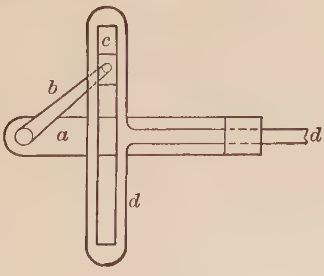


Fig. 119*b*.

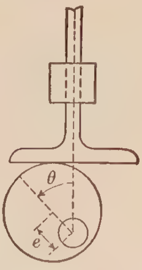


Fig. 119*c*.

4. A cam in the form of a cylindrical disk keyed to an excentric shaft drives a "lifting toe" on its upward stroke (Fig. 119*c*). The weight of the toe keeps it in contact with the cam on its downward stroke. If the

shaft revolves with constant angular velocity ω, show that the toe has simple harmonic motion.

5. If the cam in Problem 4 has a throw $e = 1$ in. and makes 40 r.p.m. what is the velocity of the toe when $\theta = 45°$?

6. What is the velocity-space curve in simple harmonic motion? What property must the subnormal of this curve possess?

7. Prove that if a particle is given a constant acceleration a_0 in addition to the acceleration $-n^2x$, it will oscillate in simple harmonic motion about the new central position $x = a_0/n^2$ with the same period as before.

120. Summary, Chapter IX. In rectilinear motion we choose an origin on the line and write $\mathbf{r} = x\mathbf{i}$. The velocity and acceleration are then $\mathbf{v} = v\mathbf{i}$, $\mathbf{a} = a\mathbf{i}$ where

$$v = \frac{dx}{dt}, \quad a = \frac{dv}{dt}; \quad \text{moreover} \quad a = v\frac{dv}{dx}.$$

For brevity the scalars v and a are also called velocity and acceleration.

The motion may be represented graphically in various ways. In the v-t curve (abscissa t) the slope at any point represents a; and the area between the curve, the t-axis and two ordinates gives the space passed over in this interval. In the v-x curve (abscissa x) the subnormal at any point represents a.

When the acceleration is given as a function of t alone, of x alone, or of v alone, we may find the relations between v-t, x-t, and v-x by integrating the differential equation

$$\frac{d^2x}{dt^2} = a, \quad \left(\frac{dv}{dt} = a, \quad v\frac{dv}{dx} = a\right).$$

The initial conditions determine the constants of integration.

When a is *constant* and $x = x_0$, $v = v_0$ when $t = 0$, we integrate $d^2x/dt^2 = a$ twice; thus

$$v = v_0 + at \qquad \text{or} \quad \Delta v = a\,\Delta t,$$
$$x = x_0 + v_0t + \tfrac{1}{2}\,at^2 \quad \text{or} \quad \Delta x = \frac{v_0 + v}{2}\,\Delta t;$$

and from $v\,dv = a\,dx$

$$v^2 - v_0^2 = 2\,a(x - x_0) \quad \text{or} \quad \Delta(v^2) = 2\,a\,\Delta x.$$

These equations apply to the free vertical motion of bodies under gravity near the surface of the earth. Then $a = g$ (approxi-

mately 32 ft./sec.^{2}) or $-g$ according as the positive direction is downward or upward.

If x and a are replaced by s and a_t in the preceding equations, they apply to curvilinear motion in which the speed is changing uniformly (a_t constant). Again, if x, v, a are replaced by θ, ω, α, the equations apply to uniformly accelerated rotation about a fixed axis (α constant).

If a particle P moves in a straight line Ox with an acceleration always directed toward O and proportional to the distance OP, P is said to have a *simple harmonic motion*. The defining equation for simple harmonic motion is therefore

$$\frac{d^2x}{dt^2} = -n^2x \quad \text{where} \quad n^2 \quad \text{is a positive constant.}$$

The integral of this differential equation is

$$x = h \cos (nt + \epsilon)$$

where h and ϵ are constants to be determined from the initial conditions. The motion consists of oscillations of *amplitude* h (the extreme displacement from O) and of *period* $2\pi/n$ (the time of an oscillation to and fro). The simple harmonic motion may be regarded as the projection of uniform circular motion on a diameter; the circle is of radius h about O and the angular speed is n rad./sec.

CHAPTER X

KINEMATICS OF PLANE MOTION

121. Plane Motion of a Rigid Body. A body is said to have *plane motion* when the velocity vectors of all of its points remain parallel to a fixed plane a during the motion. Such motion is evidently determined by the motion of any cross-section of the body parallel to a. Hence, instead of dealing directly with the body, we may study the motion of such a cross-section.

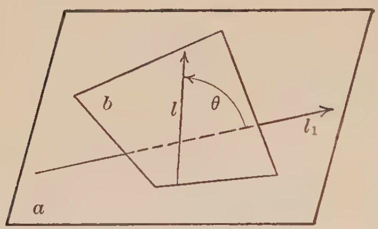


FIG. 121a.

Consider, then, a plane figure b moving in a fixed plane a (Fig. 121a); and let l_1 and l denote two directed lines, the first in the plane a, the second in the figure b and moving with it. If we denote the angle (l_1, l) by θ, the scalar angular velocity and acceleration of the figure are defined by

$$(1) \qquad \omega = \frac{d\theta}{dt}, \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2},$$

respectively. The values of ω and α are independent of the choice of l_1 and l. For if k_1, k are another pair of reference lines (not shown in the figure), the angle ϕ from k_1 to k is given by

$$\phi = (k_1, k) = (k_1, l_1) + (l_1, l) + (l, k) = \theta + \text{constant},$$

since (k_1, l_1) and (l, k) do not vary with the time. The time derivatives of ϕ and θ are therefore identical.

In the definitions (1) we suppose that the positive sense of θ has been chosen. Let $\mathbf{k}$ be a unit vector perpendicular to the plane a and pointing in the direction of advance of a right-hand screw having a positive rotation. Then the vector angular velocity and acceleration are defined by

$$(2) \qquad \boldsymbol{\omega} = \omega\mathbf{k}, \quad \boldsymbol{\alpha} = \alpha\mathbf{k}.$$

In Fig. 121a, $\mathbf{k}$ points up from the plane of the paper.

Example 1. At a certain instant the speed of a wheel of radius r, rolling over a straight track without slipping, is v ft./sec. Suppose that the wheel moves forward a distance x while a certain radius turns through an angle θ; then the condition for no slipping is that $x = r\theta$, θ being measured in radians. The angular velocity of the wheel is therefore

$$\omega = \frac{d\theta}{dt} = \frac{1}{r}\frac{dx}{dt} = \frac{v}{r} \text{ rad./sec.}$$

Example 2. In Fig. 121*b*, a and b represent two cylinders, of radii r_a, r_b, held in contact by the arm c joining their shafts. If a is held fast and the arm revolves with an angular velocity ω_c, what is the angular velocity of b if it rolls upon a without slipping?

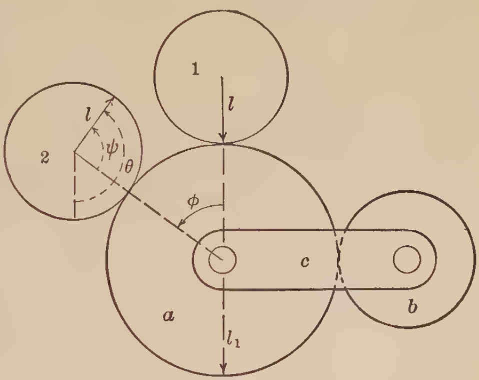


FIG. 121*b*.

Let us choose l_1 in a, l in b, as lines of reference. Then as b rolls from the position 1 to the position 2, the arm turning through an angle ϕ, we see from the figure that l turns through an angle $\theta = \phi + \psi$. Since there is no slipping the arcs rolled over on a and b must be equal; hence

$$r_a\phi = r_b\psi, \qquad \text{and} \qquad \theta = \phi + \frac{r_a}{r_b}\phi = \frac{r_a + r_b}{r_b}\phi.$$

The angular velocity of b is therefore

$$\omega_b = \frac{d\theta}{dt} = \frac{r_a + r_b}{r_b}\frac{d\phi}{dt} = \frac{r_a + r_b}{r_b}\omega_c.$$

For example, if the cylinders have equal radii, $\omega_b = 2\,\omega_c$; the angular velocity of the rolling cylinder is then twice that of the arm, at first sight a rather paradoxical result. The reason is apparent when we reflect that in a complete revolution of the arm c, b makes one revolution by virtue of its attachment to the arm, and another in its motion relative to the arm.

PROBLEMS

1. Two parallel shafts are centered 3 ft. 9 in. apart. If the driving shaft makes 80 r.p.m., find the diameters of wheels to work by rolling contact so that the following shaft will make 100 r.p.m.

Solve this problem in general terms.

2. Six spur gears a, b, c, d, e, f, mounted upon parallel shafts, engage as shown in Fig. 121c. If $n_a, n_b, \ldots, n_f$ denote the numbers of their teeth, find the angular velocity ratios ω_b/ω_a, ω_d/ω_a, ω_f/ω_a.

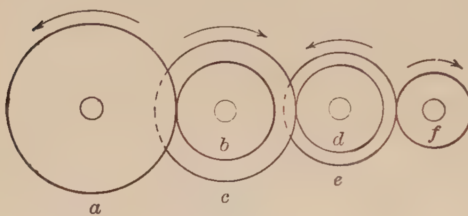


FIG. 121c.

If $a, b, \ldots, j$ are any even number of spur gears arranged as above, show that $|\omega_j/\omega_a|$ (called the *value of the train*) is equal to the product of the numbers of teeth in all the drivers divided by the product of the numbers of the teeth in all the followers. How must this result be interpreted if on some of the intermediate shafts there is but a single gear (instead of two) that serves both as driver and follower? What effect have such single gears on the value of ω_j/ω_a?

122. Translation and Rotation. We have seen that the plane motion of a body is determined by the motion of a cross-section b in a fixed plane a. If any line of b, such as l of Fig. 121a, maintains a constant direction, the motion is called a *translation.* Then the angle $\theta = (l_1, l)$ is constant and $\omega = 0$. Moreover all points of b have the same velocity at any instant. For if P, Q are points of b and O_1 an origin fixed in space,

$$\overrightarrow{PQ} = \overrightarrow{O_1Q} - \overrightarrow{O_1P} \text{ is constant during the motion;}$$

hence on differentiating with respect to the time,

$$0 = \mathbf{v}_Q - \mathbf{v}_P \quad \text{or} \quad \mathbf{v}_P = \mathbf{v}_Q,$$

where $\mathbf{v}_P$ denotes the velocity of P. In a translation all points of the moving figure describe congruent paths. According as these paths are straight or curved the translation is *rectilinear* or *curvilinear.*

Rotation about a fixed axis is a case of plane motion, the plane a being perpendicular to the axis. Then the angular velocity and acceleration as defined in § 121 are the same as in § 108. If O is a point on the axis, the velocity of any point P of the body is given by

$$\mathbf{v}_P = \boldsymbol{\omega} \times \overrightarrow{OP} \qquad (\S\ 109, 4).$$

If at any instant the *velocities* of the points of a rigid body are the same as in a translation or rotation about a fixed axis, the body is said to have an *instantaneous translation* or an *instantaneous rotation* respectively. This does not mean that the real motion is a translation or a rotation. Indeed the *accelerations* of the body in the instantaneous motion may be entirely different from those in the corresponding real motion.

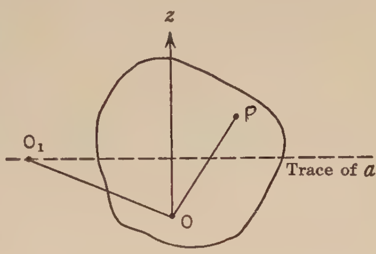


FIG. 123.

123. Fundamental Kinematic Equations. Let O and P be any two points of a rigid body moving parallel to fixed plane a. To find the time rate at which $\overrightarrow{OP}$ is changing we may imagine the point O held fast (§ 83) and the body revolving about an axis Oz normal to a with its angular velocity $\boldsymbol{\omega}$. Then since

$$\frac{d}{dt}\overrightarrow{OP} = \text{Velocity of } P \text{ when } O \text{ is fixed (§ 83),}$$

we have from (§ 109, 4)

$$\text{(1)} \qquad \frac{d}{dt}\overrightarrow{OP} = \boldsymbol{\omega}\times\overrightarrow{OP}.$$

This equation is valid whether or not O and P lie in the same plane of motion.

If O_1 is an origin fixed in space,

$$\overrightarrow{O_1P} = \overrightarrow{O_1O} + \overrightarrow{OP}.$$

On differentiating this equation with respect to the time t and making use of (1) we find

$$\text{(2)} \qquad \mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega}\times\overrightarrow{OP}.$$

This gives the velocity distribution in the body.

Now differentiate (2) with respect to t, again making use of (1); then

$$\text{(3)} \qquad \mathbf{a}_P = \mathbf{a}_O + \boldsymbol{\alpha}\times\overrightarrow{OP} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\overrightarrow{OP}).$$

This gives the acceleration distribution in the body.

We proceed to interpret these equations.

124. Velocities in Plane Motion. In the equation

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega}\times\overrightarrow{OP} \tag{1}$$

the term $\boldsymbol{\omega}\times\overrightarrow{OP}$ gives the velocities due to an instantaneous rotation $\boldsymbol{\omega}$ about a normal axis through O. The content of (1) is therefore stated in the

THEOREM I. *If O is any point of a rigid body in plane motion with the angular velocity $\boldsymbol{\omega}$, the velocities of its points may be compounded of an instantaneous translation $\mathbf{v}_O$ and an instantaneous rotation $\boldsymbol{\omega}$ about an axis through O.*

Now let P and O be any two points *in the same plane of motion.* If $\mathbf{v}_P = \mathbf{v}_O$, then $\boldsymbol{\omega}\times\overrightarrow{OP} = 0$, and since ω is perpendicular to OP, $\omega = 0$. Hence the motion is an instantaneous translation of velocity $\mathbf{v}_O$.

If $\omega \neq 0$, there is always a single point I in this plane, called the *instantaneous center*, whose velocity is zero. Assuming for the moment that I exists, let P in (1) coincide with I; then since $\mathbf{v}_I = 0$,

$$0 = \mathbf{v}_O + \boldsymbol{\omega}\times\overrightarrow{OI}.$$

To obtain $\overrightarrow{OI}$ multiply this equation by $\mathbf{k}\times$; then since $\mathbf{k}\cdot\overrightarrow{OI} = 0$ and $\mathbf{k}\cdot\boldsymbol{\omega} = \omega$, we have from (§ 19, 2)

$$0 = \mathbf{k}\times\mathbf{v}_O - \omega\overrightarrow{OI}, \quad \text{or}$$

$$\overrightarrow{OI} = \frac{\mathbf{k}\times\mathbf{v}_O}{\omega}. \tag{2}$$

We may now verify that $\mathbf{v}_I = 0$ for the point thus determined; for from (1)

$$\mathbf{v}_I = \mathbf{v}_O + \frac{\boldsymbol{\omega}\times(\mathbf{k}\times\mathbf{v}_O)}{\omega} = \mathbf{v}_O - \mathbf{v}_O = 0 \qquad (§ 19, 2).$$

For a translation $\omega = 0$ and I does not exist. However for the sake of uniformity we shall say that I in this case is at infinity in a direction perpendicular to $\mathbf{v}_O$.

If we choose O at I, (1) becomes

$$\mathbf{v}_P = \boldsymbol{\omega}\times\overrightarrow{IP}. \tag{3}$$

The velocities of the body are therefore the same as in an instantaneous rotation about an axis through I normal to the plane a.

In view of the above results we now state

THEOREM II. *At every instant the motion of a rigid body in plane motion is either an instantaneous translation or an instantaneous rotation.*

Example. Rolling Wheel. In Fig. 124 the wheel, of radius r, is rolling to the left with the angular velocity ω. The speed of its center is then $r\omega$ (§ 121, Example 1) and $\mathbf{v}_O = -r\omega\mathbf{i}$. The velocity of any point P of the wheel is now given by (1). Thus $\mathbf{v}_P$ is the sum of $\mathbf{v}_O$ and the velocity of P in a rotation $\boldsymbol{\omega}$ about O as a fixed point. For example

$$\mathbf{v}_A = -r\omega\mathbf{i} - r\omega\mathbf{i} = 2\,\mathbf{v}_O,$$
$$\mathbf{v}_B = -r\omega\mathbf{i} + r\omega\mathbf{j} = r\omega(\mathbf{j} - \mathbf{i}),$$
$$\mathbf{v}_C = -r\omega\mathbf{i} + r\omega\mathbf{i} = 0.$$

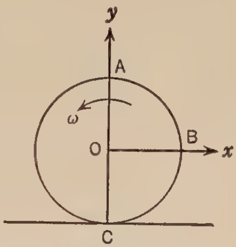


FIG. 124.

The last equation shows that the instantaneous center of the wheel is at its point of contact with the track. Of course we might have found $\overrightarrow{OI}$ at once from (2):

$$\overrightarrow{OI} = \frac{\mathbf{k}\times(-r\omega\mathbf{i})}{\omega} = -r\mathbf{j} = \overrightarrow{OC}.$$

125. Instantaneous Center. If I is the instantaneous center in the plane of motion of the point P of the body,

$$\mathbf{v}_P = \boldsymbol{\omega}\times\overrightarrow{IP};$$

hence

$$(1) \qquad \mathbf{v}_P \perp IP, \qquad v_P = \omega\cdot IP.$$

If Q is any other point of b,

$$(2) \qquad v_P : v_Q = IP : IQ.$$

The speeds of two points of b are proportional to their distances from the instantaneous center.

From (1) we see that a line perpendicular to $\mathbf{v}_P$ at P passes through I. Similarly a line perpendicular to $\mathbf{v}_Q$ at Q passes through I. We therefore have the following simple construction for I:

If a figure in plane motion has two points A, B moving in different directions, its instantaneous center is at the point of intersection of the normals to these directions at A and B (Fig. 125*a*).

Example 1. A ladder AB rests on a horizontal floor at A and against a vertical wall at B (Fig. 125*b*). If it begins to slip down, $\mathbf{v}_A$ is horizontal and $\mathbf{v}_B$ vertical; its instantaneous center I is then at the intersection of the normals to these velocities at A and B. The speeds of A and B are in the ratio $IA:IB$.

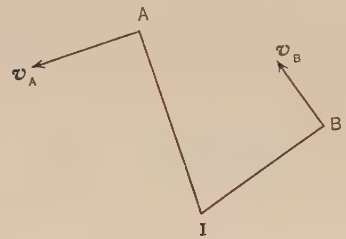


Fig. 125*a*.

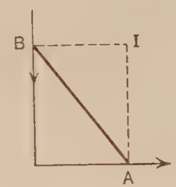


Fig. 125*b*.

Example 2. *Connecting-rod.* Let AB represent the connecting-rod of a direct-acting engine, the end A moving with the cross-head in the straight line O_1A, while the end B revolves with the crank O_1B about O_1 (Fig. 125*c*). Then $\mathbf{v}_A$ is along the line O_1A and $\mathbf{v}_B$ is perpendicular to the crank O_1B. The instantaneous center of the connecting-rod is therefore at the intersection of the normals to these velocities at A and B.

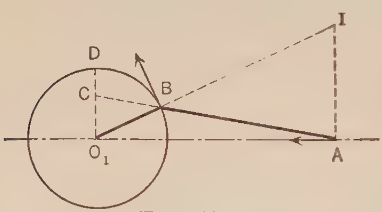


Fig. 125*c*.

The speeds of the cross-head and crank-pin are in the ratio

$$v_A:v_B = IA:IB = O_1C:O_1B = O_1C:O_1D.$$

Thus if O_1D represents the speed of the crank-pin, O_1C represents the speed of the cross-head to the same scale.

126. Centrodes. As a plane figure moves in its plane the instantaneous center describes a certain curve Γ_1 in space, called the *space centrode*, and a curve Γ in the figure itself, called the *body centrode*. Otherwise expressed, the space and body centrodes are the loci of I in the fixed and moving planes respectively.

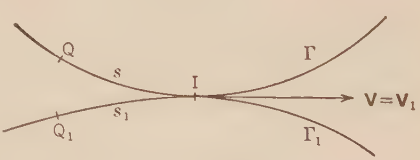


Fig. 126*a*.

Now regard I as a point describing the curves Γ_1 and Γ with

the velocities $\mathbf{V}_1$ and $\mathbf{V}$ respectively. Then $\mathbf{V}_1$ is the absolute velocity of I and $\mathbf{V}$ is the velocity of I relative to the moving figure. The velocity $\mathbf{v}_I$ of that point of the body with which I coincides for the instant is zero. Now, from § 110, the absolute velocity of I is equal to the sum of its body velocity and its relative velocity; hence

$$\mathbf{V}_1 = \mathbf{v}_I + \mathbf{V} = \mathbf{V}.$$

The instantaneous center therefore describes both centrodes with the same velocity.

From this property we can show that *the motion of the plane figure is reproduced by rolling the body centrode* (fixed in the figure) *over the space centrode.* First, since the velocity vector is always tangent to the path, Γ_1 and Γ have a common tangent at I, and are themselves tangent at this point. Moreover if the points Q_1, Q of the centrodes coincide when $t = 0$, and s_1, s denote the arcs Q_1I and QI described by I in the time t, the relation $V = V_1$ requires that

$$\frac{ds}{dt} = \frac{ds_1}{dt}, \qquad \text{and hence} \qquad s = s_1 + C,$$

where C is a constant of integration. But as $s = s_1 = 0$ when $t = 0$, $C = 0$ and $s = s_1$; that is, I describes equal arcs of the centrodes during any interval. The motion therefore entails a rolling, without slipping, of Γ over Γ_1.

Conversely, if a curve Γ on a figure in plane motion rolls without slipping over a fixed curve Γ_1, the point of contact I of the curves is the instantaneous center. For, with the same notation as above, the condition of pure rolling requires that

$$s = s_1 \qquad \text{and hence} \qquad \frac{ds}{dt} = \frac{ds_1}{dt};$$

and since the curves have a common tangent at I, $\mathbf{V} = \mathbf{V}_1$. But since $\mathbf{V}_1 = \mathbf{v}_I + \mathbf{V}$, $\mathbf{v}_I$ must be zero and I is the instantaneous center.

When one plane curve rolls without slipping upon another, the normals to the paths of all points in the moving plane pass through I, the point of contact. This is a general property of all *roulettes* — curves traced by points invariably connected to one curve rolling upon another. For example, when a circle rolls on a straight line, any point on its circumference describes

a *cycloid;* the normal to a cycloid at any point therefore passes through the corresponding point of contact of the circle and line.

Example. The Trammel. A link AB is pivoted at A and B to two blocks that slide in straight grooves perpendicular to each other (Fig. 126*b*). The instantaneous center I of the link is at the intersection of the normals to the grooves at A and B. If the center lines of the grooves meet at O, the distance $OI = AB$, and is constant; hence the space centrode is a circle with center at O and of radius AB. Again, since AIB is a right angle in all positions of AB, the locus of I relative to the link is a circle on AB as diameter. This circle is the body centrode. The motion of the link may therefore be reproduced by rolling the smaller circle, with the link rigidly attached, within the larger circle.

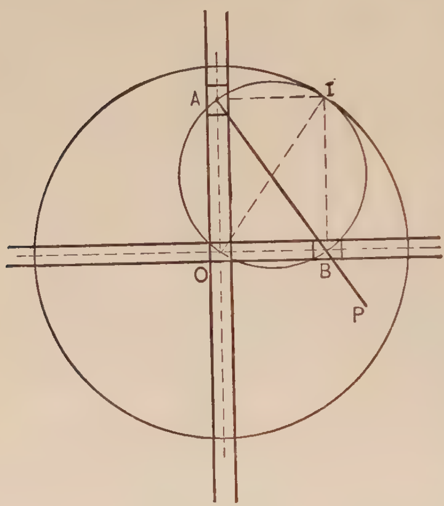


FIG. 126*b*.

It is well known that any point P of the link describes an ellipse whose semiaxes are of length PA, PB respectively. Hence when a circle rolls within a circle of twice its size, a point P on the diameter of the rolling circle describes an ellipse. In particular, if P coincides with A or B, one semiaxis is zero, and the ellipse degenerates into a diameter of the large circle, described twice.

PROBLEMS

1. If the grooves of the trammel are not at right angles, what are the two centrodes?

2. A circular disk is suspended by two strings attached to points on its rim. When one string is cut, the initial motion of its center is vertically downwards. Find the instantaneous center.

3. Four links a, b, c, d are jointed together to form a plane quadrilateral, the link a being fixed, but the others capable of motion (see Fig. 136*a*). Find the instantaneous center of the coupler, c, and construct the two centrodes graphically when the proportions of the linkage are $a:b:c:d = 4:4:5:3$. (This linkage is called the *four-bar chain.*)

4. Determine the nature of the centrodes for the four-bar chain when $a = c$, $b = d$, *and the links* b, d *are crossed.* (This linkage is called the *crossed parallelogram* since the same links form a parallelogram when b, d are not crossed.)

5. A point of a plane figure moves on a fixed straight line, while a line of the figure always passes through a fixed point. Find the instantaneous center in any position and plot the space centrode graphically.

6. A body, having a plane face channelled by two straight grooves, moves over two blocks that fit the grooves and are pivoted on *fixed* pins (Fig. 126*c*). If the pin centers are A, B, and the center lines of the grooves meet at O, show that the space centrode is a circle through A, B and any position of O; and that the body centrode is a circle with center at O and double the size of the first.

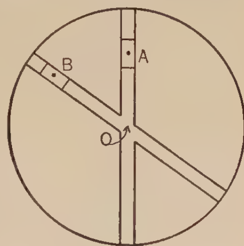


FIG. 126*c*.

Note the relation of this problem to Problem 1 above.

7. If a wheel of radius r, traveling on a straight track with a speed v, is skidding so that its lowest point is moving backward at a constant speed u, prove that its body centrode is a circle of radius $vr/(v + u)$ concentric with the rim. What is the space centrode? [In Fig. 124, $\mathbf{v}_O = -v\mathbf{i}$, $\mathbf{v}_C = u\mathbf{i}$; hence find ω.]

8. If two points A, B of a plane figure are moving in the same direction, *not perpendicular to* AB, show that all points of the figure have the same velocity.

9. A ladder 13 feet long rests on a level floor and against a vertical wall. If the lower end of the ladder, distant 5 feet from the wall, is slipping at the rate of 2 ft./sec., how fast is its upper end falling?

10. When the crank-pin B in Fig. 125*c* is at D, show that the motion of the connecting-rod AB is an instantaneous translation.

127. The Accelerations in Plane Motion. The acceleration of any point P of the moving body is given by (§ 123, 3):

$$\text{(1)} \qquad \mathbf{a}_P = \mathbf{a}_O + \boldsymbol{\alpha}\times\overrightarrow{OP} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\overrightarrow{OP}).$$

Now $\boldsymbol{\alpha}\times\overrightarrow{OP}$ and $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\overrightarrow{OP})$ are the tangential and normal projections of the acceleration in a rotation about a fixed axis through O (§ 109, 5). The content of (1) is therefore stated in

THEOREM. *If O is any point of a rigid body in plane motion, the accelerations of its points may be compounded of the acceleration $\mathbf{a}_O$ of O and the acceleration due to a rotation about a fixed axis through O with the $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ of the body.*

If O and P are points *in the same plane of motion*, $\boldsymbol{\omega}\cdot\overrightarrow{OP} = 0$ and

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\overrightarrow{OP}) = -\omega^2\overrightarrow{OP};$$

this is the usual expression for the normal acceleration towards O. Then (1) becomes

$$\mathbf{a}_P = \mathbf{a}_O + \boldsymbol{\alpha}\times\overrightarrow{OP} - \omega^2\overrightarrow{OP}. \tag{2}$$

If $\mathbf{a}_P = \mathbf{a}_O$,

$$\boldsymbol{\alpha}\times\overrightarrow{OP} - \omega^2\overrightarrow{OP} = 0 \quad \text{and} \quad \text{hence} \quad \omega = 0,\ \alpha = 0;$$

for the vectors in the last equation are perpendicular. Therefore two points in a plane of motion can have the same acceleration only when both the angular velocity and acceleration of the body are zero. In this case all points of the body have the same velocity and acceleration.

When ω and α are not *both* zero, there is always a single point J in the plane, called the *center of acceleration* whose acceleration is zero. Assuming for the moment that J exists, let P in (2) coincide with J; then since $\mathbf{a}_J = 0$,

$$0 = \mathbf{a}_O + \boldsymbol{\alpha}\times\overrightarrow{OJ} - \omega^2\overrightarrow{OJ}.$$

To find $\overrightarrow{OJ}$ multiply this equation in turn by $\boldsymbol{\alpha}\times$ and ω^2:

$$0 = \boldsymbol{\alpha}\times\mathbf{a}_O - \alpha^2\overrightarrow{OJ} - \omega^2\boldsymbol{\alpha}\times\overrightarrow{OJ},$$

$$0 = \omega^2\mathbf{a}_O + \omega^2\boldsymbol{\alpha}\times\overrightarrow{OJ} - \omega^4\overrightarrow{OJ}.$$

On adding these equations we find that

$$\overrightarrow{OJ} = \frac{\omega^2\mathbf{a}_O + \boldsymbol{\alpha}\times\mathbf{a}_O}{\omega^4 + \alpha^2}. \tag{3}$$

We may now verify from (2) that $\mathbf{a}_J = 0$ for the point thus determined.

If we choose O at J, (2) becomes

$$\mathbf{a}_P = \boldsymbol{\alpha}\times\overrightarrow{JP} - \omega^2\overrightarrow{JP}. \tag{4}$$

The accelerations of the body are the same as if it were revolving for the instant about a fixed axis through the center of acceleration.

128. Center of Acceleration. If P is any point in the plane of motion of the center of acceleration J, its acceleration is

$$\mathbf{a}_P = \boldsymbol{\alpha}\times\overrightarrow{JP} - \omega^2\overrightarrow{JP}.$$

From Fig. 128*a* we see that the magnitude and direction of $\mathbf{a}_P$ are given by

$$(1) \qquad |\mathbf{a}_P| = \sqrt{\omega^4 + \alpha^2}\cdot JP, \quad \tan(\overrightarrow{JP}, \mathbf{a}_P) = -\frac{\alpha}{\omega^2}\cdot$$

Hence $|\mathbf{a}_P|$ is proportional to JP, and the angle $(\overrightarrow{JP}, \mathbf{a}_P)$ is the same for all points of the moving figure. These results may be stated as follows:

At any instant the accelerations of all points of a figure in plane motion are equally inclined to the position vectors from the center of acceleration, and their magnitudes are proportional to the lengths of these position vectors.

In particular, all points on a circle about J as center have accelerations of equal magnitude; and all points on a straight line through J have parallel accelerations.

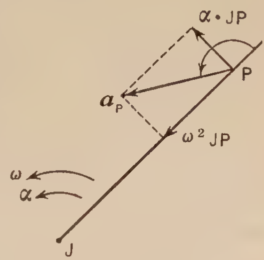


FIG. 128*a*.

If the lines of the acceleration vectors at two points A, B of the moving figure intersect at the point X, we must have either

$$\angle JAX = \angle JBX \qquad \text{or} \qquad \angle JAX + \angle JBX = 180^\circ$$

in view of the above theorem. Hence, from a well-known proposition in plane geometry, the circle through J, X, A will also pass through B. Otherwise expressed, *the circle through any two points of a figure in plane motion and the point of intersection of their localized accelerations also passes through the center of acceleration.*

This property enables us to locate the center of acceleration when the acceleration directions of three points of the moving figure are known. For if the lines of the acceleration vectors at A and B, B and C, C and A, intersect at X, Y, Z respectively, J lies on each of the three circles ABX, BCY, CAZ (Fig. 128*b*).

The center of acceleration may also be located if the directions of the accelerations and the ratio of their magnitudes are known for *two* points of the figure. One method of procedure is suggested by Problem 5 below; another will be given in § 129.

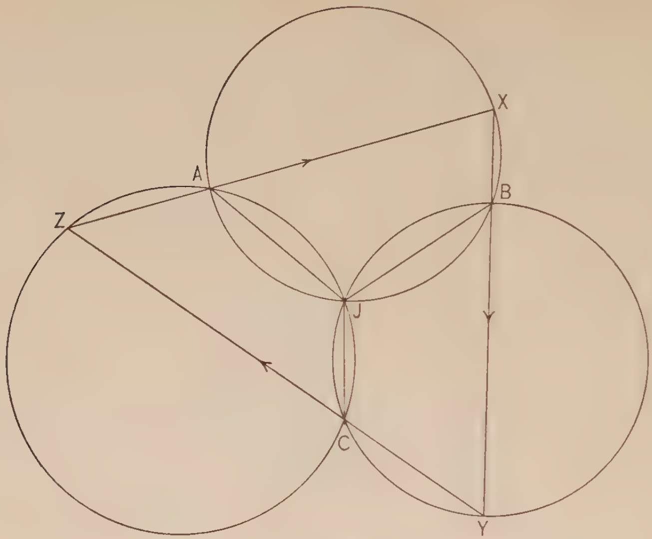


FIG. 128*b*.

Example 1. *Rolling Wheel.* Let the wheel in Fig. 128*c* be rolling to the left with the angular velocity ω and angular acceleration α. Then

$$\mathbf{v}_O = -\omega r\mathbf{i} \quad \text{and} \quad \mathbf{a}_O = \frac{d\mathbf{v}_O}{dt} = -\alpha r\mathbf{i},$$

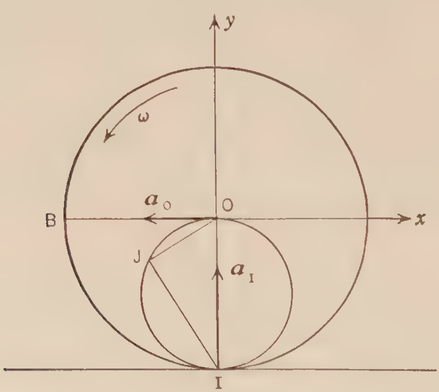


FIG. 128*c*.

and, from (§ 127, 3), its center of acceleration is given by

$$\overrightarrow{OJ} = \frac{\omega^2(-\alpha r\mathbf{i}) + \mathbf{k}\alpha\times(-\alpha r\mathbf{i})}{\omega^4 + \alpha^2} = \frac{-\alpha r}{\omega^4 + \alpha^2}(\omega^2\mathbf{i} + \alpha\mathbf{j}).$$

Hence OJ has the slope α/ω^2. Moreover

$$\overrightarrow{IJ} = \overrightarrow{IO} + \overrightarrow{OJ} = r\mathbf{j} + \overrightarrow{OJ} = \frac{\omega^2 r}{\omega^4 + \alpha^2}(-\alpha\mathbf{i} + \omega^2\mathbf{j})$$

and has the slope $-\omega^2/\alpha$. Hence IJ and OJ are perpendicular and J must lie on the circle having OI as diameter. Thus J is at the point where this circle cuts the line OJ of slope α/ω^2.

When ω is constant ($\alpha = 0$), $OJ = 0$ and $J \equiv 0$.

When $\omega = 0$ but $\alpha \neq 0$, $\overrightarrow{OJ} = -r\mathbf{j}$ and $J \equiv I$. This is the case at the instant the wheel starts or stops.

Example 2. Let us find the acceleration $\mathbf{a}_I$ of that point of the body which coincides for the instant with its instantaneous center of velocity I. If O_1 is an origin fixed in space,

$$\mathbf{v}_P = \boldsymbol{\omega}\times\overrightarrow{IP} = \boldsymbol{\omega}\times(\overrightarrow{O_1P} - \overrightarrow{O_1I}) \qquad (\S\ 124, 3),$$

and on differentiating with respect to the time,

$$\mathbf{a}_P = \boldsymbol{\alpha}\times\overrightarrow{IP} + \boldsymbol{\omega}\times(\mathbf{v}_P - \mathbf{V})$$

where $\mathbf{V}$ denotes the velocity of I along the centrodes (§ 126). When P coincides with I this equation reduces to

$$\mathbf{a}_I = \mathbf{V}\times\boldsymbol{\omega}.$$

Hence $\mathbf{a}_I$ is normal to $\mathbf{V}$ and is of magnitude $V\omega$.

Thus for the rolling wheel above

$$\mathbf{V} = -r\omega\mathbf{i}, \quad \mathbf{a}_I = (-r\omega\mathbf{i})\times(\omega\mathbf{k}) = r\omega^2\mathbf{j}.$$

PROBLEMS

1. Show that $J \equiv I$ in Fig. 125*b* at the instant the ladder begins to slip.

2. A wheel 4 ft. in diameter starts from rest and rolls down an inclined plane. If its axis has a uniform acceleration of 2 ft./sec.^{2}, locate the center of acceleration $\frac{1}{2}$ sec., 1 sec., and 2 sec. after the beginning of the motion.

3. Show that when ω is constant (but not zero), the accelerations of all points of the moving figure are directed toward J; and that when $\omega = 0$, $\alpha \neq 0$, the accelerations of all points are normal to their position vectors from J.

4. Prove that all points of the moving figure lying on a circle through J have accelerations directed toward one and the same point of this circle.

5. If the accelerations of two points A, B of a figure in plane motion are given in magnitude and direction by the vectors AA', BB' whose lines intersect at the point X, prove that J must lie on the circle $A'B'X$.

6. Show that when the crank in Fig. 125c is revolving with uniform angular velocity, the center of acceleration of the connecting-rod AB lies on a circle through A, B, and O_1.

What can be said about the position of J when the crank is passing a dead center?

Locate J when the crank is midway between the dead centers. (See Problem 3.)

129. Velocity and Acceleration Images. If the vectors of the system $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, . . . , issuing from a common point O, yield the system $\overrightarrow{OA'}$, $\overrightarrow{OB'}$, $\overrightarrow{OC'}$, . . . when multiplied by the same scalar k, the figures ABC . . . , $A'B'C'$ . . . formed by their end-points are similar. For since

$$\overrightarrow{A'B'} = \overrightarrow{OB'} - \overrightarrow{OA'} = k\cdot\overrightarrow{OB} - k\cdot\overrightarrow{OA} = k\,(\overrightarrow{OB} - \overrightarrow{OA}) = k\cdot\overrightarrow{AB},$$

we see that the corresponding sides of the two figures are parallel and proportional. In particular, if the points A,B,C lie on the same straight line, the same will be true of A',B',C', and $AB : BC = A'B' : B'C'$.

Again, if the vectors of the new system are all turned through the same angle, the figure formed by their end-points will obviously remain similar to ABC . . . .

We may state these results as follows:

Principle. *If the vectors forming a system issuing from a common point are all stretched or shortened in the same ratio and then turned through the same angle, the figures formed by the corresponding end-points of the two systems are similar.*

Consider, now, the velocity vectors $\mathbf{v}_A$, $\mathbf{v}_B$, . . . of the points A, B, . . . of a body in plane motion. Denoting as usual the instantaneous center of velocity by I, we have from § 125

$$\mathbf{v}_A \perp IA, \qquad |\mathbf{v}_A| = |\omega|\cdot IA,$$

and similarly for the other vectors. Hence if the vectors $\overrightarrow{IA}$, $\overrightarrow{IB}$, . . . are all multiplied by $|\omega|$ and then turned through 90° in the sense of ω, we shall obtain the system of velocity vectors issuing from I (Fig. 129a). In view of the Principle above stated we have proved

Theorem 1. *If the velocity vectors of the points A, B, C, . . . of a rigid body in plane motion are all drawn from a common pole, their end-points will form a figure similar to ABC . . . .*

The figure $a'b'c'$ formed by the end-points of $\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C$, . . . when these vectors are drawn from a common point or *pole* is called a *polar velocity image* of ABC . . . . Note that since I has zero velocity, the pole must correspond to the point I.

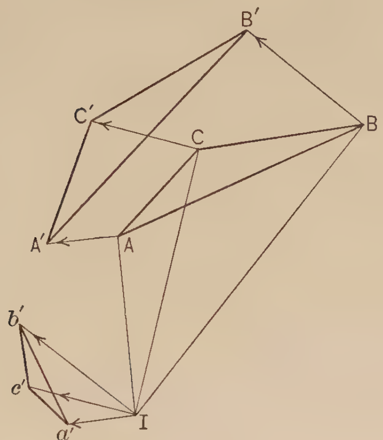


FIG. 129*a*.

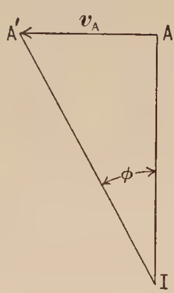


FIG. 129*b*.

Now suppose that the velocity vectors $\mathbf{v}_A$, $\mathbf{v}_B$, . . . are all drawn from the respective points A, B, . . .; and write

$$\overrightarrow{IA} + \mathbf{v}_A = \overrightarrow{IA'}, \text{ etc.}$$

We then have (Fig. 129*b*)

$$\tan\phi = \frac{AA'}{IA} = \frac{\omega\cdot IA}{IA} = \omega, \quad IA' = IA\sec\phi = \sqrt{1+\omega^2}\cdot IA.$$

The vectors $\overrightarrow{IA'}$, $\overrightarrow{IB'}$, . . . may therefore be obtained from the vectors $\overrightarrow{IA}$, $\overrightarrow{IB}$, . . . by stretching them all in the ratio of $\sqrt{1+\omega^2}$: 1 and then turning them through an angle $\phi = \tan^{-1}\omega$ in the sense of ω. The above Principle now gives

THEOREM 2. *If the velocity vectors of the points A, B, C, . . . of a rigid body in plane motion are drawn localized at these points, their end-points will form a figure similar to ABC . . . .*

The figure $A'B'C'$ . . . formed by the end-points of the localized velocity vectors is called a *velocity image* of ABC . . . .

We next turn our attention to the acceleration vectors $\mathbf{a}_A$, $\mathbf{a}_B$, . . .

of the points A, B, . . . of the body. Denoting the center of acceleration by J, we have from (§ 128, 1)

$$|\mathbf{a}_A| = \sqrt{\omega^4 + \alpha^2}\cdot JA, \qquad \angle\,(\overrightarrow{JA}, \mathbf{a}_A) = \tan^{-1}(-\alpha/\omega^2),$$

and similarly for the other vectors. Hence if the vectors $\overrightarrow{JA}$, $\overrightarrow{JB}$, . . . are all multiplied by $\sqrt{\omega^4 + \alpha^2}$ and then turned through the angle $\tan^{-1}(-\alpha/\omega^2)$, we shall obtain the system of acceleration vectors issuing from J (Fig. 129c). Thus we have

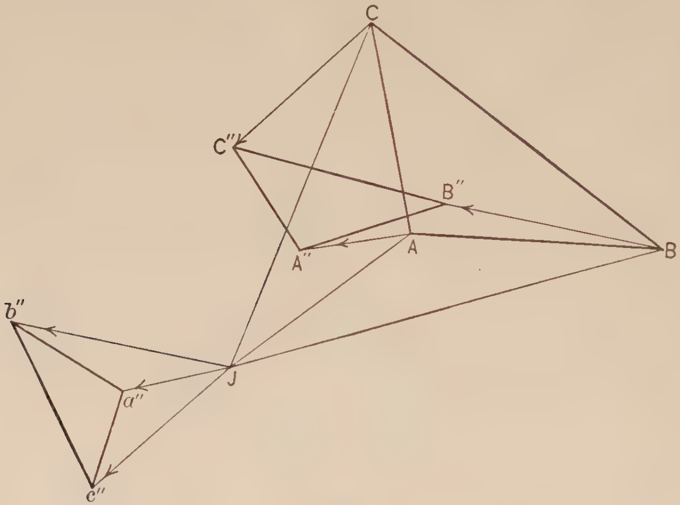


Fig. 129c.

Theorem 3. *If the acceleration vectors of the points A, B, C, . . . of a rigid body in plane motion are all drawn from a common pole, their end-points will form a figure similar to ABC . . . .*

The figure $a''b''c''$ . . . formed by the end-points of $\mathbf{a}_A$, $\mathbf{a}_B$, $\mathbf{a}_C$, . . . when these vectors are all drawn from a common pole is called the *polar acceleration image* of ABC . . . . Note that since J has zero acceleration, the pole must correspond to the point J.

Lastly, suppose that acceleration vectors $\mathbf{a}_A$, $\mathbf{a}_B$, . . . are all drawn from the respective points A, B, . . .; and write

$$\overrightarrow{JA} + \mathbf{a}_A = \overrightarrow{JA''}, \quad \text{etc.}$$

Then since the components of $\mathbf{a}_A$ parallel and perpendicular to JA are respectively $-\omega^2 \cdot JA$ and $\alpha \cdot JA$, we have (Fig. 129d)

$$\tan\psi = \frac{LA''}{JL} = \frac{LA''}{JA - LA} = \frac{\alpha \cdot JA}{(1-\omega^2)\cdot JA} = \frac{\alpha}{1-\omega^2},$$

$$JA'' = \sqrt{JL^2 + LA^2} = \sqrt{(1-\omega^2)^2 + \alpha^2}\cdot JA.$$

These equations show that when the vectors of the system $\overrightarrow{JA}$, $\overrightarrow{JB}$,. . . are multiplied by $\sqrt{(1-\omega)^2 + \alpha^2}$ and then turned through the angle $\tan^{-1}\dfrac{\alpha}{1-\omega^2}$, the system $\overrightarrow{JA''}$, $\overrightarrow{JB''}$, . . . is obtained. Again applying the Principle, we have the

THEOREM 4. *If the acceleration vectors of the points A, B, C, . . . of a rigid body in plane motion are drawn localized at these points, their end-points will form a figure similar to ABC . . . .*

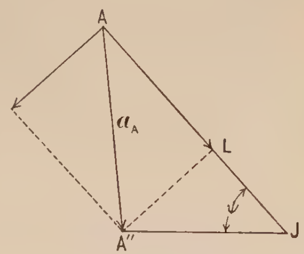


FIG. 129d.

The figure $A''B''C''$ . . . formed by the end-points of the localized acceleration vectors is called an *acceleration image* of ABC . . . .

Now let abc be one of the images of ABC mentioned above. Then as abc is formed from ABC by a process of stretching and turning, it is clear that abc must be similar to ABC *in the same sense;* that is, the sense of the circuits abc and ABC must be the same.

Theorem 3 supplies a simple method of locating the center of acceleration J when the accelerations of two points are known in direction and relative magnitude. For if we draw from an arbitrary point j the vectors $\overrightarrow{ja}$, $\overrightarrow{jb}$ to represent the accelerations of the points A, B, the triangle abj will be the polar acceleration image of ABJ. J is therefore the vertex of a triangle on AB as base, drawn similar to abj in such a manner that the circuits ABJ, abj have the same sense.

130. Résumé. The essential results concerning the motion of a plane figure in its plane may be expressed as follows: If A, B are points of the figure,

(1), (2) $\qquad \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA}, \quad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA},$

where $\mathbf{v}_{BA}$ and $\mathbf{a}_{BA}$ denote the velocity and acceleration that B would have if the figure were revolving about A as a fixed point with its instantaneous angular velocity and acceleration: that is

(3), (4) $$\mathbf{v}_{BA} = \boldsymbol{\omega}\times\overrightarrow{AB}, \quad \mathbf{a}_{BA} = \boldsymbol{\alpha}\times\overrightarrow{AB} - \omega^2\overrightarrow{AB}.$$

Thus if $\overrightarrow{AB} = r\mathbf{R}$ and $\mathbf{P} = \mathbf{k}\times\mathbf{R}$ (Fig. 130a)

$$\mathbf{v}_{BA} = \omega r\mathbf{P}, \quad \mathbf{a}_{BA} = \alpha r\mathbf{P} - \omega^2 r\mathbf{R}.$$

Since the speed of B in its rotation about A is $v_{BA} = \omega r$, the radial component of $\mathbf{a}_{AB}$ may also be expressed as $-v^2_{BA}/r$. We shall call $\mathbf{v}_{BA}$ and $\mathbf{a}_{BA}$ the *velocity* and *acceleration of B relative to A*.*

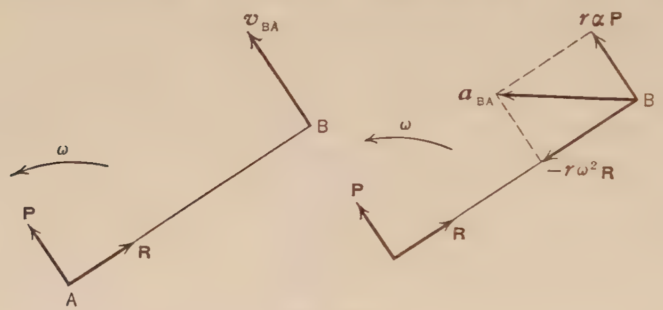


Fig. 130a.

If ω is not zero, there is a single point I of the figure, its *instantaneous center*, whose velocity is zero. If the normals to $\mathbf{v}_A$ at A and $\mathbf{v}_B$ at B are not coincident, they will intersect at I. Since $\mathbf{v}_I = 0$, we have from (1) and (3)

$$\mathbf{v}_B = \mathbf{v}_{BI} = \boldsymbol{\omega}\times\overrightarrow{IB}.$$

If ω and α are not *both* zero, there is a single point J of the figure, its *center of acceleration*, whose acceleration is zero. Since $\mathbf{a}_J = 0$, we have from (2) and (4)

$$\mathbf{a}_B = \mathbf{a}_{BJ} = \boldsymbol{\alpha}\times\overrightarrow{JB} - \omega^2\overrightarrow{JB}.$$

If the vectors

$$\overrightarrow{ia} = \mathbf{v}_A, \quad \overrightarrow{ib} = \mathbf{v}_B, \quad \overrightarrow{ic} = \mathbf{v}_C, \ldots$$

* Strictly speaking, $\mathbf{v}_{BA}$ and $\mathbf{a}_{BA}$ are the velocity and acceleration of B relative to a figure having a translation of velocity $\mathbf{v}_A$ and acceleration $\mathbf{a}_A$ (§ 134).

are drawn from a point i, the polygon $iabc\ldots$ is similar to $IABC\ldots$ in the same sense.

If the vectors

$$\overrightarrow{ja'} = \mathbf{a}_A, \quad \overrightarrow{jb'} = \mathbf{a}_B, \quad \overrightarrow{jc'} = \mathbf{a}_C, \ldots$$

are drawn from a point j, the polygon $ja'b'c'\ldots$ is similar to $JABC\ldots$ in the same sense.

The polygons $abc\ldots$ and $a'b'c'\ldots$ are both similar to $ABC\ldots$ in the same sense; they are called the *polar velocity* and *acceleration images* of $ABC\ldots$ respectively.

Example. Rolling Wheel. Consider again a wheel of radius r rolling to the left with the angular velocity and acceleration ω, α. Since $\mathbf{v}_O = -\omega r\mathbf{i}$, $\mathbf{a}_O = -\alpha r\mathbf{i}$ (§ 128, Example 1), for any point E on the rim

$$\mathbf{v}_E = \mathbf{v}_O + \mathbf{v}_{EO} = -\omega r\mathbf{i} + \omega r\mathbf{P},$$
$$\mathbf{a}_E = \mathbf{a}_O + \mathbf{a}_{EO} = -\alpha r\mathbf{i} + \alpha r\mathbf{P} - \omega^2 r\mathbf{R}.$$

In particular, $\mathbf{a}_I = \mathbf{a}_O + \mathbf{a}_{IO} = -\alpha r\mathbf{i} + \alpha r\mathbf{i} + \omega^2 r\mathbf{j} = \omega^2 r\mathbf{j}.$

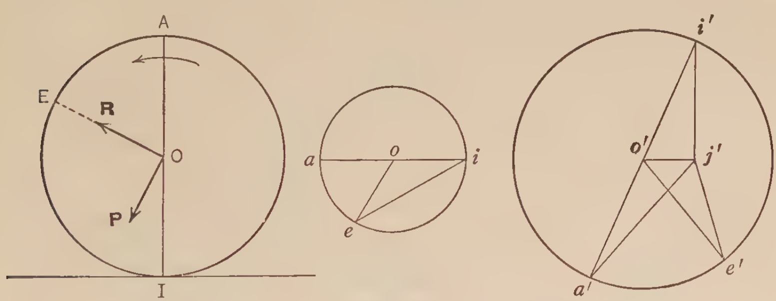


FIG. 130*b*.

If we draw the vector $\overrightarrow{io} = \mathbf{v}_O$, the velocity image of the wheel is a circle with o as center and oi as radius. To obtain $\mathbf{v}_E$, construct $\angle\, aoe = \angle\, AOE$ in the same sense; then $\overrightarrow{oe} = \mathbf{v}_{EO}$, $\overrightarrow{ie} = \mathbf{v}_E$.

From an arbitrary point j' draw the vectors $\overrightarrow{j'o'} = \mathbf{a}_O$, $\overrightarrow{j'i'} = \mathbf{a}_I$; then a circle with o' as center and $o'i'$ as radius is the acceleration image of the wheel. If we construct $\angle\, a'o'e' = \angle\, AOE$ in the same sense, then $\overrightarrow{o'e'} = \mathbf{a}_{EO}$, $\overrightarrow{j'e'} = \mathbf{a}_E$.

131. Construction of the Polar Velocity Image. Let A and B be any two points of a rigid body in plane motion. Then if the

vectors $\overrightarrow{ia}$ and $\overrightarrow{ib}$ are drawn from a common pole i to represent the velocities of A and B (Fig. 131a), the vector

$$\overrightarrow{ab} = \overrightarrow{ib} - \overrightarrow{ia} = \mathbf{v}_B - \mathbf{v}_A = \mathbf{v}_{BA}$$

represents the velocity of B relative to A, and is perpendicular to AB (§ 130, 3). Hence if $\mathbf{v}_A$ is given, $\mathbf{v}_B$ may be determined graph-

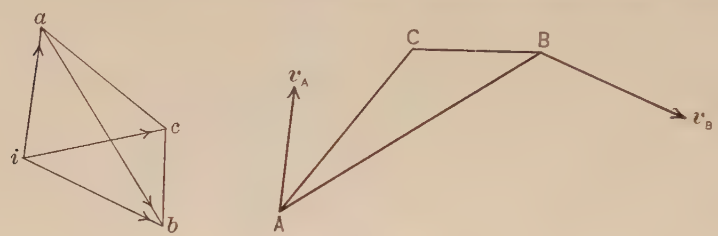


Fig. 131a.

ically if its direction is known. For if we draw $\overrightarrow{ia} = \mathbf{v}_A$, a line through a perpendicular to AB will cut a line through i parallel to $\mathbf{v}_B$ in the point b; and $\overrightarrow{ib} = \mathbf{v}_B$.

The segment ab is the polar velocity image of AB; and if I marks the position of the instantaneous center of the body, the triangle iab is the polar velocity image of the triangle IAB. The velocity of any other point C of the body may be found by constructing a triangle abc similar to ABC in the same sense; then $\overrightarrow{ic} = \mathbf{v}_C$. Or we may reason as follows: Since $\mathbf{v}_{CA}$ is perpendicular to AC, c must lie on a line through a perpendicular to AC; similarly c must lie on a line through B perpendicular to BC. Hence c is the point of intersection of the lines ac, bc perpendicular to AC, BC respectively.

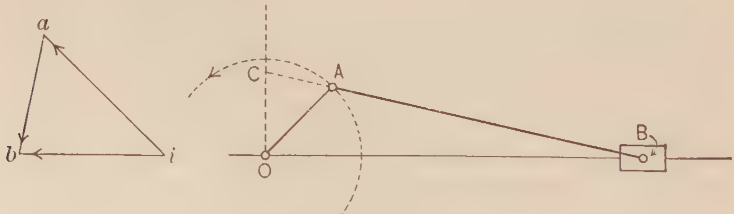


Fig. 131b.

Example 1. *Direct-acting Engine.* In Fig. 131b, OA and AB represent the crank and connecting-rod of a direct-acting engine. Then if $\overrightarrow{ia}$,

drawn perpendicular to OA, represents the velocity of the crank-pin A, we may find the velocity of the piston by drawing the line ib parallel to BO, and the line ab perpendicular to AB. The point b in which these lines meet determines the vectors $\overrightarrow{ab} = \mathbf{v}_{BA}$ and $\overrightarrow{ib} = \mathbf{v}_B$.

As the triangles iab and OAC have their sides mutually perpendicular they are similar, and

$$\frac{ia}{OA} = \frac{ib}{OC} = \frac{ab}{AC} \qquad \text{or} \qquad \frac{v_A}{OA} = \frac{v_B}{OC} = \frac{v_{BA}}{AC}.$$

Since $v_A = \omega_1 \cdot OA$, where ω_1 is the angular velocity of the crank-pin in radians per second, the common value of the above ratios is ω_1. Hence

$$v_A = \omega_1 \cdot OA, \qquad v_B = \omega_1 \cdot OC, \qquad v_{BA} = \omega_1 \cdot AC.$$

Example 2. *The Four-bar Chain.* The mechanism represented in Fig. 131*c* consists of four links connected by pin joints. The link AD is fixed. Since the links AB and DC revolve about fixed centers, $\mathbf{v}_B \perp AB$, $\mathbf{v}_C \perp DC$; moreover $\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{CB}$, where $\mathbf{v}_{CB} \perp BC$. To construct a polar velocity image of the coupler BC, choose a pole i at pleasure, and draw $\overrightarrow{ib}$ to represent $\mathbf{v}_B$. Then a line through b perpendicular to BC will cut a line through i perpendicular to DC in the point c, and $\overrightarrow{bc} = \mathbf{v}_{CB}$, $\overrightarrow{ic} = \mathbf{v}_C$; bc is a polar velocity image of BC.

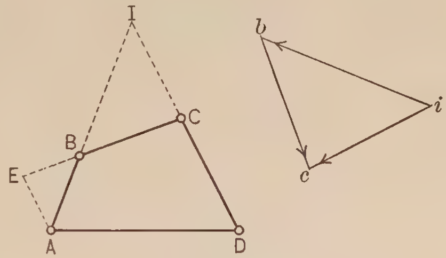


FIG. 131*c*.

If I is the point of intersection of AB and DC produced, the triangles ibc and IBC are similar as their sides are mutually perpendicular; hence

$$\frac{ib}{IB} = \frac{ic}{IC} = \frac{bc}{BC} \qquad \text{or} \qquad \frac{v_B}{IB} = \frac{v_C}{IC} = \frac{v_{CB}}{BC} = \omega,$$

where ω is the angular velocity of the coupler BC (§ 125, 1).

If AE is drawn parallel to DC, meeting CB produced at E, the triangles IBC and ABE are similar; hence

$$\frac{v_B}{AB} = \frac{v_C}{AE} = \frac{v_{CB}}{BE}.$$

Now $v_B = \omega_1 \cdot AB$, where ω_1 denotes the angular velocity of AB. Thus ω_1 is the common value of the ratios just written, and

$$v_B = \omega_1 \cdot AB, \qquad v_C = \omega_1 \cdot AE, \qquad v_{CB} = \omega_1 \cdot BE.$$

132. Construction of Polar Acceleration Image. As before let A and B denote points of a rigid body in plane motion. Then if the vectors $\overrightarrow{j'a'}$ and $\overrightarrow{j'b'}$ are drawn from a common pole j' to represent the accelerations of A and B (Fig. 132a), the vector

$$\overrightarrow{a'b'} = \overrightarrow{j'b'} - \overrightarrow{j'a'} = \mathbf{a}_B - \mathbf{a}_A = \mathbf{a}_{BA}$$

represents the acceleration of B relative to A. Now $\overrightarrow{a'b'}$ may be resolved into the sum of its radial and transverse projections:

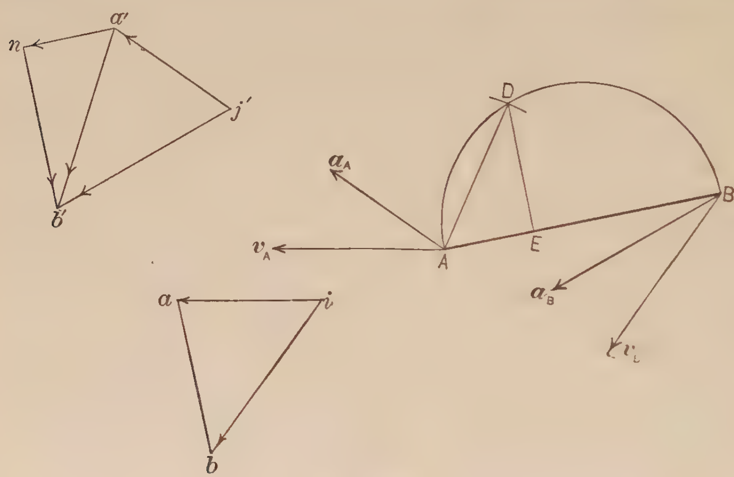


FIG. 132a.

$\overrightarrow{a'n}$ parallel to BA, and $\overrightarrow{nb'}$ perpendicular to BA. The lengths of these vectors are, according to (§ 130),

$$a'n = \frac{v^2_{BA}}{AB}, \qquad nb' = \alpha \cdot AB.$$

When $\mathbf{v}_A$ and $\mathbf{a}_A$ are given, $\mathbf{a}_B$ may be determined graphically if the directions of $\mathbf{v}_B$ and $\mathbf{a}_B$ are known. First construct ab, the polar velocity image of AB, as explained in the preceding article; then $ab = v_{BA}$. We can now construct $a'n$, the radial component of $\mathbf{a}_{BA}$. Draw a semicircle on AB as diameter, and strike an arc with A as center and ab as radius, cutting the semicircle at D; then if E is the foot of the perpendicular from D upon AB, we have

$$AE \cdot AB = AD^2 = ab^2 = v^2_{BA}, \qquad AE = \frac{v^2_{BA}}{AB} = a'n.$$

Now draw $\overrightarrow{j'a'} = \mathbf{a}_A$, and $\overrightarrow{a'n}$ in the direction of $\overrightarrow{BA}$, laying off $a'n = AE$. Then a line through n perpendicular to BA will cut a line through j' parallel to $\mathbf{a}_B$ in the point b'; and $\overrightarrow{j'b'} = \mathbf{a}_B$.

The segment $a'b'$ is the polar acceleration image of AB; and if J marks the position of the center of acceleration of the body, the triangle $j'a'b'$ is the polar acceleration image of the triangle JAB. The acceleration of any other point C of the rigid body may be found by constructing a triangle $a'b'c'$ similar to ABC in the same sense; then $\overrightarrow{j'c'} = \mathbf{a}_C$.

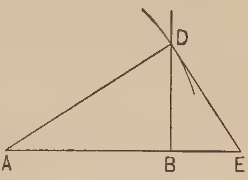


FIG. 132*b*

The above construction for $v^2{}_{BA}/AB$ can only be applied when $ab < AB$. When $ab > AB$ we may proceed as follows (Fig. 132*b*). Draw a perpendicular to AB at B, and strike an arc with A as center and ab as radius cutting this perpendicular at D; then a line perpendicular to AD at D will cut AB produced in a point E such that

$$AB \cdot AE = AD^2 = ab^2 = v^2{}_{BA}, \qquad AE = \frac{v^2{}_{BA}}{AB}.$$

Thus far we have tacitly assumed that the lengths of the velocity and acceleration vectors were equal to the magnitudes of the quantities represented. In practice, however, such a representation is rarely possible, owing to the diverse magnitudes of the distances, velocities, and accelerations involved. Let us suppose therefore that all lengths are measured with the same unit, but that the magnitude of a velocity is given by k times the length of its representative vector, e.g. $\mathbf{v}_A = k \cdot \overrightarrow{ia}$. Then since the magnitude of the radial component of $\mathbf{a}_{BA}$ is

$$\frac{v^2{}_{BA}}{AB} = \frac{k^2 \cdot ab^2}{AB} = k^2 \cdot AE,$$

AE may be used as the length of an acceleration vector provided that the magnitudes of *all* accelerations are given by k^2 times the lengths of their representative vectors. In brief, when the above construction is employed,

$$\mathbf{v}_A = k \cdot \overrightarrow{ia} \qquad \text{requires that} \qquad \mathbf{a}_A = k^2 \cdot \overrightarrow{j'a'}.$$

It will be observed that this condition is fulfilled in the examples that follow.

Example 1. *Klein's Construction for the Piston Acceleration.* The construction just described may be used to determine the piston acceleration of a direct-acting engine. We shall suppose that the crank OA (Fig. 132*c*) is revolving *uniformly* with the angular velocity of ω_1 radians per second. Then the acceleration of A is entirely radial, and from (§ 109), $\mathbf{a}_A = \omega_1^2\overrightarrow{AO}$. The construction is performed most simply by drawing all of the acceleration vectors reversed in direction. Thus, choosing O as pole, we may take $\overrightarrow{OA}$ to represent $-\mathbf{a}_A$. From § 131, Example 1, $v_{BA} = \omega_1 \cdot AC$; hence if a circle is drawn with A as center and AC as radius, cutting the circle on AB as diameter in the points D, D', we have

$$\frac{v^2_{BA}}{AB} = \omega_1^2 \frac{AC^2}{AB} = \omega_1^2 \frac{AD^2}{AB} = \omega_1^2 \cdot AE,$$

where E is the point in which the common chord DD' cuts AB. Now the radial acceleration of B relative to A is directed from B towards A. Hence the *negative* of this radial acceleration is given by $\omega_1^2\overrightarrow{AE}$, and may be represented by $\overrightarrow{AE}$ to the same scale that $\overrightarrow{OA}$ represents $-\mathbf{a}_A$. To complete the construction we need only prolong the common chord DD' until it meets OB in a point F. Then since EF is perpendicular to AB, and OB is parallel to $\mathbf{a}_B$, the transverse projection of $-\mathbf{a}_{BA}$ is given by $\omega_1^2\overrightarrow{EF}$, and $-\mathbf{a}_B = \omega_1^2\overrightarrow{OF}$. Upon reversing these vectors we have finally:

$$\mathbf{a}_A = \omega_1^2\overrightarrow{AO}, \qquad \mathbf{a}_{BA} = \omega_1^2(\overrightarrow{FE} + \overrightarrow{EA}) = \omega_1^2\overrightarrow{FA},$$

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} = \omega_1^2(\overrightarrow{FA} + \overrightarrow{AO}) = \omega_1^2\overrightarrow{FO}.$$

Thus $\overrightarrow{AO}$ and $\overrightarrow{FO}$ represent respectively the accelerations of the crank pin and piston to the scale of $1 : \omega_1^2$.

In Fig. 132*c* a length of 1 inch represents 1 foot in the actual engine (of 18 inch stroke). If the crank makes 150 r.p.m.,

$$\omega_1 = \frac{2\pi \times 150}{60} = 5\pi \text{ rad./sec.}^2$$

Hence

$$a_B = 25\pi^2 \cdot FO = 247 \cdot FO \text{ ft./sec.}^2$$

when FO is given the value in feet corresponding to the full size diagram. In the position shown in the figure, $FO = 0.65$ ft., and $a_B = 247 \times 0.65 = 160$ ft./sec.^{2}

Example 2. *The Four-bar Chain.* Suppose that the crank AB of the four-bar chain $ABCD$ (Fig. 132*d*) is driven with the constant angular velocity of ω_1 radians per second. The link DC will then oscillate between certain limiting positions for each revolution of the crank.

If AE is drawn parallel to DC, we know from § 131, Example 2, that

$$v_B = \omega_1 \cdot AB \qquad v_C = \omega_1 \cdot AE, \qquad v_{CB} = \omega_1 \cdot BE.$$

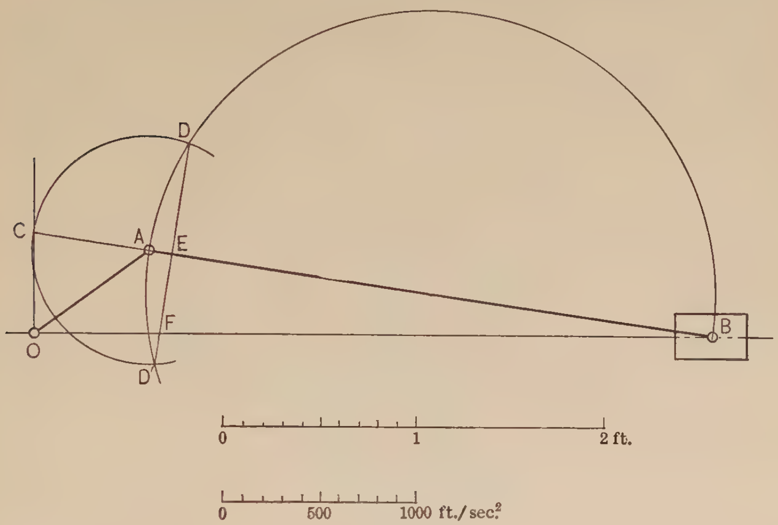


FIG. 132c.

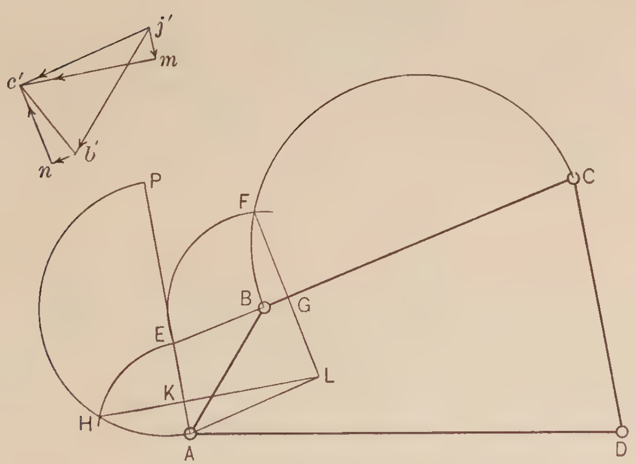


FIG. 132d.

Since ω_1 is constant, the acceleration of B is entirely radial, and $a_B = \omega_1^2\overrightarrow{BA}$. In order to obtain a_C graphically by means of the construction given above we may use BE to represent v_{CB}, and $\overrightarrow{BA}$ to represent

$\mathbf{a}_B$. We shall perform the construction by drawing all of the acceleration vectors reversed in direction. Thus choosing A as pole, we take $\overrightarrow{AB}$ to represent $-\mathbf{a}_B$. The next step is to determine the radial projection of $\mathbf{a}_{CB}$. To this end draw a semicircle on BC as diameter, and strike an arc with B as center and BE as radius, cutting the semicircle at F. Then if FG is perpendicular to BC, we have

$$\frac{v^2_{CB}}{BC} = \omega_1^2 \frac{BE^2}{BC} = \omega_1^2 \frac{BF^2}{BC} = \omega_1^2 \cdot BG,$$

and $\overrightarrow{BG}$ may be used to represent the radial projection of $-\mathbf{a}_{CB}$. The transverse projection of $-\mathbf{a}_{CB}$ will therefore be represented by a vector issuing from G and perpendicular to BG.

The direction of $\mathbf{a}_C$ is not known, but its radial projection, of magnitude v^2_C/DC, may be constructed as follows: Lay off $AP = DC$ on the line AE parallel to DC, and draw a circle on AP as diameter. Strike an arc with A as center and AE as radius, cutting this circle at H; draw the perpendicular HK to AP. Then from (1),

$$\frac{v^2_C}{DC} = \omega_1^2 \frac{AE^2}{DC} = \omega_1^2 \frac{AH^2}{AP} = \omega_1^2 \cdot AK,$$

and $\overrightarrow{AK}$ may be used to represent the radial projection of $-\mathbf{a}_C$. The transverse projection of $-\mathbf{a}_C$ will therefore be represented by a vector issuing from K and perpendicular to AK.

We thus see that when $\mathbf{a}_C$ is represented by a vector issuing from A, its end-point must lie on both of the lines GL and KL, perpendicular to BG and AK respectively. Hence $-\mathbf{a}_C$ is represented by $\overrightarrow{AL}$ to the same scale $(1 : \omega_1^2)$ that $-\mathbf{a}_B$ is represented by $\overrightarrow{AB}$. Upon reversing these vectors, we have

$$\mathbf{a}_B = \omega_1^2 \overrightarrow{BA}, \qquad \mathbf{a}_C = \omega_1^2 \overrightarrow{LA}.$$

The figure also shows the acceleration vectors drawn in their proper directions from the pole j'; $b'c'$ is the acceleration image of BC.

Example 3. *Watt's Parallel Motion.* In the four-bar chain $ABCD$ of Fig. 132c the arms AB and CD are parallel. Let it be required to find a point P in the center line of the coupler BC whose path is most nearly a straight line in the vicinity of its present position.

The points B and C are both moving normal to the parallel arms; the instantaneous center of BC is therefore at infinity in the direction of the arms (§ 124). The instantaneous motion of the coupler is thus a translation normal to the arms. In particular B and C are moving with the same speed v.

Let us now construct the acceleration image of BC. From j as pole

draw the vectors $\overrightarrow{jm}$ and $\overrightarrow{jn}$ to represent the radial projections of $\mathbf{a}_B$ and $\mathbf{a}_C$ respectively, taking their lengths in the ratio

$$jm : jn = \frac{v^2}{AB} : \frac{v^2}{DC} = DC : AB.$$

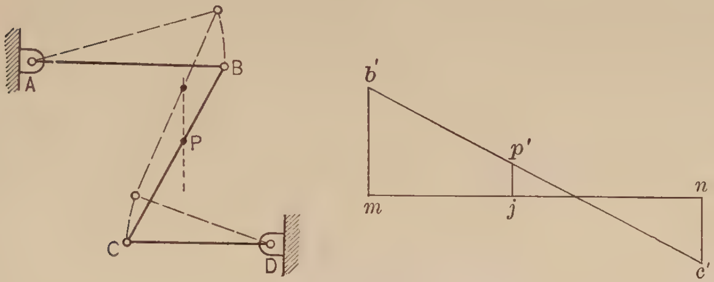


FIG. 132e.

Let $\overrightarrow{mb'}$ represent the transverse projection of $\mathbf{a}_B$; then $\overrightarrow{jb'}$ represents $\mathbf{a}_B$. Now

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{CB},$$

where $\mathbf{a}_{CB}$ is entirely transverse (normal to BC) since its radial component vanishes; for

$$\frac{v^2_{CB}}{BC} = \frac{(v_C - v_B)^2}{BC} = 0.$$

Hence draw $b'c'$ perpendicular to BC and nc' perpendicular to jn; their intersection c' determines the vector $\overrightarrow{jc'}$ which represents $\mathbf{a}_C$.

To find the point P of BC whose acceleration has the same direction as its velocity (vertical), draw jp' perpendicular to jm; then since $b'c'$ is the acceleration image of BC,

$$\frac{PB}{PC} = \frac{p'b'}{p'c'} = \frac{jm}{jn} = \frac{DC}{AB}.$$

Consequently P *divides the coupler into segments inversely proportional to the lengths of the adjacent arms.* Since the velocity and acceleration of P are both vertical, the motion of P will evidently persist in approximately this direction longer than any other point of BC. In other words, P is the point of BC whose path is most nearly a straight line in the vicinity of the position considered. The entire path of P has the shape of a distorted 8; it can readily be plotted by drawing the mechanism in a number of different positions. The range of approximate linearity may be increased by choosing the proportions of the linkage so that the coupler is perpendicular to the arms in its mean position. For $b'c'$ is horizontal in this case, and the acceleration of P ($\overrightarrow{jp'}$) is less than that of any other point of BC.

This mechanism for obtaining approximate rectilinear from rotatory motion was first applied by James Watt, the inventor of the condensing steam engine. Watt also used the modification in which the arms are on the same side of the coupler. It can then be shown precisely as above that the point P will divide the coupler (prolonged beyond the longer arm) *externally* in the inverse ratio of the arms.

PROBLEM

1. In the *Tchebicheff parallel motion* a four-bar chain $ABCD$ is used in which the arms AB, DC are crossed and of equal length, and the tracing point P is taken at the middle point of the coupler BC (Fig. 132*f*). Taking

$$AD = 4, \quad AB = DC = 5, \quad BC = 2,$$

with any convenient unit of length, construct the velocity and acceleration images of the coupler when it is horizontal, and when it is vertical, assuming that the angular velocity of AB is constant. Draw the vectors representing the velocity and acceleration of P in both cases. Is P a suitable tracing point for an approximate straight line motion?

With the above proportions of the mechanism, show that P is at the same distance above AD when the coupler is horizontal or vertical.

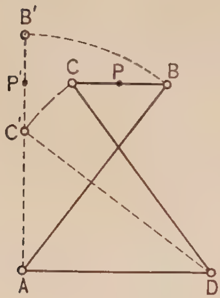


FIG. 132*f*.

133. Relative Time Rates. Let the motion of a vector $\mathbf{u}$ be referred to a rigid body of reference b which is itself in motion. At the instant t let $\mathbf{u}$ coincide with the vector $\mathbf{w}$ fixed in the body. At a later instant $t' = t + \Delta t$, $\mathbf{u}$ becomes $\mathbf{u}'$ and $\mathbf{w}$ becomes $\mathbf{w}'$. Then, since $\mathbf{u} = \mathbf{w}$,

$$\frac{\mathbf{u}' - \mathbf{u}}{\Delta t} = \frac{\mathbf{u}' - \mathbf{w}'}{\Delta t} + \frac{\mathbf{w}' - \mathbf{w}}{\Delta t}.$$

Now pass to the limit $\Delta t \to 0$ in this equation; then

$$\frac{\mathbf{u}' - \mathbf{u}}{\Delta t} \to \frac{d\mathbf{u}}{dt},$$ the absolute rate at which $\mathbf{u}$ is changing,

$$\frac{\mathbf{u}' - \mathbf{w}'}{\Delta t} \to \frac{\partial \mathbf{u}}{\partial t},$$ the rate at which $\mathbf{u}$ is changing relative to b,*

$$\frac{\mathbf{w}' - \mathbf{w}}{\Delta t} \to \frac{d\mathbf{w}}{dt},$$ the absolute rate at which $\mathbf{w}$ is changing.

* Note that $\partial \mathbf{u}/\partial t$ is *not* a partial derivative, but merely a temporary notation to denote a relative time-rate.

Hence

$$(1) \qquad \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \frac{d\mathbf{w}}{dt}.$$

If the body b has plane motion of angular velocity ω,

$$(2) \qquad \frac{d\mathbf{w}}{dt} = \omega\times\mathbf{w} = \omega\times\mathbf{u} \qquad (\S\ 123,\ 1);$$

hence in this case (1) becomes

$$(3) \qquad \frac{d\mathbf{u}}{dt} = \omega\times\mathbf{u} + \frac{\partial \mathbf{u}}{\partial t}.$$

We shall prove later on (§ 213) that (2) holds for the most general motion of the rigid body b. Consequently (3) is valid for any motion of the body of reference.

134. Theorem of Coriolis. We shall now consider the motion of a particle P relative to a rigid body in plane motion with the angular velocity ω and angular acceleration α. Let O_1 and O be origins fixed in space and in the body respectively; then if $\mathbf{r} = \overrightarrow{OP}$,

$$\overrightarrow{O_1P} = \overrightarrow{O_1O} + \overrightarrow{OP} = \overrightarrow{O_1O} + \mathbf{r}.$$

On differentiating this equation with respect to the time we obtain the velocity of P:

$$\mathbf{v} = \mathbf{v}_O + \frac{d\mathbf{r}}{dt} = \mathbf{v}_O + \omega\times\mathbf{r} + \frac{\partial \mathbf{r}}{\partial t}.$$

Here $\partial\mathbf{r}/\partial t$, the time rate at which $\overrightarrow{OP}$ is changing relative to the body, is the relative velocity, $\mathbf{v}_r$, of P; hence

$$(1) \qquad \mathbf{v} = \mathbf{v}_O + \omega\times\mathbf{r} + \mathbf{v}_r.$$

On differentiating again with respect to the time we obtain the acceleration of P:

$$\mathbf{a} = \mathbf{a}_O + \alpha\times\mathbf{r} + \omega\times\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{v}_r}{dt}.$$

But from § 133,

$$\frac{d\mathbf{r}}{dt} = \omega\times\mathbf{r} + \frac{\partial \mathbf{r}}{\partial t} = \omega\times\mathbf{r} + \mathbf{v}_r,$$

$$\frac{d\mathbf{v}_r}{dt} = \omega\times\mathbf{v}_r + \frac{\partial \mathbf{v}_r}{\partial t} = \omega\times\mathbf{v}_r + \mathbf{a}_r,$$

where $\mathbf{a}_r = \partial\mathbf{v}_r/\partial t$ denotes the relative acceleration of P. Therefore

(2) $$\mathbf{a} = \mathbf{a}_O + \boldsymbol{\alpha}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\,\boldsymbol{\omega}\times\mathbf{v}_r + \mathbf{a}_r.$$

The velocity and acceleration of that point of the moving body with which P coincides for the instant are called the *body velocity* and *body acceleration* of P. Denoting these vectors by $\mathbf{v}_b$ and $\mathbf{a}_b$, we have

$$\mathbf{v}_b = \mathbf{v}_O + \boldsymbol{\omega}\times\mathbf{r} \qquad (\S\ 123,\ 2),$$
$$\mathbf{a}_b = \mathbf{a}_O + \boldsymbol{\alpha}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \qquad (\S\ 123,\ 3).$$

These results also follow from (1) and (2) above on putting $\mathbf{v}_r = 0$, $\mathbf{a}_r = 0$. The equations for $\mathbf{v}$ and $\mathbf{a}$ may therefore be written

(1) $$\mathbf{v} = \mathbf{v}_b + \mathbf{v}_r,$$
(2) $$\mathbf{a} = \mathbf{a}_b + 2\,\boldsymbol{\omega}\times\mathbf{v}_r + \mathbf{a}_r.$$

Equation (1) gives the theorem on the composition of velocities, already proved in § 110. Equation (2) shows that an analogous theorem for the composition of accelerations is not in general true; we have in fact the additional term

(3) $$\mathbf{a}_c = 2\,\boldsymbol{\omega}\times\mathbf{v}_r$$

known as the *complementary acceleration* or the *acceleration of Coriolis.* We state (2) as the

THEOREM OF CORIOLIS. *If the motion of a particle is referred to a moving body, its absolute acceleration is equal to the vector sum of the body acceleration, the complementary acceleration, and the relative acceleration:**

$$\mathbf{a} = \mathbf{a}_b + \mathbf{a}_c + \mathbf{a}_r$$

If a particle P moving in a plane α is referred to a body b in plane motion parallel to α, $\boldsymbol{\omega}$ is perpendicular to the plane and therefore to $\mathbf{v}_r$. Then from (3) $\mathbf{a}_c$ is numerically equal to $2\,\omega v_r$ and is turned 90° from $\mathbf{v}_r$ in the sense of b's rotation. Clearly $\mathbf{a}_c = 0$ in two cases only:

(1) $\omega = 0$; the instantaneous motion of b is then a translation;

(2) $v_r = 0$; the particle P is then momentarily at rest relative to b.

* This is the general Theorem of Coriolis for a rigid body of reference having any motion whatever. We have only proved the theorem in the case when the body has plane motion. With a suitable interpretation of $\boldsymbol{\omega}$, however, the above proof applies also to the general case.

Example 1. The motion of a particle P along a plane curve may be referred to a system of rectangular axes b revolving about a fixed origin O, so that the x-axis always passes through P (Fig. 134a). We shall denote the positive unit vectors along these moving axes by R, P (instead of $\mathbf{i}$, $\mathbf{j}$), and instead of x, we shall write $OP = r$. The x-axis thus becomes the r-axis.

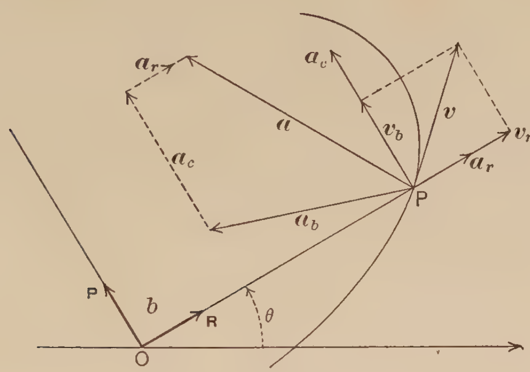


FIG. 134a.

Relative to the moving system of reference b, the particle simply travels along the r-axis. Hence from (§ 112, 1) we have

$$\mathbf{v}_r = \frac{dr}{dt}\,\mathrm{R}, \qquad \mathbf{a}_r = \frac{d^2r}{dt^2}\,\mathrm{R}.$$

The point of the moving system which momentarily coincides with the particle is describing a circle of radius r with the angular velocity $\omega = d\theta/dt$; hence from (§ 109)

$$\mathbf{v}_b = r\omega\mathrm{P}, \qquad \mathbf{a}_b = -\,\omega^2 r\,\mathrm{R} + r\frac{d\omega}{dt}\,\mathrm{P}.$$

Lastly, the complementary acceleration is

$$\mathbf{a}_c = 2\,\boldsymbol{\omega}\times\mathbf{v}_r = 2\,(\omega\mathbf{k})\times\left(\frac{dr}{dt}\,\mathrm{R}\right) = 2\,\omega\frac{dr}{dt}\,\mathrm{P}.$$

The absolute velocity and acceleration of P are therefore

$$\mathbf{v} = \mathbf{v}_b + \mathbf{v}_r = \frac{dr}{dt}\,\mathrm{R} + r\omega\mathrm{P},$$

$$\mathbf{a} = \mathbf{a}_b + \mathbf{a}_c + \mathbf{a}_r = \left(\frac{d^2r}{dt^2} - \omega^2 r\right)\mathrm{R} + \left(r\frac{d\omega}{dt} + 2\,\omega\frac{dr}{dt}\right)\mathrm{P}.$$

We may verify these results by differentiating $\overrightarrow{OP} = r\mathbf{R}$ twice with respect to the time. From § 85 we have

$$\frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{d\theta}\frac{d\theta}{dt} = \omega\mathbf{P}, \qquad \frac{d\mathbf{P}}{dt} = \frac{d\mathbf{P}}{d\theta}\frac{d\theta}{dt} = -\omega\mathbf{R}, \quad \text{and hence}$$

$$\mathbf{v} = \frac{dr}{dt}\mathbf{R} + r\omega\mathbf{P},$$

$$\mathbf{a} = \frac{d^2r}{dt^2}\mathbf{R} + \frac{dr}{dt}\omega\mathbf{P} + \left(\frac{dr}{dt}\omega + r\frac{d\omega}{dt}\right)\mathbf{P} - r\omega^2\mathbf{R}.$$

Example 2. As a numerical illustration of the Law of Coriolis we shall compute the acceleration of the point P of the cylinder *2*, which is constrained to roll over the cylinder *1* by means of the arm *3* connecting

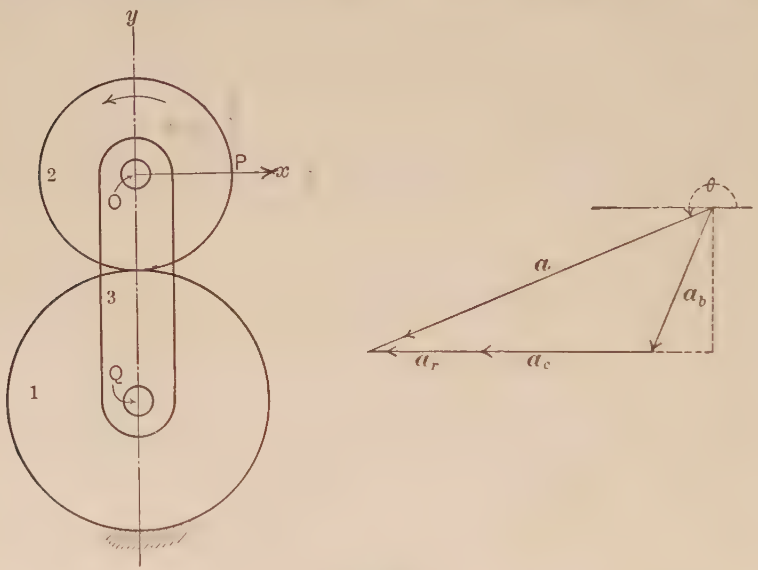


FIG. 134*b*.

their axes (Fig. 134*b*). Assume that the arm is driven counterclockwise with the angular velocity $\omega_3 = 10$ rad./sec.; and that the radii of the cylinders are $r_1 = 7$ in., $r_2 = 5$ in. In § 121, Example 2 we have shown that the angular velocity of *2* is

$$\omega_2 = \frac{r_1 + r_2}{r_2}\omega_3 = \frac{12}{5} \times 10 = 24 \text{ rad./sec.}$$

Hence the angular velocity of *2* relative to the arm is

$$\omega_r = \omega_2 - \omega_3 = 14 \text{ rad./sec.}$$

Choose the rectangular axes Ox, Oy, fixed in *2*, in the position shown. As usual, the positive unit vectors on the axes will be denoted by **i**, **j**.

The point in the plane of the arm that coincides with P has the acceleration

$$\mathbf{a}_b = \omega_3^2 \overrightarrow{PQ} = 100\,(\overrightarrow{PO} + \overrightarrow{OQ}) = 100\,(-5\,\mathbf{i} - 12\,\mathbf{j})$$
$$= -500\,\mathbf{i} - 1200\,\mathbf{j}\ \text{in./sec.}^2$$

The acceleration of P relative to the arm is

$$\mathbf{a}_r = \omega_r^2 \overrightarrow{PO} = 14^2\,(-5\,\mathbf{i}) = -980\,\mathbf{i}\ \text{in./sec.}^2$$

The velocity of P relative to the arm is

$$\mathbf{v}_r = r_2\omega_r\,\mathbf{j} = 5 \times 14\,\mathbf{j} = 70\,\mathbf{j}\ \text{in./sec.}$$

The complementary acceleration is

$$\mathbf{a}_c = 2\,\boldsymbol{\omega}_3 \times \mathbf{v}_r = 2\,(10\,\mathbf{k}) \times (70\,\mathbf{j}) = -1400\,\mathbf{i}\ \text{in./sec.}^2$$

The total acceleration of P is now given by

(i) $$\mathbf{a} = \mathbf{a}_b + \mathbf{a}_c + \mathbf{a}_r = -(500 + 1400 + 980)\,\mathbf{i} - 1200\,\mathbf{j}$$
$$= -2880\,\mathbf{i} - 1200\,\mathbf{j}\ \text{in./sec.}^2$$
$$= -\ \ 240\,\mathbf{i} - \ \ 100\,\mathbf{j}\ \text{ft./sec.}^2$$

Its magnitude is

$$|\mathbf{a}| = \sqrt{240^2 + 100^2} = 260\ \text{ft./sec.}^2,$$

and it makes with the positive x-axis an angle of

$$\theta = \tan^{-1}\frac{10}{24} = 180° + 22°\,37' = 202°\,37'.$$

In the above calculation P was regarded as a point moving in the plane of the arm. We may approach the problem, however, from a different point of view by regarding P as a point fixed in *2*, and computing its acceleration from the equation (§ 130, 2) written in the form

(ii) $$\mathbf{a}_P = \mathbf{a}_O + \mathbf{a}_{PO}.$$

We first compute the acceleration of O, which is revolving about Q with the constant angular velocity ω_3:

$$\mathbf{a}_O = \omega_3^2 \overrightarrow{OQ} = 100\,(-12\,\mathbf{j}) = -1200\,\mathbf{j}\ \text{in./sec.}^2$$

Relative to O, P is moving in a circle of radius $r_2 = 5$ in. with the constant angular velocity of $\omega_2 = 24$ rad./sec. The acceleration of P relative to O is therefore

$$\mathbf{a}_{PO} = \omega_2^2 \overrightarrow{PO} = 24^2\,(-5\,\mathbf{i}) = -2800\,\mathbf{i}\ \text{in./sec.}^2$$

The acceleration of P computed from (ii) thus agrees with our former value.

PROBLEMS

1. A particle is moving with the velocity $\mathbf{v}_r$ and acceleration $\mathbf{a}_r$ relative to a disk which is revolving uniformly about its axis with the angular velocity of ω rad./sec. The path of the particle is along a diameter of the disk. Draw vector diagrams showing $\mathbf{v}_b$, $\mathbf{v}_r$, $\mathbf{v}$, and $\mathbf{a}_b$, $\mathbf{a}_c$, $\mathbf{a}_r$, $\mathbf{a}$ in the following cases:

(*a*) $v_r = 4$, $a_r = 2$, $\omega = 3$ when r (distance from axis) $= 1$;
(*b*) $v_r = 4$, $a_r = 2$, $\omega = -3$ when $r = 1$;
(*c*) $v_r = -4$, $a_r = 2$, $\omega = 3$ when $r = 1$.

2. A particle describes a circle of radius r with the constant angular velocity ω_r relative to a disk which is revolving uniformly about its axis with the angular velocity ω. The center of the circle is on the axis of the disk. Draw vector diagrams as in Problem 1 for the following cases:

(*a*) $r = 2$, $\omega_r = 2$, $\omega = 3$;
(*b*) $r = 2$, $\omega_r = -2$, $\omega = 3$;
(*c*) $r = 2$, $\omega_r = 2$, $\omega = -2$.

3. In Problem 2 show that the absolute acceleration of the particle is directed towards the center, and that its magnitude is $r(\omega + \omega_r)^2$.

4. A train is running from west to east in the latitude of the equator with the speed of 60 mi./hr. Find the magnitude of its absolute acceleration (neglecting the orbital motion of the earth). What percentage of this is due to $\mathbf{a}_c$? to $\mathbf{a}_r$? Take the radius of the earth as 4000 miles.

What is the absolute acceleration when the train is running from east to west?

135. Instantaneous Center in Relative Motion. We now consider two plane figures, a and b, moving in a plane regarded as fixed; and for the sake of simplifying our statements, we shall imagine both a and b attached to infinite planes that share their motion. Let O, P be any two points of a and O', P' the points of b with which they coincide at a certain instant. Then if ω_a and ω_b are the angular velocities of a and b, the velocities of P and P' are, from (§ 123, 2),

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega}_a \times \overrightarrow{OP}, \quad \mathbf{v}_{P'} = \mathbf{v}_{O'} + \boldsymbol{\omega}_b \times \overrightarrow{OP}.$$

If these velocities are equal,

$$\mathbf{v}_O - \mathbf{v}_{O'} + (\boldsymbol{\omega}_a - \boldsymbol{\omega}_b) \times \overrightarrow{OP} = 0, \quad \text{or} \quad \mathbf{k} \times \overrightarrow{OP} = -\frac{\mathbf{v}_O - \mathbf{v}_{O'}}{\omega_a - \omega_b}.$$

When $\omega_a \neq \omega_b$ this equation locates a pair of coincident points in a and b that have the same velocity at the instant considered. The velocity of each point relative to the figure that carries the other is therefore zero (§ 110). The point of a is called the instantaneous center of a relative to b, and is denoted by I_{ab}. Similarly, the point of b which coincides with I_{ab} is called the instantaneous center of b relative to a, and is denoted by I_{ba}. On multiplying the last equation by $\mathbf{k}\times$ and replacing P by I_{ab} we obtain

$$(1) \qquad \overrightarrow{OI_{ab}} = \frac{\mathbf{k}\times(\mathbf{v}_O - \mathbf{v}_{O'})}{\omega_a - \omega_b}.$$

When $\omega_a = \omega_b$ the figures are either relatively at rest, or their relative motion is a pure translation. In the latter case their relative instantaneous center is said to be at infinity.

136. Kinematic Chains. A *chain* is an assemblage of bodies coupled in such a manner that each body is capable of motion relative to its neighbors. A *kinematic chain* is a chain in which the relative motion of any two of the bodies determines the relative motion of all of the others. Such motion is described as *completely constrained.* When one body of a kinematic chain is fixed in position and a certain motion is impressed upon a second, the motions of the remaining bodies are perfectly definite.

The several bodies forming a kinematic chain are called its *links;* and the portions of two connecting links that form the articulation between them are called a *kinematic pair of elements.* The various links of a kinematic chain will be denoted by small italic letters (a, b, etc.) or by numerals; and we shall speak of the pair of elements connecting the links a and b as the elements ab.

In the study of kinematic chains it is convenient to classify pairs of kinematic elements according to the nature of their contact. Thus a pair of elements having surface contact is called a *lower pair,* and a pair having line contact, a *higher pair.* For example shaft and bearing, or cylinder and piston, form a lower pair; while a pair of gear wheels or cams in driving contact form a higher pair.

We shall only deal with kinematic chains constrained so that all the links move parallel to a fixed plane. The lower pairs may then be of two kinds: *turning pairs* or *sliding pairs.* In a turning pair the contact is over surfaces of revolution, commonly cylinders; and the relative motion of the elements is one of pure rotation. In a sliding pair the contact is over plane or cylindrical

surfaces* so arranged that the only possible relative motion of the elements is a pure translation. Thus shaft and bearing form a turning pair; while piston and cylinder, or cross-head and guides, form a sliding pair.

The *relative* motion of any two links p, q of a kinematic chain is, by definition, perfectly definite, and will not be altered if any link of the chain is fixed. Hence the position of the instantaneous center I_{pq} (of p relative to q) is not altered by fixing any one link of the chain. In particular it is permissible to regard q as fixed in locating I_{pq}.

If the elements pq form a turning pair, I_{pq} lies on the common axis of their contact surfaces of revolution; for all points of this axis have the same velocity whether considered as points of p or as points of q. If the elements pq form a sliding pair, I_{pq}, strictly speaking, does not exist. In this case, however, we shall say that I_{pq} is at infinity in a direction perpendicular to the relative translation of p and q.

We shall now consider two important kinematic chains.

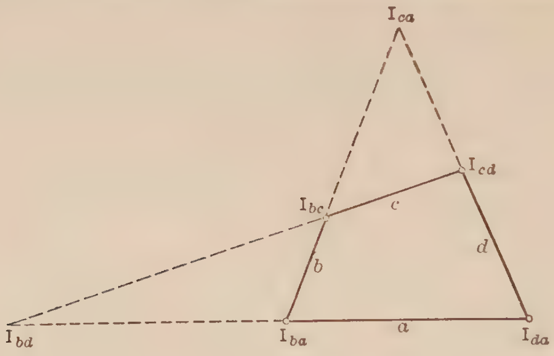


FIG. 136a.

The Four-bar Chain. This chain, shown conventionally in Fig. 136a, consists of four links a, b, c, d, connected by turning pairs

* A cylindrical surface is generated by a straight line which moves parallel to itself and always intersects a fixed curve. A plane may thus be regarded as a cylindrical surface for which the fixed curve is a straight line. When the contact surfaces of a sliding pair are other than planes, the relative motion of the elements is parallel to their generating lines.

Note that if a block c slides in a slot formed in a body d by the surfaces of two concentric circular cylinders, the elements cd form a turning pair, *not* a sliding pair (see Fig. 137).

ab, *bc*, *cd*, *da*. The axes of these pairs are all parallel, so that the relative motion of the links is plane. The relative instantaneous center of two directly connected links is on the axis of their turning pair. To locate the relative instantaneous center of two links not directly connected, such as *a* and *c*, regard *a* as fixed; the points I_{cb} and I_{cd} of *c* then move normal to the center lines of *b* and *d* respectively and I_{ca} is at the intersection of these lines. Similarly I_{bd} is the point of intersection of the center-lines of *a* and *c*.

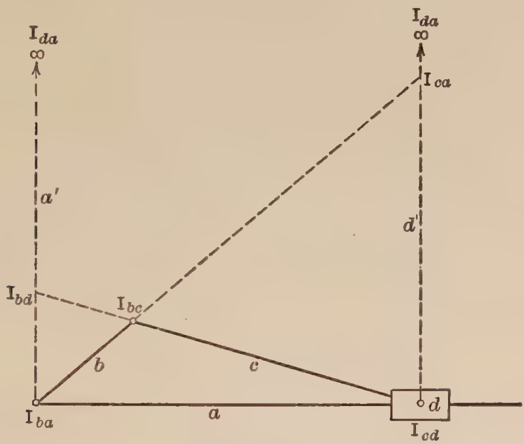


FIG. 136*b*.

The Slider-crank Chain. This chain (Fig. 136*b*) consists of four links *a*, *b*, *c*, *d*, connected by three turning pairs *ab*, *bc*, *cd*, and a sliding pair *da*. The instantaneous centers I_{ba}, I_{bc}, I_{cd} are situated on the axes of the corresponding pairs, while I_{da} is at infinity in the direction perpendicular to the relative motion of *d* and *a*. To locate I_{ca} regard *a* as fixed; then I_{ca} is the point of intersection of the center-line of *b* with the line through I_{cd} perpendicular to the direction of *d*'s motion (§ 125, Ex. 2). To locate I_{bd} regard *d* as fixed; then *c* revolves about the permanent center I_{cd}, and *a* is constrained to slide in a fixed direction. We therefore know the directions in which I_{bc} and I_{ba} are moving. The normals to these directions at I_{bc} and I_{ba} meet in the point I_{bd}.

In the figure the line I_{cd} I_{ba} gives the direction of relative translation. This arrangement of the chain, which is usually adopted in practice, is not one of its essential characteristics. When the

line in which I_{cd} moves relative to a does not pass through I_{ba}, the chain is said to be *crossed*.

The slider-crank chain may be regarded as a limiting case of the four-bar chain in which two of the links, a and d, are infinitely long. Thus the slider-crank chain in the figure is equivalent to the four-bar chain a', b, c, d', in which a' and d' are connected at infinity. If we adopt this point of view, the constructions given above for the four-bar chain supply all of the instantaneous centers.

137. The Criterion of Constraint. We shall now consider the conditions under which a chain, *in which lower pairs only occur*, is completely constrained. For the sake of simplicity we shall assume that all of the pairs are turning pairs. Each joint of the chain then corresponds to a point which represents its axis; and if there are j joints in the chain with distinct axes, they will be represented by j distinct points $P_1, P_2, \ldots, P_j$. To fix the relative position of these points we may take one point P_1 as the origin of a system of rectangular axes, and pass the x-axis through a second point P_2. The relative position of the j points will then be determined by giving the abscissa of P_2 and both coördinates of the $j - 2$ remaining points. Thus

$$C = 1 + 2(j - 2) = 2j - 3 \tag{1}$$

conditions are required to fix the relative position of j points or of the j joints which they represent (cf. § 56).

We shall now find the number of conditions imposed on these j joints by the rigidity of the links on which they lie. For this purpose the links of the chain are classified according to the number of joint elements that each contains. Thus suppose that there are n_2 links with 2 elements, n_3 with 3, and, in general, n_k with k elements. Since the elements on any link are represented by distinct points, we see from the above that each link with k elements imposes $2k - 3$ conditions on these elements to fix their relative positions. For example, a link with 2 elements imposes $4 - 3 = 1$ condition on them, namely, that the distance between them remain constant. Hence the n_k links with k elements impose $(2k - 3)n_k$ conditions; and the total number of conditions imposed on the joints by all of the links forming the chain is given by

$$C' = \sum(2k - 3)n_k = n_2 + 3n_3 + 5n_4 + 7n_5 + \cdots. \tag{2}$$

The difference $C - C'$ indicates the number of additional conditions required to fix the relative positions of the joints and thus determine the form of the chain.

If $C - C' = 0$, the form of the chain is prescribed and its links can not be set in relative motion. The chain is then said to be *locked.*

If $C - C' = 1$, one extra condition will determine the form of the chain. Hence, if one link is fixed and a movable joint P is placed in a definite position, the extra condition will be imposed and the form of the chain determined. This is precisely the case of completely constrained motion. We have then a *kinematic chain.*

If $C - C' \geqq 2$, two or more extra conditions are required to determine the form of the chain. If one link is fixed as before, and a movable joint placed in a definite position, the form of the chain is still undetermined. In this case the chain is said to be *loose* or *unconstrained* and obviously can not be used as a mechanism.

A kinematic chain must therefore satisfy the *Criterion of Constraint* $C' = C - 1$, that is

$$\sum (2k - 3)n_k = 2j - 4. \tag{3}$$

As a general rule, the criterion (3) also applies to kinematic chains containing sliding pairs. This is due to the fact that the straight line representing a sliding pair is determined by two conditions (for example its inclination to the x-axis and its x-intercept) just as the point representing a turning pair. The criterion, however, does not apply to certain chains that contain sliding pairs — for example, when closed circuits of sliding pairs are present.* These exceptional cases occur rather infrequently and we shall not stop to consider them here.†

If n denotes the total number of links in the chain, we have

$$n = \sum n_k.$$

Moreover if just two links are coupled at each joint, the number of joint elements is twice the number of joints: $2j = \sum kn_k$. In

* Thus a kinematic chain may be formed by three links coupled by sliding pairs. In this case we have $C = 3$, $C' = 3$, and the chain is apparently locked. Actually, the chain is constrained.

† See Klein, A. W., *Kinematics of Machinery*, p. 22 et seq.

this particular case (3) assumes the simple form

$$4j - 3n = 2j - 4, \text{ or}$$

(4) $$3n = 2j + 4 \quad \text{(all binary joints).}$$

Example 1. Let us apply the Criterion of Constraint to the chain of Fig. 144*d*. Here we have 10 joints, 4 links with 2 joint elements, and 4 links with 3 joint elements. In our notation

$$j = 10, \quad n_2 = 4, \quad n_3 = 4;$$

hence

$$C = 2 \times 10 - 3 = 17, \quad C' = 4 + 3 \times 4 = 16.$$

Since $C - C' = 1$, the chain is constrained.

Example 2. In the mechanism of Fig. 139*i* we have $j = 10$, $n_2 = 5$ (links *3, 4, 6, 7, 8*), $n_3 = 2$ (links *1, 2*), $n_4 = 1$ (link *5*); hence

$$C = 2 \times 10 - 3 = 17, \quad C' = 5 + 3 \times 2 + 5 \times 1 = 16,$$

and the chain is constrained.

Since this chain has only binary joints, the simpler criterion (4) may be applied: $3 \times 8 = 2 \times 10 + 4$ This also applies in Example 1.

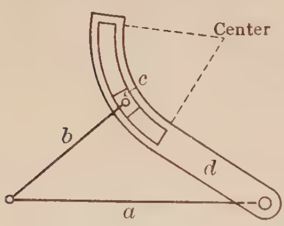


FIG. 137.

PROBLEMS

1. Test the chains of Figs. 139*f*, 139*g* and 139*h* for constraint.

2. Sketch a constrained chain consisting of three links coupled by sliding pairs.

3. Locate all of the relative instantaneous centers for the chain shown in Fig. 137.

4. If a chain satisfies the criterion (3), show that the number of its links is necessarily even.

5. Show that the criterion (4) may be applied to chains in which joints involving three or more links occur, provided that every joint at which i links are coupled is reckoned as $i - 1$ binary joints.

138. Inversion. When one of the links of a kinematic chain is fixed (relative to the earth or other body of reference, as a boat or locomotive) the chain is called a *mechanism* or a *machine*. The term *mechanism* is usually employed when the relative motions of the links are of principal importance, the term *machine* when the forces transmitted or the energy transformations are being considered.

By fixing in turn each of the links of a kinematic chain we obtain as many mechanisms as there are links in the chain. The various mechanisms thus formed are called the *inversions* of the chain;

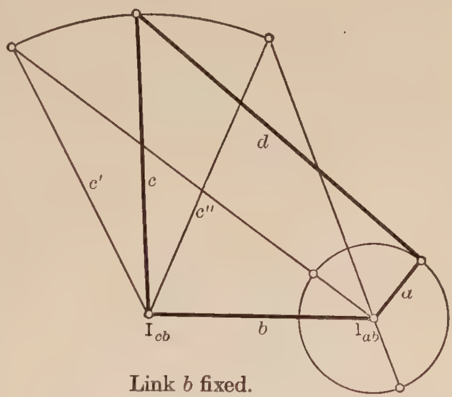
they may or may not be distinct mechanisms. The relative motions of any two links of a kinematic chain are the same in each of its inversions. Moreover, in any given configuration of a chain, the positions of the relative instantaneous centers of the various pairs of links are the same irrespective of the particular link which is fixed.

Inversions of the Four-bar Chain. Let us consider a four-bar chain in which the lengths of the links a, b, c, d are such that

$$l_a < l_b < l_c < l_d, \qquad \text{and} \qquad l_a + l_d < l_b + l_c.$$

Here l_a denotes the length of the link a, that is, the distance between the axes of the turning pairs ab, ad. In the figure l_a, l_b, l_c, l_d are proportional to 1, 3, 4, 5 respectively.

By fixing in turn each of the links we obtain the four inversions of this chain. In these mechanisms a link directly connected to the fixed link is called a *crank* if it is capable of complete revolutions, a *lever*, if it can only oscillate between certain limiting positions. The link not directly connected to the fixed link is usually called the *coupler.*



Link b fixed.

FIG. 138*a*.

1. When b is fixed (Fig. 138a) the link a is capable of complete revolutions, but c can only oscillate between certain limiting positions c' and c''. These positions are determined by the points of intersection of circles of radii $l_d - l_a$ and $l_d + l_a$ about I_{ab} as center with a circle of radius l_c about I_{cb}. Thus c is a lever and a is a crank; and the mechanism is therefore called a *lever-crank mechanism.*

Note also that *relative to the link d, a* performs complete revolutions, while c merely swings in partial revolutions. The relative motions of the directly connected links are therefore as follows:

a makes complete revolutions relative to b or d;

c makes partial revolutions relative to b or d.

These *relative* motions remain the same in each of the inversions of the chain.

2. When d is fixed we see from the relative motions listed above that a is again a crank, c a lever, and we obtain a second lever-crank mechanism.

3. When a is fixed both b and d are capable of complete revolutions about their permanent centers; for the relative motions of b and a and of d and a must be the same as in case 1. The links b, d are both cranks, and we have a *double-crank mechanism.*

4. When c is fixed both b and d are limited to oscillations about their permanent centers and are therefore levers. The chain now becomes a *double-lever mechanism.*

Inversions of the Slider-crank Chain. By fixing in turn each of the links a, b, c, d of the slider-crank chain (Fig. 136b), four different mechanisms are formed.

1. When a is fixed the mechanism is that of the ordinary direct-acting engine; a represents the bed-plate and its rigid connections, such as cylinder and cross-head guides; b, the crank and shaft;

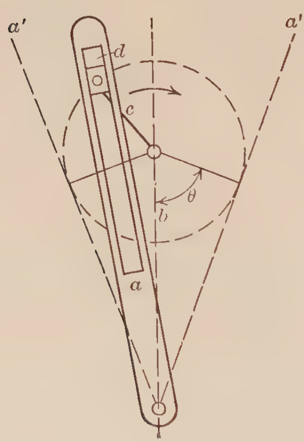


Fig. 138b.

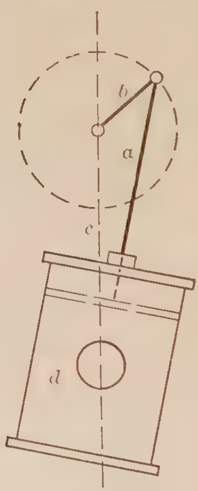


Fig. 138c.

c, the connecting-rod; and d, the cross-head, piston-rod and piston. This inversion may be regarded as a *mechanism* for transforming the reciprocating motion of the piston into the continuous rotation of the crank-shaft, or as a *machine* for transforming energy.

2. When b is fixed the resulting mechanism is exemplified by the *crank and slotted lever* (Fig. 138*b*), utilized in some quick return motions. The block d slides in a straight slot in the link a, which oscillates to and fro between the positions a' and a'' for every revolution of the crank c. If c is driven with a uniform angular velocity ω, the forward and backward oscillations of a are performed in the times $2(\pi - \theta)/\omega$ and $2\theta/\omega$ respectively, where $\theta = \cos^{-1}(l_c/l_b)$.

3. When c is fixed the mechanism is exemplified by the *oscillating cylinder engine* (Fig. 138*c*). The block d has suffered a change "from solid to hollow" and is now the cylinder of the engine mounted upon trunnions. The link a consists of the piston and piston-rod.

4. The mechanism formed when d is fixed has found but slight application in practice. It has been employed however in a small steam pump, known as the *pendulum pump*. For a description of this pump the reader is referred to Durley, *Kinematics of Machines*, § 40.

PROBLEMS

1. When the sum of the longest and shortest links of a four-bar chain is less than the sum of the other two, prove that the chain becomes (1) a double–crank mechanism when the shortest link is fixed; (2) a lever-crank mechanism when either link adjoining the shortest is fixed; (3) a double–lever mechanism when the link opposite the shortest is fixed.

2. When the sum of the longest and shortest links of a four-bar chain is greater than the sum of the other two, prove that all the inversions of the chain give double-lever mechanisms.

3. The kinematic chain shown in Fig. 119*b* is called a *double slider-crank chain*. Show that it may be regarded as a slider-crank chain in which the connecting-rod is infinitely long. Locate the six relative instantaneous centers of its links.

When the link d is fixed show that the resulting mechanism is essentially a trammel (§ 126, Example).

4. If in Fig. 119*b* the link a is fixed, what curve will a point on the center-line of the crank b trace on a plane attached to d?

5. Study the inversion of the double slider-crank chain formed by fixing the link b (Oldhams's coupling; the elliptic chuck). Reference: Durley, *Kinematics of Machines*, § 44; Dunkerley, *Mechanism*, §§ 99, 139.

6. The *crossed-slide chain* consists of four links coupled by two turning and two sliding pairs, each link having an element of a turning and a

sliding pair. The mechanism of Fig. 142c is derived from a crossed-slide chain by fixing the link d (a plate with a rectangular slot in its face). Locate all of the relative instantaneous centers of its links.

In what essential respect does the crossed-slide chain differ from double slider-crank chain?

7. The *Scott Russell* parallel motion may be obtained as follows: in the slider-crank mechanism of Fig. 131b take $OA = AB$, and let C be a point on the prolongation of the connecting-rod such that $AC = AB$. Show (by use of the instantaneous center of the connecting-rod) that C will describe a straight line.

139. The Theorem of Three Centers. Suppose that a, b, c are three figures moving in a fixed plane; and let the instantaneous centers I_{ab}, I_{bc}, I_{ca} be all distinct at the instant considered (Fig. 139a). Then, by the defining property of relative instantaneous centers (§ 135), the coincident points I_{ab} and I_{ba}, in a and b respectively, have the same velocity relative to c. But, relative to c, these points are moving for the instant about the instantaneous centers I_{ac} and I_{bc} respectively; their common velocity vector is consequently perpendicular to *both* instantaneous radii $I_{ac}I_{ab}$ and $I_{bc}I_{ba}$. This can only be the case when the points I_{ab}, I_{ac}, I_{bc} lie in the same straight line. We have thus proved the important property known as the

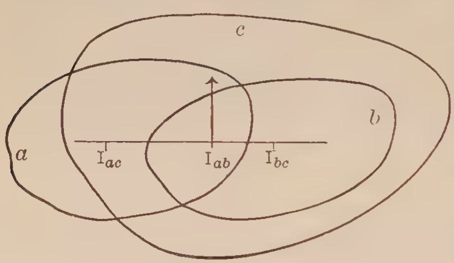


FIG. 139a.

THEOREM OF THREE CENTERS. *If three figures a, b, c are moving in a plane, the relative instantaneous centers of the three pairs ab, bc, ca, in any configuration, lie in the same straight line.*

This theorem may also be established analytically. Choose, at the instant considered, three coincident points O, O', O'' in a, b, c. Then from (§ 135, 1)

$$(\omega_a - \omega_b)\,\overrightarrow{OI}_{ab} + (\omega_b - \omega_c)\,\overrightarrow{OI}_{bc} + (\omega_c - \omega_a)\,\overrightarrow{OI}_{ca} = \mathbf{k}\times(\mathbf{v}_O - \mathbf{v}_{O'} + \mathbf{v}_{O'} - \mathbf{v}_{O''} + \mathbf{v}_{O''} - \mathbf{v}_O) = 0.$$

Since the sum of the scalar coefficients in the left member is zero, the points I_{ab}, I_{bc}, I_{ca} lie in a straight line (§ 7, Example 2).

Example 1. The six instantaneous centers of the four-bar chain (Fig. 136*a*) lie in sets of three upon four straight lines. It will be observed that the subscripts of each set of three upon a line refer to but three of the four links.

Example 2. The six instantaneous centers of the slider-crank chain (Fig. 136*b*) lie in sets of three upon four straight lines if we regard the parallel lines $I_{ba}I_{bd}$, $I_{cd}I_{ca}$ as meeting at infinity in the point I_{ad}.

Any four plane figures a, b, c, d have

$${}_4C_2 = \frac{4 \cdot 3}{1 \cdot 2} = 6\,*$$

relative instantaneous centers, namely

$$I_{ab},\ I_{bc},\ I_{cd},\ I_{da},\quad I_{ac},\ I_{bd}$$

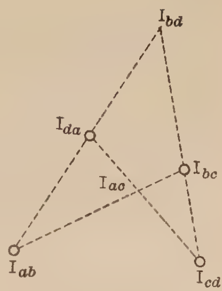


Fig. 139*b*.

If the first four of these are known (note the cyclical order in the subscripts) the other two may be found by means of the Theorem of Three Centers. For I_{ac} lies on the lines $I_{ab}I_{bc}$ and $I_{cd}I_{da}$, and is therefore at their point of intersection (Fig. 139*b*). Similarly I_{bd} is at the point of intersection of the lines $I_{bc}I_{cd}$ and $I_{da}I_{ab}$. We shall indicate these constructions by the scheme:

$$I_{ac}\begin{cases} I_{ab}I_{bc} \\ I_{cd}I_{da} \end{cases}, \qquad I_{bd}\begin{cases} I_{bc}I_{cd} \\ I_{da}I_{ab} \end{cases}.$$

We shall call any group of four instantaneous centers whose subscripts may be arranged in cyclical order a *four-cycle;* and we shall denote a four-cycle such as

$$I_{ab},\ I_{bc},\ I_{cd},\ I_{da},\ \text{by } (abcd).$$

The results of the preceding paragraph may now be stated as follows:

If four plane figures have a known four-cycle, their remaining instantaneous centers may be determined by the Theorem of Three Centers.

The above method of finding instantaneous centers from a four-cycle may be remembered as follows. For example, to find I_{bd} from the four-cycle ($abcd$), pass from b to d, then from d to b, following the cyclical order $abcd$:

$$\overset{\frown}{b\,c}\overset{\frown}{c\,d}\,\underset{\smile}{d\,a}\underset{\smile}{a\,b}.$$

* ${}_nC_m$ denotes the number of combinations of n things taken m at a time.

The two routes bc, cd and da, ab indicate the subscripts of the centers whose joining lines $I_{bc}I_{cd}$, $I_{da}I_{ab}$ determine I_{bd}. A similar process will locate I_{ac}: $\quad a\ b\ c\ d\ a.$

Six plane figures a, b, c, d, e, f have ${}_6C_2 = 6 \cdot 5/1 \cdot 2 = 15$ relative instantaneous centers. If two four-cycles involving together all six of the figures are known, all the remaining instantaneous centers may be found by the Theorem of Three Centers.

For example, suppose that the 7 instantaneous centers of the two four-cycles

$$(abcd), \quad (abef)$$

are known. We may determine I_{ac}, I_{bd} from the first cycle and I_{ae}, I_{bf} from the second as shown above; 11 instantaneous centers are then known. From these we now form four other known four-cycles,

$$(acbe), \quad (acbf), \quad (adbe), \quad (adbf),$$

by alternating the two letters a, b common to the given cycles with two letters *not* common to them. Each of these cycles determines one of the remaining instantaneous centers, namely that one with the letters other than a, b as subscripts; taken in order, these are I_{ce}, I_{cf}, I_{de}, I_{df}. These centers together with the 11 already known make up the total number of 15.

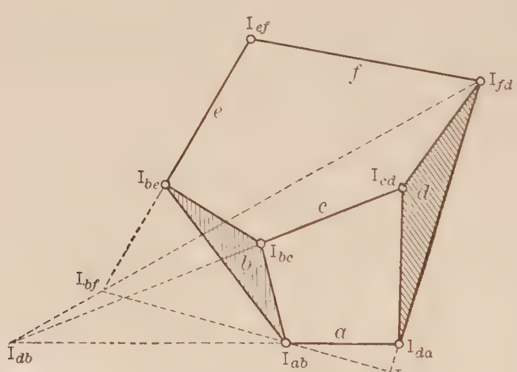


FIG. 139c.

Sometimes only one four-cycle $(abcd)$ is given directly, together with three additional centers such as I_{be}, I_{ef}, I_{fd}. Then I_{db} must first be determined from the given cycle before the second cycle

(*befd*) can be regarded as known. This case is exemplified in the chain shown diagrammatically in Fig. 139*c*. If I_{fa} is required, we may proceed as follows. From (*abcd*) we locate

$$I_{db}\begin{cases} I_{da}I_{ab} \\ I_{bc}I_{cd} \end{cases}.$$

The four-cycle (*befd*) is now completely determined, and

$$I_{bf}\begin{cases} I_{be}I_{ef} \\ I_{fd}I_{db} \end{cases}.$$

Finally we form a four-cycle by alternating the letters b, d common to (*abcd*) and (*befd*) with the letters f, a, forming the subscripts of the required instantaneous center, I_{fa}. In this cycle (*bfda*) we locate

$$I_{fa}\begin{cases} I_{fd}I_{da} \\ I_{ab}I_{bf} \end{cases}.$$

Example 3. The mechanism of the *Crosby Indicator* is shown conventionally in Fig. 139*d*. This consists of six links: a, b, c, d, e, f; a is the fixed indicator cylinder, and d consists of the indicator piston and piston-rod. The point P on the link f traces the indicator diagram.

Suppose that we wish to find the direction of P's motion in the position shown. This can be given at once when I_{fa} is known; we therefore proceed to find this instantaneous center.

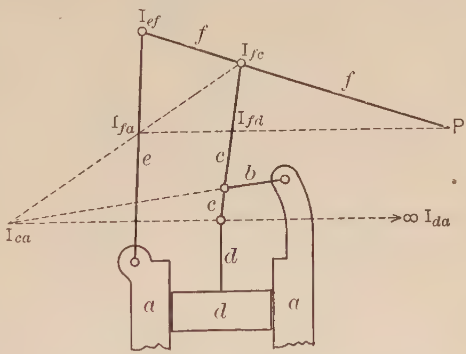


FIG. 139*d*.

The slider-crank chain a, b, c, d gives the known four-cycle (*abcd*). Since the centers I_{ae}, I_{ef}, I_{fc} are also given, we shall have a second known four-cycle (*aefc*) after locating

$$I_{ca}\begin{cases} I_{cd}I_{da} \\ I_{ab}I_{bc} \end{cases}.$$

(Note that I_{da} is at infinity in the direction normal to d's motion.) From the cycle (*aefc*) we now determine

$$I_{fa}\begin{cases} I_{fc}I_{ca} \\ I_{ae}I_{ef} \end{cases}.$$

The instantaneous motion of P is normal to the line $I_{fa}P$.

Let v_P and v_d denote the speeds of the tracing point and of the piston d respectively. In order to find the ratio v_P/v_d we shall next locate I_{fd}. With this in view, form the four-cycle ($fadc$) by alternating the letters a, c, common to ($abcd$) and ($aefc$), with f, d; then

$$I_{fd}\begin{cases} I_{fa}I_{ad} \\ I_{dc}I_{cf} \end{cases}.$$

Now I_{fd} is the point of f which has the same velocity as the piston d. Hence if ω_f denotes the angular velocity of f, we have

$$v_P = \omega_f \cdot I_{fa}P, \qquad v_d = \omega_f \cdot I_{fa}I_{fd}, \qquad \frac{v_P}{v_d} = \frac{I_{fa}P}{I_{fa}I_{fd}}.$$

Example 4. The *Peaucellier Cell* (Fig. 139*e*) is the most familiar mechanism for generating a true straight line. This consists of eight links, including the fixed link a, with pin connections that allow of plane motion. The links e f, g, h are all of equal length and form a rhombus in all po-

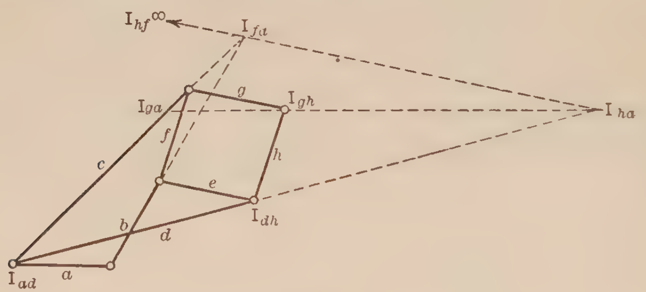


FIG. 139*e*.

sitions. The links c, d are equal, as are also the remaining links a, b. When the link b revolves about its fixed pin, the point I_{gh} describes a straight line perpendicular to the center line of a. This property may be verified graphically by showing that I_{ha} (or I_{ga}) always lies on a line through I_{gh} parallel to the center line of a.*

In order to locate I_{ha} consider the known four-cycles ($abfc$) and ($efgh$):

$$\text{from } (abfc), \quad I_{fa}\begin{cases} I_{fc}I_{ca} \text{ (center line of } c) \\ I_{ab}I_{bf} \text{ (center line of } b) \end{cases};$$

$$\text{from } (efgh), \quad I_{hf}\begin{cases} I_{he}I_{ef} \text{ (center line of } e) \\ I_{fg}I_{gh} \text{ (center line of } g) \end{cases}.$$

* For a geometric proof of this property see McKay, *The Theory of Machines*, (London, 1915) p. 161.

The cycle $(adhf)$ is now known; therefore

$$I_{ha}\begin{cases} I_h I_{fa} \text{ (parallel to } g) \\ I_{ad} I_{dh} \text{ (center line of } d). \end{cases}$$

To locate I_{ga} consider the cycle $(acgh)$; thus

$$I_{ga}\begin{cases} I_{gh} I_{ha} \\ I_{ac} I_{cg} \text{ (center line of } c). \end{cases}$$

PROBLEMS

1. Locate I_{62} in the toggle-press mechanism shown in Fig. 142*b*. Show how the speed of the block *6* may be found when the speed of *A* is known.

2. Give a series of four-cycles that will determine the location of the instantaneous centers I_{35}, I_{71}, I_{64}, I_{32} of the chain of Fig. 139*f*.

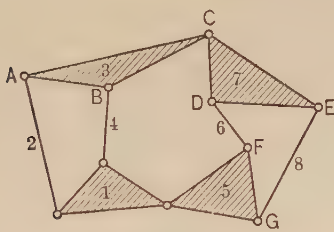


FIG. 139*f*.

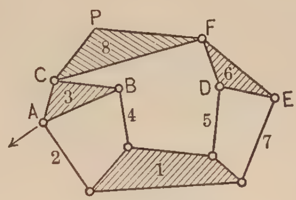


FIG. 139*g*.

3. In the mechanism of Fig. 139*g* the link *1* is fixed. Find graphically the direction in which the point *P* is moving when *A* moves in the direction shown.

4. Bricard's linkage for an exact straight line motion consists of six links arranged as in Fig. 139*h* and proportioned as follows:

$$AE = DF = x, \quad GE = GF = y, \; AD = z,$$

$$BE = CF = \frac{y^2}{x}, \; BC = \frac{yz}{x}.$$

(Note that *BE* and *CF* are prolongations of the links *2* and *4*.) When the link *1* is fixed, the point *G* will describe a segment of the straight line bisecting *AD* at right angles. Test this property by finding I_{61} for two positions of the linkage, taking $x = 2''$, $y = 1''$, $z = 3''$.

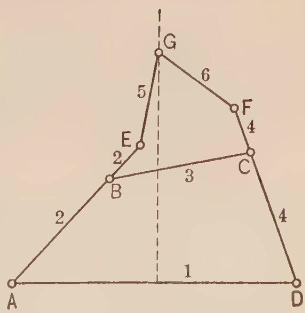


FIG. 139*h*.

5. Show that besides I_{13} and I_{24}, no additional instantaneous centers of the chain of Fig. 144*d* can be found by means of the Theorem of Three Centers.

6. Find the center I_{82} in the *Stephenson Link* mechanism shown

schematically in Fig. 139*i*. Show how the speed of the valve *8* may be found when ω_2 is known.

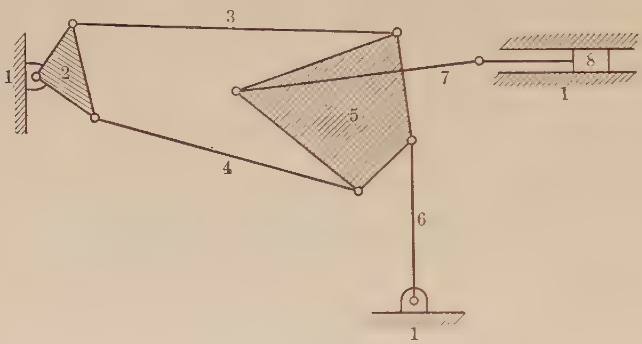


FIG. 139*i*.

140. Relative Angular Velocity. Let a and b be two plane figures in plane motion and l_a, l_b, directed lines in these figures (Fig. 140*a*). Then the time rate of change of the angle (l_a, l_b) is called the *angular velocity of b relative to a*, and is denoted by ω_{ba}; in symbols,

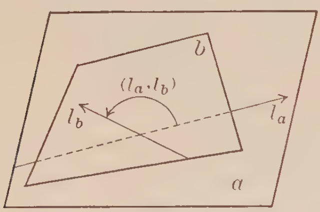


FIG. 140*a*.

(1) $$\omega_{ba} = \frac{d}{dt}(l_a, l_b).$$

Just as in § 121 we may show that the value of ω_{ba} is independent of the choice of l_a and l_b.

Since $(l_b, l_a) = -(l_a, l_b)$, we have

(2) $$\omega_{ab} = \frac{d}{dt}(l_b, l_a) = -\frac{d}{dt}(l_a, l_b) = -\omega_{ba}.$$

If c is a third figure moving in the same plane as a and b, and l_c is a directed line in c, then

$$(l_a, l_b) = (l_a, l_c) + (l_c, l_b) = (l_c, l_b) - (l_c, l_a).$$

Upon differentiation with respect to the time, these equations give

(3) $$\omega_{ba} = \omega_{bc} + \omega_{ca} = \omega_{bc} - \omega_{ac}.$$

The relations (3) may be readily remembered by noting that the double subscripts follow the same order as if they denoted vectors.

Now let $a, b, \ldots, p, q$ denote the links of a kinematic chain, and suppose that the angular velocities of all the links relative to

a certain link a are known. Then the angular velocities of the links relative to another link, as b, may be computed by means of (3); thus $\omega_{pb} = \omega_{pa} - \omega_{ba}$, etc. Again, if the *ratios* of the angular velocities of the links relative to a link a are known, the ratios of the angular velocities relative to another link b may also be obtained; for example:

$$\text{(4)} \qquad \frac{\omega_{pb}}{\omega_{qb}} = \frac{\omega_{pa} - \omega_{ba}}{\omega_{qa} - \omega_{ba}} = \frac{\dfrac{\omega_{pa}}{\omega_{ba}} - 1}{\dfrac{\omega_{qa}}{\omega_{ba}} - 1}.$$

This calculation may be exhibited in tabular form as follows:

	Link a	Link b	$\cdots$	Link p	Link q
(a fixed)	0	ω_{ba}	$\cdots$	ω_{pa}	ω_{qa}
(b fixed)	$-\omega_{ba}$	0	$\cdots$	$\omega_{pa} - \omega_{ba}$	$\omega_{qa} - \omega_{ba}$.

The angular velocities of the various links relative to a are given in the first row, a being considered at rest. Now if the same quantity is added to or subtracted from each angular velocity it is clear that the *relative* angular velocities of the links remain unaltered, a condition that must always be fulfilled in any kinematic chain. Hence to obtain all the angular velocities relative to b, we may reduce b to rest by subtracting ω_{ba} from every entry in the first row, as shown in the second row. The *ratio* of any two of the quantities in the second row then gives the angular velocity ratio of the corresponding links when b is fixed; thus

$$\frac{\omega_{pb}}{\omega_{qb}} = \frac{\omega_{pa} - \omega_{ba}}{\omega_{qa} - \omega_{ba}}, \qquad \frac{\omega_{pb}}{\omega_{ab}} = \frac{\omega_{pa} - \omega_{ba}}{-\omega_{ba}} = 1 - \frac{\omega_{pa}}{\omega_{ba}}.$$

Equation (4) fails when $\omega_{ba} = 0$, that is, when the relative motion of a and b is a translation. In this case, however,

$$\text{(5)} \qquad \omega_{pb} = \omega_{pa} - \omega_{ba} = \omega_{pa}, \qquad \omega_{qb} = \omega_{qa} - \omega_{ba} = \omega_{qa}.$$

Example 1. Let *1* and *2* be two friction wheels having parallel shafts mounted upon a fixed link *0* (Fig. 140*b*). If the wheels roll without slipping, their points of contact have the same numerical speed v. Since *1* and *2* revolve in opposite directions, we have

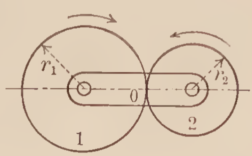


FIG. 140*b*.

$$\frac{\omega_{20}}{\omega_{10}} = -\frac{v/r_2}{v/r_1} = -\frac{r_1}{r_2}.$$

Now suppose that the wheel *1* is fixed and that the link *0* revolves about *1*'s shaft. Then *2* will roll over *1*, and the ratio ω_{21}/ω_{01} may be obtained by expressing ω_{21} and ω_{01} in terms of angular velocities relative to *0*. We thus obtain

$$\frac{\omega_{21}}{\omega_{01}} = \frac{\omega_{20} - \omega_{10}}{-\omega_{10}} = 1 - \frac{\omega_{20}}{\omega_{10}} = 1 + \frac{r_1}{r_2},$$

in agreement with the result of § 121, Example 2.

If *1* and *2* represent two spur gears that transmit the same angular velocity ratio as the friction wheels, they are said to have the circles in the figure as *pitch circles*. Their teeth are then formed partly above and partly below the pitch circles, and the numbers of their teeth, n_1, n_2, are proportional to the circumferences of these circles; hence $n_1/n_2 = r_1/r_2$, and the above angular velocity ratios may be written

$$(6), (7) \qquad \frac{\omega_{20}}{\omega_{10}} = -\frac{n_1}{n_2}, \qquad \frac{\omega_{21}}{\omega_{01}} = 1 + \frac{n_1}{n_2}.$$

The *circular pitch* (or simply the *pitch*) of the teeth of a gear is defined as the circumference of the pitch circle divided by the number of teeth:

$$\text{Circular Pitch} = \frac{2\pi r}{n}.$$

The circular pitch is thus equal to the arc of the pitch circle occupied by a tooth and a space. In order that two gears mesh properly they must have the same circular pitch.

The method of tabulation may also be applied to deduce (7) from (6). Thus when the link *0* is fixed let us suppose that $\omega_{10} = -1$; this is permissible since we are only concerned with the angular velocity *ratios* and may therefore choose any convenient value for one of the angular velocities. Then from (6) $\omega_{20} = n_1/n_2$, and we may form the following table:

	Link *0*	Gear *1*	Gear *2*
(*0* fixed)	0	-1	$\frac{n_1}{n_2}$
(*1* fixed)	1	0	$1 + \frac{n_1}{n_2}$

In the second row gear *1* is reduced to rest by adding unity to each angular velocity in the first row. We thus find $\omega_{21}/\omega_{01} = 1 + n_1/n_2$.

Example 2. Let the gear *1* drive the gear *3* through the interposed "idle wheel" *2* (Fig. 140*c*). If the frame or link *0* carrying the gears is fixed, we have from (6)

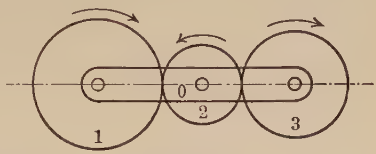


FIG. 140*c*.

$$\frac{\omega_{30}}{\omega_{10}} = \frac{\omega_{20}}{\omega_{10}} \cdot \frac{\omega_{30}}{\omega_{20}} = \left(-\frac{n_1}{n_2}\right)\left(-\frac{n_2}{n_3}\right) = \frac{n_1}{n_3}.$$

The gears *1* and *3* revolve in the same sense and their angular velocity ratio is numerically the same as if they were directly in mesh.

If the gear *1* of this train is fixed and the link *0* driven about *1*'s shaft, we have

$$\frac{\omega_{31}}{\omega_{01}} = \frac{\omega_{30} - \omega_{10}}{-\omega_{10}} = 1 - \frac{\omega_{30}}{\omega_{10}} = 1 - \frac{n_1}{n_3}.$$

For example if $n_1 = 45$, $n_3 = 50$,

$$\frac{\omega_{31}}{\omega_{01}} = 1 - \frac{45}{50} = \frac{1}{10};$$

that is, gear *3* will make one complete revolution for every ten revolutions of the arm *0*.

Applying the tabular method to this case we have:

	Link *0*	Gear *1*	Gear *2*	Gear *3*
(*0* fixed)	0	-1	$\frac{n_1}{n_2}$	$-\frac{n_1}{n_3}$
(*1* fixed)	1	0	$1 + \frac{n_1}{n_2}$	$1 - \frac{n_1}{n_3}$

Example 3. *Epicyclic Gearing.* A train of wheel gears in which one gear is fixed and the remaining gears and the frame carrying them are free to move is called an *epicyclic train.* The mechanisms of Examples 1 and 2 formed by fixing the gear *1* are examples of epicyclic trains. The methods employed above will give the velocity ratios in any epicyclic train.

Let the arm of an epicyclic train be denoted by *0*, the fixed gear by *1*, and the last gear by *k*. If we suppose that the arm is fixed and all of the gears movable, the ratio ω_{k0}/ω_{10} can be readily computed from the numbers of the teeth on the several gears. Denote this ratio by ϵ; then ϵ is positive or negative according as the gears *1* and *k* revolve in the same or opposite senses.

Now suppose that the gear *1* is fixed and the arm *0* movable. Then

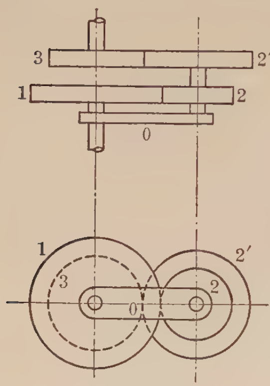


Fig. 140*d*.

$$(7)\qquad \frac{\omega_{k1}}{\omega_{01}} = \frac{\omega_{k0} - \omega_{10}}{-\omega_{10}} = 1 - \frac{\omega_{k0}}{\omega_{10}} = 1 - \epsilon.$$

The last gear therefore makes $1 - \epsilon$ revolutions for each revolution of the arm. By designing the gear train so that ϵ is very nearly equal to *1*, the ratio ω_{k1}/ω_{01} may be made exceedingly small.

Special cases of this general result are given in Examples 1 and 2. As a further illustration let us consider a "reverted" epicyclic train (Fig. 140*d*), that is, a train in which the axes of the first and last gears are coincident. The gears *1* and *3* are on different but coaxial shafts. The gears *2* and *2′* are fastened together or keyed to the same shaft, and thus form but a single link in the kinematic chain. Gear *1* meshes with gear *2*, gear *2′* with *3*; hence denoting the numbers of their teeth by n_1, n_2, n_2', n_3, respectively, and regarding the link *0* as fixed,

$$\epsilon = \frac{\omega_{30}}{\omega_{10}} = \frac{\omega_{20}}{\omega_{10}} \cdot \frac{\omega_{30}}{\omega_{20}} = \left(-\frac{n_1}{n_2}\right)\left(-\frac{n_2'}{n_3}\right) = \frac{n_1 n_2'}{n_2 n_3}.$$

When the gear *1* is fixed and the link *0* movable we have an epicyclic train for which

$$\frac{\omega_{31}}{\omega_{01}} = 1 - \frac{n_1 n_2'}{n_2 n_3}.$$

The arrangement of this mechanism requires that

$$r_1 + r_2 = r_2' + r_3.$$

Hence if the teeth of both pairs of gears have the same pitch we must have

$$n_1 + n_2 = n_2' + n_3.$$

To show the possibilities of this train in producing a great reduction in angular velocity, let us assume that $n_1 = 101$, $n_2 = 100$, $n_2' = 99$, $n_3 = 100$; then

$$\frac{\omega_{31}}{\omega_{01}} = 1 - \frac{101 \times 99}{100 \times 100} = \frac{1}{10000},$$

and the gear *3* will complete one revolution for every 10000 revolutions of the arm. In this case the pitch of the teeth of gears *1* and *2* is different from that in gears *3* and *2′*.

The preceding methods may also be applied to trains containing bevel gears. Directions of rotation, however, can only be indicated by + or − signs when the gears under comparison revolve in parallel planes. The directions in which the various gears revolve may usually be determined by inspection.

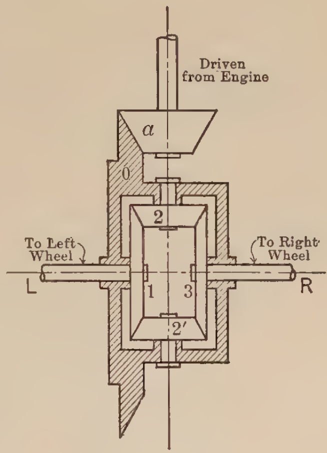


Fig. 140*e*.

Example 4. *Differential Mechanism for Automobiles.* When an automobile turns a corner, the outside wheel must revolve faster than the inner as it travels the greater distance. In order to meet this requirement in machines driven from the rear axle some sort of "differential" mechanism must be employed. Figure 140*e* shows schematically the arrangement of a *bevel gear differential.* The gear *a*, keyed to a shaft driven from the engine, meshes with the gear *0* rigidly attached to the differential housing (cross-hatched) which turns loosely on the rear axle. The housing carries pins on which the bevel pinions *2*, *2′* ride freely; in practice three or more pinions are used in order to distribute the load. These pinions in turn mesh with the "master" gears *1* and *3*, keyed respectively to the two parts *L* and *R* of the rear axle.

Let us first suppose that the gear *0* and attached housing is held fast. It is then clear that a rotation of *1* will cause *3* to rotate with the same angular speed but in the opposite sense; that is

$$\omega_{30} = -\omega_{10}.$$

Since the *relative* motion of the links of a mechanism is not altered by fixing one of the links, this relation must also hold when the gear *0* is driven by the engine. Hence, denoting the angular velocities of *0*, *1*, *3* by ω_0, ω_1, ω_3, we have

$$\omega_3 - \omega_0 = -(\omega_1 - \omega_0) \qquad \text{or} \qquad \omega_0 = \frac{\omega_1 + \omega_3}{2};$$

that is, the angular velocity of the housing is always equal to the arithmetic mean of the angular velocities of the wheels.

If the car is going straight ahead, both wheels revolve at the same rate and $\omega_0 = \omega_1 = \omega_3$. Since *1* and *3* do not revolve relative to the housing, the same is true of the pinions *2*, *2′*; thus the gears *0*, *1*, *2*, *2′*, *3* all revolve as a unit. But as soon as the car moves on a curve the outer wheel turns faster than the inner. The pinions *2* and *2′* are then set in motion rela-

tive to *0* and ω_1 and ω_3 differ from ω_0 by equal amounts. If the engine speed, and therefore ω_0, is constant, an increase of ω_1 entails a decrease of ω_3 of the same amount.

PROBLEMS

1. Suppose that in Fig. 140*c* a thick gear *2* meshes with three gears *3*, *4*, *5* (all represented by the same circle *3* in the figure) which ride loosely upon the same spindle. If $n_1 = 50$, $n_3 = 49$, $n_4 = 50$, $n_5 = 51$, describe the motions of the gears *3*, *4*, *5* when *1* is fixed and the arm *0* makes a single turn. (This mechanism is known as *Ferguson's Paradox.*)

2. Design a four-wheel reverted train (Fig. 140*d*) by which a velocity ratio of $\omega_{31}/\omega_{01} = 1/430$ can be exactly transmitted by wheels having not over 50 teeth.

3. Design a four-wheel epicyclic train so that the last wheel will make 1 rev./hr. when the arm revolves at the rate of 1 rev./sec.

4. In Fig. 140*f* the driving arm *0* carries two equal planet wheels *2* that mesh with the fixed gear *1* and with the annular gear *3*, coaxial with *1*. Find the ratio ω_{31}/ω_{01}, given n_1, n_2, n_3.

What relation connects n_1, n_2, n_3? If the above ratio is to be 1.5 and $n_1 = 30$, find n_2 and n_3.

5. If in Fig. 140*f* the gear *3* were fixed, what would be the value of ω_{13}/ω_{03} when $n_1 = 50$, $n_3 = 80$?

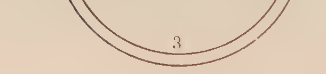


Fig. 140*f*.

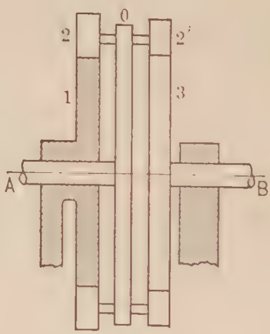


Fig. 140*g*.

6. In the reduction gear of Fig. 140*g*, *1* is a fixed gear, *2* and *2'* are pinions keyed to a common spindle carried by the arm *0* which revolves with the driving shaft *A*, and *3* is a gear keyed to the shaft *B*. The gears *1*, *2* and *2'*, *3* mesh together as shown, four pinions being used for the sake of balance. If the numbers of their teeth are $n_1 = 60$, $n_2 = 16$, $n_2' = 15$, $n_3 = 61$, find the number of revolutions of the shaft *B* for every 100 revolutions of the shaft *A*.

141. Theorem on Angular Velocity Ratios. Let a, b, c be three figures moving in a plane; and suppose that at a given instant the instantaneous centers I_{ab}, I_{bc}, I_{ca} are all *finite* and *distinct*. Then the relative angular velocities ω_{ab}, ω_{bc}, ω_{ca} must all be different from zero. If, for example, ω_{ab} were zero, the instantaneous motion of a relative to b would be a pure translation, and I_{ab} would be at infinity, contrary to our hypothesis. We shall now show that the ratio of the angular velocities of any two of the figures relative to the third may be obtained when the positions of I_{ab}, I_{bc}, I_{ca} are known.

Consider the motion of the coincident points I_{ab}, I_{ba}, of a and b respectively. The velocity of I_{ab} relative to c is the same as if it were moving in a circle about I_{ac} with the angular velocity ω_{ac}. Similarly the velocity of I_{ba} relative to c is the same as if it were moving in a circle about I_{bc} with the angular velocity ω_{bc}. But I_{ab} and I_{ba} have, by definition, the same velocity relative to c; its magnitude is therefore given by either member of the equation:

$$|\omega_{ac}\cdot I_{ac}I_{ab}| = |\omega_{bc}\cdot I_{bc}I_{ba}|.$$

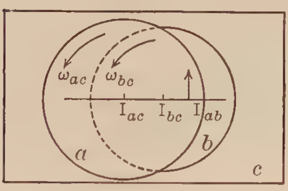


Fig. 141a.

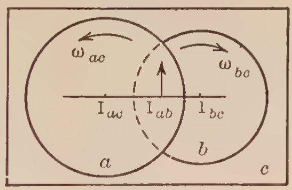


Fig. 141b.

This equation is still true when the absolute value bars are removed, *provided that* $I_{ac}I_{ab}$ *and* $I_{bc}I_{ba}$ *are regarded as segments of a directed line.* For these segments have the same sign or opposite signs according as ω_{ac} and ω_{bc} have the same or opposite signs (Figs. 141a, b). The ratio of the angular velocities is therefore

$$\text{(1)} \qquad \frac{\omega_{ac}}{\omega_{bc}} = \frac{I_{bc}I_{ba}}{I_{ac}I_{ab}};$$

or on reversing the signs of both segments,

$$\text{(2)} \qquad \frac{\omega_{ac}}{\omega_{bc}} = \frac{I_{ba}I_{bc}}{I_{ab}I_{ac}}.$$

The form (2) is perhaps easier to remember than (1). When the subscripts in the left member of (2) have been written, those in

the right member may be obtained from the following scheme:

$$\text{Numerator: } \begin{matrix} a & c \\ \uparrow & \nearrow \\ b & c \end{matrix} (ba,\ bc); \qquad \text{Denominator: } \begin{matrix} a & c \\ \downarrow & \searrow \\ b & c \end{matrix} (ab,\ ac).$$

It is preferable, however, to remember the theorem in words:

The angular velocities of a and b relative to c are inversely proportional to the directed segments from I_{ab} to the instantaneous centers of a and b relative to c.

If the link c is fixed the notation of (2) may be simplified by writing

$$\omega_a\,,\ \omega_b\,,\ I_a\,,\ I_b\,,\ I\,,$$

instead of

$$\omega_{ac},\ \omega_{bc},\ I_{ac},\ I_{bc},\ I_{ab}.$$

Equation (2) then becomes

$$\frac{\omega_a}{\omega_b} = \frac{II_b}{II_a}. \tag{3}$$

Example 1. The relative instantaneous center I_{12} of the friction wheels of Fig. 140*b* lies on their line of contact (§ 126); hence from (2):

$$\frac{\omega_{20}}{\omega_{10}} = \frac{I_{12}I_{10}}{I_{21}I_{20}} = -\frac{r_1}{r_2} \quad \text{and} \quad \frac{\omega_{21}}{\omega_{01}} = \frac{I_{02}I_{01}}{I_{20}I_{21}} = \frac{r_1 + r_2}{r_2} = 1 + \frac{r_1}{r_2}.$$

Example 2. In the four-bar chains of Figs. 141*c* and 141*d* let us regard *a* as the fixed link. The link *c* is then called the *coupler*. The angular

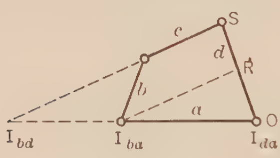


FIG. 141*c*.

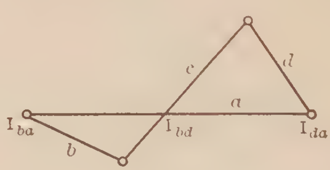


FIG. 141*d*.

velocity ratio of the links *b* and *d*, which revolve about the fixed centers, is, from (2),

$$\frac{\omega_{ba}}{\omega_{da}} = \frac{I_{db}I_{da}}{I_{bd}I_{ba}}.$$

Since I_{bd} is at the point of intersection of the center lines of *a* and *c*, this relation may be stated as follows:

The coupler (produced if necessary) divides the center line of the fixed link into segments which are inversely proportional to the angular velocities of the adjacent links.

If in Fig. 141c the coupler is but slightly inclined to a, I_{bd} may be inaccessible in the drawing. We may then obtain a ratio of segments equal to that above by drawing a line $I_{ba}R$ parallel to the center line of c; for from the similar triangles,

$$I_{ab}I_{da} : I_{bd}I_{ba} : = \mathrm{SO} : \mathrm{SR}.$$

Example 3. In the slider-crank chain (Fig. 141e) we have

$$\frac{\omega_{ca}}{\omega_{ba}} = \frac{I_{bc}I_{ba}}{I_{cb}I_{ca}} = \frac{RO}{RS} = \frac{I_{bc}I_{bd}}{I_{bc}S}.$$

The last ratio is most convenient since its denominator, the length of the connecting-rod, is constant.

When a is the fixed link, the chain becomes the mechanism of the direct-acting engine. The ratio of the angular velocities of the connecting-rod and crank is then given by

$$\frac{\omega_c}{\omega_b} = \frac{I_{bc}I_{bd}}{l},$$

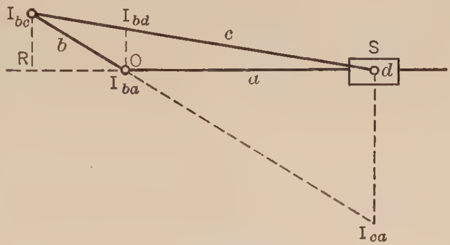


FIG. 141e.

where l denotes the length of the connecting-rod. If ω_b is constant, ω_c is numerically greatest at the dead centers, where $I_{bc}I_{bd}$ attains its maximum value; at these points $\omega_c = \omega_b r/l$, where r denotes the length of the crank. When the crank is midway between the dead centers $\omega_c = 0$.

PROBLEMS

1. In the mechanism of Fig. 137, are the links b and d moving in the same or opposite directions? The link a is fixed.

2. In the mechanism of Fig. 139c, find graphically the value of the ratio ω_e/ω_f. The link a is fixed.

3. In the mechanism of Fig. 139h, find graphically the value of the ratio ω_5/ω_6, (a) when the link *1* is fixed; (b) when the link *3* is fixed.

4. Find the angular velocity ratio of the arms in the Tchebicheff parallel motion (Fig. 132f) when the coupler is horizontal; when the coupler is vertical.

5. If I_{ab} is at infinity, prove that $\omega_{ac} = \omega_{bc}$. Show that this equation may be regarded as a limiting case of (2).

6. If in the double slider-crank chain of Fig. 119b the link b is fixed, find the ratio ω_{ab}/ω_{cb}. (This is the angular velocity ratio of the shafts in Oldham's coupling. See § 138, Problem 5.)

7. If ω_a, ω_b, ω_c are the absolute angular velocities of the three links a, b, c, show that (2) may be expressed in the symmetrical form:

$$\omega_a \cdot I_{ab}I_{ac} + \omega_b \cdot I_{bc}I_{ba} + \omega_c \cdot I_{ca}I_{cb} = 0.$$

142. Translatory Motions. We shall now consider the cases excluded in the preceding article, namely, when the relative motions of the three figures a, b, c involve translation.

We shall first assume that the relative motion of but two of the three figures, say a and c, is a translation. Then $\omega_{ac} = 0$, I_{ac} is at infinity, and every point of a has the same velocity $\mathbf{v}_{ac}$ relative to c. Now the point I_{ab} of a has the same velocity relative to c as the point I_{ba} of b with which it coincides; and since

Speed $I_{ab} = v_{ac}$, Speed $I_{ba} = \omega_{bc} \cdot I_{bc}I_{ba}$ (relative to c),

we must have

$$v_{ac} = \omega_{bc} \cdot I_{bc}I_{ba} \tag{1}$$

if all the quantities involved are regarded as positive.

If c is the fixed link, the above notation may be simplified by dropping the subscript c and writing I for I_{ba}. Equation (1) then becomes

$$v_a = \omega_b \cdot I_b I. \tag{2}$$

Next suppose that both a and b have motions of translation relative to c, and let the velocities of translation be $\mathbf{v}_{ac}$, $\mathbf{v}_{bc}$. Then unless $\mathbf{v}_{ac} = \mathbf{v}_{bc}$ (in which case a and b are relatively at rest) no point of a has the same velocity as a point of b, and I_{ab} does not exist — or as we usually say, I_{ab} is "at infinity." Hence the relative motion of a and b is also translatory; and it is readily seen that

$$\mathbf{v}_{ab} = \mathbf{v}_{ac} - \mathbf{v}_{bc}. \tag{3}$$

Example 1. In the direct-acting engine (Fig. 141c, link a fixed), let us compare the speeds of the piston and the crank-pin. The point I_{db} in the plane attached to the piston has the same speed as the point I_{bd} in the plane attached to the crank; hence

$$\text{Piston Speed} = v_d = \omega_b \cdot I_{ba}I_{bd}.$$

As the point I_{bc} lies on the axis of the crank-pin,

$$\text{Crank-pin Speed} = \omega_b \cdot I_{ba}I_{bc};$$

hence

$$\text{Piston Speed} : \text{Crank-pin Speed} = I_{ba}I_{bd} : I_{ba}I_{bc}.$$

Example 2. In the *beam-engine* of Fig. 142*a* the speed of the piston *f* may be expressed in terms of ω_b, the angular velocity of the crank *b*. For since the points I_{fb} and I_{bf} have the same velocity,

$$v_f = \omega_b \cdot I_{ba}I_{bf}.$$

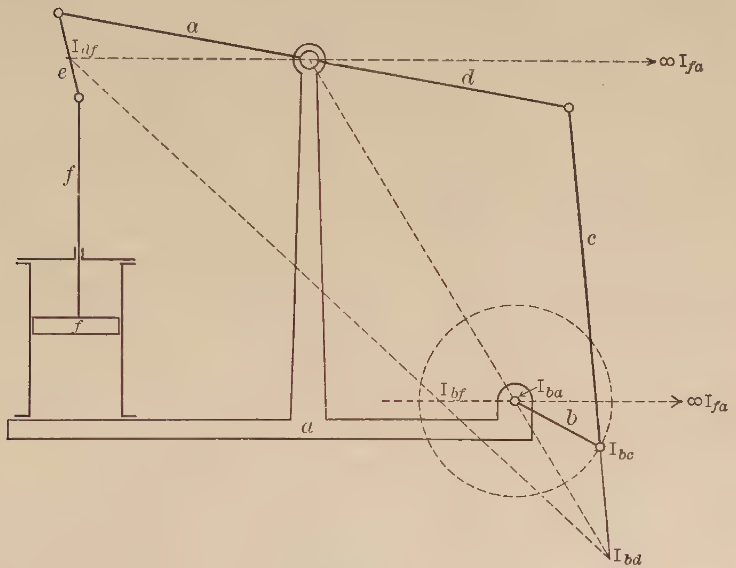


FIG. 142*a*.

The center I_{bf} may be located as follows:

from the four-cycle (*abcd*), $I_{bd}\begin{cases} I_{bc}I_{cd} \\ I_{da}I_{ab} \end{cases}$;

from the four-cycle (*adef*), $I_{df}\begin{cases} I_{de}I_{ef} \\ I_{fa}I_{ad} \end{cases}$;

from the four-cycle (*abdf*), $I_{bf}\begin{cases} I_{bd}I_{df} \\ I_{fa}I_{ab} \end{cases}$.

The crank-pin speed is evidently $\omega_b \cdot I_{ba}I_{bc}$; hence

Piston Speed : Crank-pin Speed $= I_{ba}I_{bf} : I_{ba}I_{bc}$.

PROBLEMS

1. In the toggle-press mechanism shown in Fig. 142*b*, $l_2 = 3''$, $l_3 = 10''$, $l_4 = l_5 = 8''$. Assuming that the crank makes 12 r.p.m., find graphically the speed of the block *6* when $\theta = 0°, 45°, 90°, 135°, 180°$, and plot these speeds as ordinates on (*a*) a time base, (*b*) a displacement base.

2. The mechanism of Fig. 142*c*, known by the name of *Rapson's slide*, is sometimes used to control the rudder of large ships. The fixed link *d* represents the frame of the ship, *a* is the tiller keyed to the rudder-head,

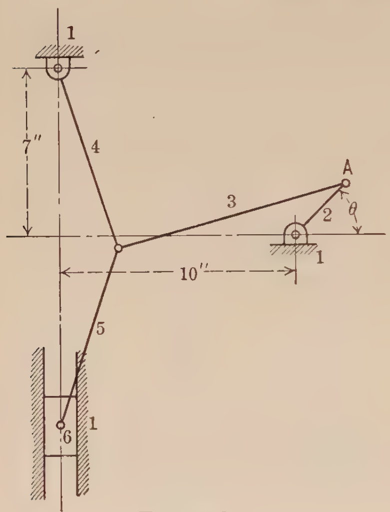


FIG. 142*b*.

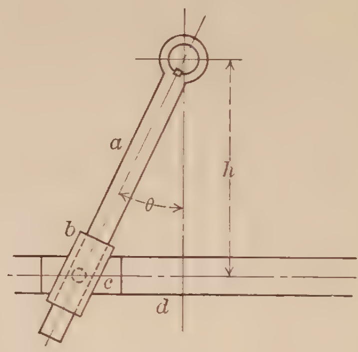


FIG. 142*c*.

b is a block sliding on the end of *a* and coupled by a turning pair to *c*, which in practice is a carriage running on guide rails athwart the ship. The steering engine pulls the carriage to the right or left by means of wire cables and thus controls the rudder. Prove that $\omega_a = v_c \cos^2 \theta / h$.

143. Speed Ratios. The theorem on angular velocity ratios enables us to compare the speeds of any two points P, Q of a mechanism in plane motion. Let P, Q be points of the links p, q respectively, and let a denote the fixed link. For simplicity we shall write I_p, I_q, I for I_{pa}, I_{qa}, I_{pq}. Then the speeds of P and Q are given by

(1) $$v_P = \omega_p \cdot I_pP, \qquad v_Q = \omega_q \cdot I_qQ,$$

and from (§ 141, 3)

$$\frac{\omega_p}{\omega_q} = \frac{I_qI}{I_pI}.$$

We therefore have the equation

(2) $$\frac{v_P}{v_Q} = \frac{I_qI \cdot I_pP}{I_pI \cdot I_qQ}$$

in which all the quantities involved are regarded as positive as it is no longer profitable to distinguish signs.

A simple construction enables us to express v_P/v_Q as the ratio of but *two* line segments. In Fig. 143a let P, Q, I_p, I_q, I represent the relative position of the points in the mechanism; I_p, I_q and I (I_{pq}) lie in a straight line. Draw circles with centers at I_p and

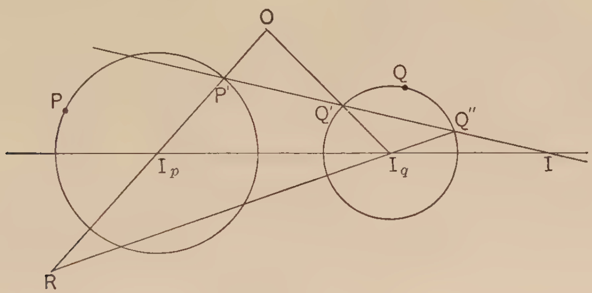


Fig. 143a.

I_q and passing through P and Q respectively. Through I draw any line cutting both circles and let P', Q' be two of the four points of intersection, chosen so that one lies on each circle. Then regarding $IQ'P'$ as a transversal of the triangle OI_pI_q, we have by the Theorem of Menelaus*

$$\frac{I_pI}{I_qI}\cdot\frac{I_qQ'}{OQ'}\cdot\frac{OP'}{I_pP'}=1,$$

and hence

$$\frac{OP'}{OQ'}=\frac{I_qI\cdot I_pP}{I_pI\cdot I_qQ}=\frac{v_P}{v_Q}.$$

This construction can be performed in such a variety of ways that a convenient figure can always be obtained. For example, by taking the point Q'' instead of Q', we see from the figure that $RP'/RQ'' = v_P/v_Q$.

Equation (2) applies whenever the points I_p, I_q, and I are all finite and distinct. However when *one* of the three points is

* If a transversal meets the sides BC, CA, AB of a triangle ABC at the points X, Y, Z respectively, then

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=-1.$$

See § 7, Example 5 for a method of proof. As we shall regard all the segments as positive, the right-hand member becomes $+1$.

infinitely distant, the corresponding equation is easily obtained from (2) as a limiting case.

I at Infinity. By keeping the finite points I_p, I_q fixed and letting I recede to infinity, the ratio I_qI/I_pI approaches 1, and (2) becomes

$$(3) \qquad \frac{v_P}{v_Q} = \frac{I_pP}{I_qQ} \qquad (I_{pq} \text{ at } \infty).$$

This equation may be deduced directly. For since I_{pq} is at infinity, the instantaneous motion of p relative to q is a translation and $\omega_{pq} = 0$. Now

$$\omega_{pq} = \omega_p - \omega_q, \qquad \text{and therefore} \qquad \omega_p = \omega_q.$$

On dividing the first of equations (1) by the second we obtain (3).

I_p at Infinity. When I, I_q are fixed and I_p recedes to infinity, the ratio I_pP/I_pI approaches 1, and (2) becomes

$$(4) \qquad \frac{v_P}{v_Q} = \frac{I_qI}{I_qQ} \qquad (I_p \text{ at } \infty).$$

This equation may also be proved directly. For since p has translatory motion, all of its points have the same velocity. Moreover I (I_{pq}) has the same velocity whether considered as a point of p or of q; hence

$$\frac{v_P}{v_Q} = \frac{v_I}{v_Q} = \frac{I_qI}{I_qQ}.$$

This case is illustrated by the examples of the preceding article.

I_p and I_q at Infinity. In this case both p and q have a translatory motion relative to a, and since I is also at infinity (see § 142) equation (2) can no longer be used. We then refer the motions of p and q to some link b for which I_{pb}, I_{pq} are finite. Remembering that all points of p, or of q, have the same speed relative to a, we have

$$v_p = \text{Speed } I_{pb} \text{ (in } p) = \text{Speed } I_{bp} \text{ (in } b) = \omega_b \cdot I_bI_{bp},$$
$$v_q = \text{Speed } I_{qb} \text{ (in } q) = \text{Speed } I_{bq} \text{ (in } b) = \omega_b \cdot I_bI_{bq},$$

and hence

$$(5) \qquad \frac{v_p}{v_q} = \frac{I_bI_{bp}}{I_bI_{bq}} \qquad (I_p,\ I_q \text{ at } \infty).$$

Example. The mechanism of the *Hanna Riveter* (Fig. 143*b*) supplies an example of the case last considered. The links *1* (fixed), *4*, *5*, *6* form

a four-bar chain, the coupler *5* being connected to the air driven piston *2* by the link *3* and to the punch *8* by the link *7*.

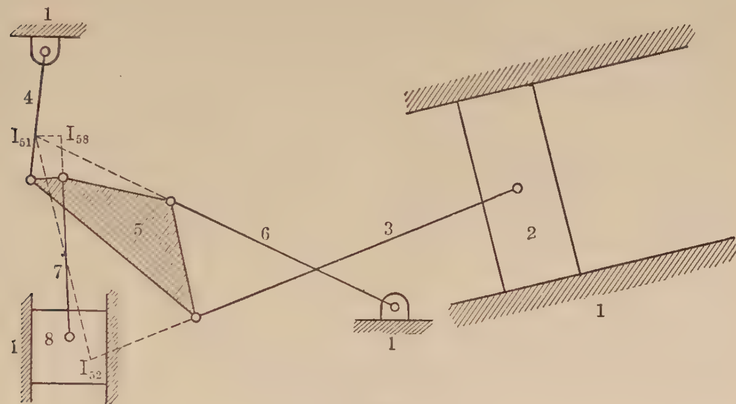


FIG. 143*b*.

Let it be required to find the speed ratio of the pistons, v_8/v_2, in the position of the mechanism shown. Since I_2, I_8, and therefore I_{28}, are at infinity, we shall refer the motions of *2* and *8* to the link *5*. From equation (5) we have

$$\frac{v_8}{v_2} = \frac{I_{51}I_{58}}{I_{51}I_{52}}.$$

We therefore seek the instantaneous centers I_{51}, I_{52}, I_{58}. Proceeding from the known four-cycle (*1456*), these centers are located as follows:

from (*1456*) $\quad I_{51}\begin{cases} I_{56}I_{61}; \\ I_{14}I_{45} \end{cases}$

from (*1235*) $\quad I_{52}\begin{cases} I_{51}I_{12}; \\ I_{23}I_{35} \end{cases}$

from (*1578*) $\quad I_{58}\begin{cases} I_{57}I_{78}. \\ I_{81}I_{15} \end{cases}$

144. Polar Diagrams of Velocity and Acceleration. The graphic methods of §§ 131, 132 may be used to construct vector diagrams that give the velocity or the acceleration of various points of a plane mechanism. Suppose, for example, that the velocity of a point P in the link p is known, and the velocity of the point Q in the link q is required. Let I denote the relative instantaneous center I_{pq} of the links. Then since I has the same velocity whether

considered as a point of p or of q, we have

$$\mathbf{v}_I = \mathbf{v}_P + \mathbf{v}_{IP} \quad (\mathbf{v}_{IP} \perp IP),$$
$$\mathbf{v}_Q = \mathbf{v}_I + \mathbf{v}_{QI} \quad (\mathbf{v}_{QI} \perp QI).$$

Hence $\mathbf{v}_Q$ may be determined graphically if the directions of $\mathbf{v}_I$ and $\mathbf{v}_Q$ are known (Fig. 144a).

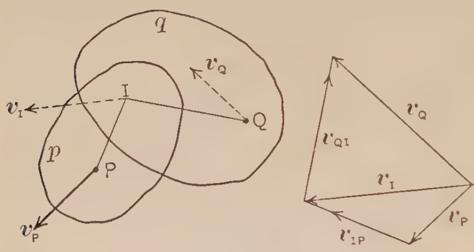


FIG. 144a.

If the relative motion of p and q is translatory, I_{pq} is at infinity and the above method fails. We may then use the point P' of q, which coincides with P in the position considered, to pass from $\mathbf{v}_P$ to $\mathbf{v}_Q$ (Fig. 144b). For from (§ 134, 1),

$$\mathbf{v}_{P'} = \mathbf{v}_P + \mathbf{v}_r,$$

where $\mathbf{v}_r$, the velocity of P' relative to p, is parallel to the direction of the relative translation; and then

$$\mathbf{v}_Q = \mathbf{v}_{P'} + \mathbf{v}_{QP'} \quad (\mathbf{v}_{QP'} \perp QP').$$

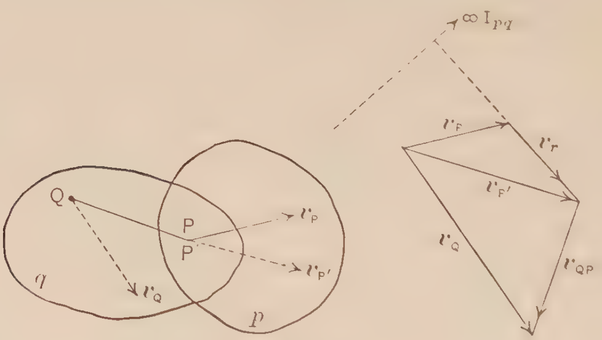


FIG. 144b.

If the links p and q are not directly connected, it is often more convenient to pass from p to q over the links that join them. Thus if p is joined to q through the links r and s ($p - r - s - q$), we may use I_{pr}, I_{rs}, I_{sq} to make the transition from $\mathbf{v}_P$ to $\mathbf{v}_Q$; or we may skip the link r by using only I_{ps}, I_{sq}. The method to be used in any case depends on the accessibility of the instantaneous centers required.

In the construction of an acceleration diagram we require the relative center of acceleration J_{pq} in passing from p to q. For this point, by definition, has the same acceleration when considered as a point of p or q. If p and q are connected by a turning pair, J_{pq} is a point on its axis. The centers of acceleration on the axes of turning pairs are the only ones used in practice; the others are not readily located.

When the links are connected by a sliding pair we may use, as before, the coincident points P, P' to make the transition between them. In this case we must use the Theorem of Coriolis (§ 134, 2),

$$\mathbf{a}_{P'} = \mathbf{a}_P + \mathbf{a}_c + \mathbf{a}_r,$$

where $\mathbf{a}_r$, the acceleration of P' relative to p, is in the direction of the relative translation, and $\mathbf{a}_c$, the complementary acceleration, is turned 90° from $\mathbf{v}_r$ in the sense of ω_p. The details of the construction are shown in the following example.

Example 1. The quick return mechanism shown conventionally in Fig. 144c consists of six links, the link *1* being fixed; *1234* and *1456* are inversions of the slider-crank chain (see § 138). We shall draw the velocity and acceleration diagrams for this mechanism when the crank *2* is driven with the angular velocity of ω rad./sec., and thus determine the motion of the sliding block *6*.

Velocity Diagram. From an arbitrary pole o draw the vector $\overrightarrow{ob}$ perpendicular to QB to represent the velocity of the crank-pin B. We shall take $ob = 2r$, where r denotes the length of the crank *2* in the diagram of the mechanism. Since the speed of B is ωr, we have

$$v_B = \omega r = \tfrac{1}{2}\omega \cdot 2r = \tfrac{1}{2}\omega \cdot ob.$$

This establishes the velocity scale; the magnitude of any velocity will be $\frac{1}{2}\omega$ times the length of its representative vector in the diagram.

To make the transition from link *3* to *4*, let C denote the point of *4* directly underneath B. The velocity of C is normal to OC; also

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_r,$$

where $\mathbf{v}_r$, the velocity of C relative to *3*, is parallel to OC. Hence draw the lines

$$bc \parallel OC, \qquad oc \perp OC;$$

their point of intersection c determines the vector $\overrightarrow{oc}$ which represents $\mathbf{v}_C$.

Now oc is a polar velocity image of OC; hence to obtain the vector $\overrightarrow{od}$ that represents $\mathbf{v}_D$, prolong oc to d so that

$$od : oc = OD : OC.$$

The direction of $\mathbf{v}_E$ is known, and

$$\mathbf{v}_E = \mathbf{v}_D + \mathbf{v}_{ED} \quad (\mathbf{v}_{ED} \perp DE).$$

Hence draw the lines

$$de \perp DE, \qquad oe \parallel \mathbf{v}_E;$$

their intersection e determines the vector $\overrightarrow{oe}$ which represents $\mathbf{v}_E$, that is, $\mathbf{v}_E = \frac{1}{2}\omega \cdot \overrightarrow{oe}$.

The velocity diagram is now complete. Note that it is lettered so that any vector $\overrightarrow{xy}$ in the diagram represents the velocity of the point Y relative to X in the mechanism. The vectors issuing from o represent "absolute" velocities.

Acceleration Diagram. We shall construct the acceleration diagram under the assumption that the angular velocity ω of the crank is *constant.* The acceleration of B is then entirely radial and of magnitude $\omega^2 r$. If we wish to use the velocity diagram in the construction, we see from § 132 that the relation

$$v_B = \tfrac{1}{2}\omega \cdot ob \quad \text{requires that} \quad a_B = \tfrac{1}{4}\omega^2 \cdot o'b';$$

hence

$$\omega^2 r = \tfrac{1}{4}\omega^2 \cdot o'b' \qquad \text{or} \qquad o'b' = 4r.$$

We therefore draw from an arbitrary pole o' the vector $\overrightarrow{o'b'}$ in the direction $\overrightarrow{BQ}$, and of length $4r$, to represent $\mathbf{a}_B$.

To pass from the link *3* to *4* we again use the point C of *4* directly underneath B. Now $\mathbf{a}_C$ has a radial projection in the direction $\overrightarrow{CO}$, whose magnitude is v^2_C/OC; and a transverse projection normal to OC. The length of the radial projection is readily obtained by the graphical method of § 132. Draw a semicircle on OC as diameter, strike an arc with center O and radius oc cutting it at H, and drop the perpendicular HK on OC. Then since oc represents v_C,

$$\frac{oc^2}{OC} = \frac{OH^2}{OC} = OK \quad \text{represents} \quad \frac{v^2_C}{OC},$$

and a vector $\overrightarrow{o'n}$ in the direction $\overrightarrow{CO}$ and of length OK will represent the radial projection of $\mathbf{a}_C$. The vector $\overrightarrow{o'c'}$ that represents $\mathbf{a}_C$ must end on the line nc' perpendicular to $o'n$.

To locate c' we must use the relation

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_c + \mathbf{a}_r,$$

where $\mathbf{a}_r$, the acceleration of C relative to *3*, is parallel to OC, and $\mathbf{a}_c$, the complementary acceleration, is turned 90° away from $\mathbf{v}_r$ (represented by $\overrightarrow{bc}$ in the velocity diagram) in the sense of ω_3 (the angular velocity of *3*).

The magnitude of $\mathbf{a}_c$ is also known; for

$$\omega_3 = \frac{v_C}{OC} = \tfrac{1}{2}\omega \cdot \frac{oc}{OC}, \qquad v_r = \tfrac{1}{2}\omega \cdot bc,$$

$$a_c = 2\omega_3 v_r = \tfrac{1}{4}\omega^2 \frac{oc \cdot bc}{\tfrac{1}{2}OC}.$$

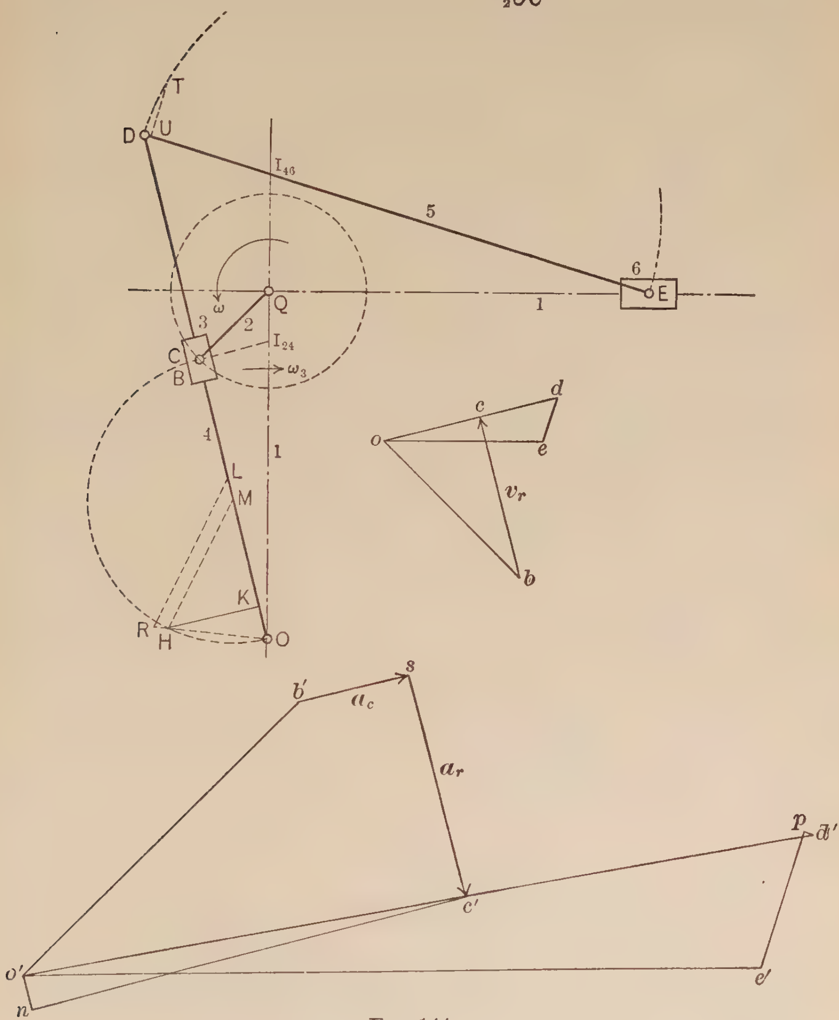


FIG. 144c.

A segment of length $oc \cdot bc/\tfrac{1}{2}OC$ will therefore represent a_c. To construct this segment join H to the middle point M of OC, lay off $OL = bc$ on OM, and draw LR parallel to MH. Then if LR cuts the line OH

at R, we have from similar triangles

$$OR = \frac{OL}{OM}\cdot OH = \frac{bc\cdot oc}{\frac{1}{2}OC}.$$

Now draw the vector $\overrightarrow{b's}$ perpendicular to OC, pointing to the *right*, and having the length of OR; it will represent $\mathbf{a}_C$. Since

$$\overrightarrow{o'c'} = \overrightarrow{o'b'} + \overrightarrow{b's} + \overrightarrow{sc'},$$

where $\overrightarrow{sc'}$ represents $\mathbf{a}_r$, c' must lie on the line sc' parallel to OC. The point c' is thus determined by the intersection of the lines nc' and sc'.

The segment $o'c'$ is a polar acceleration image of OC; hence to obtain the vector $\overrightarrow{o'd'}$ that represents $\mathbf{a}_D$, prolong $o'c'$ to d' so that

$$o'd' : o'c' = OD : OC.$$

Finally, to obtain the vector $\overrightarrow{o'e'}$ that represents $\mathbf{a}_E$, we use the equation.

$$\mathbf{a}_E = \mathbf{a}_D + \mathbf{a}_{ED}.$$

The radial projection of $\mathbf{a}_{ED}$ has the direction $\overrightarrow{ED}$ and the magnitude v^2_{ED}/DE. Describe a semicircle on DE as diameter, strike an arc with D as center and de as radius cutting it at T, and drop a perpendicular TU on DE. Then since de represents v_{ED},

$$\frac{de^2}{DE} = \frac{DT^2}{DE} = DU \quad \text{represents} \quad \frac{v^2_{ED}}{DE},$$

and a vector $\overrightarrow{d'p}$ in the direction $\overrightarrow{ED}$ and of length DU will represent the radial projection of $\mathbf{a}_{ED}$. Now

$$\overrightarrow{o'e'} = \overrightarrow{o'd'} + \overrightarrow{d'p} + \overrightarrow{pe'},$$

where $\overrightarrow{pe'}$, representing the transverse projection of $\mathbf{a}_{ED}$, is perpendicular to $d'p$. A line through o' parallel to the direction of E's motion will cut pe' in the point e'; and

$$\mathbf{a}_E = \tfrac{1}{4}\omega^2\cdot\overrightarrow{o'e'}.$$

Scales. In § 132 we have seen that

(1) x in. = 1 ft., $\quad x$ in. = k ft./sec., $\quad x$ in. = k^2 ft./sec.^{2}

form a consistent set of scales for the construction of the mechanism, velocity and acceleration diagrams. If we write $1/x = m$, $k/x = n$, then $k^2/x = n^2/m$, and the above scales may be expressed as

(2) 1 in. = m ft., $\quad$ 1 in. = n ft./sec., $\quad$ 1 in. = $\dfrac{n^2}{m}$ ft./sec.^{2}.

The scales of length and velocity may thus be chosen at pleasure; the scale of acceleration must then be chosen as indicated in (1) or (2).

Example 2. In the mechanism of Fig. 144*d*, the link *1* is fixed and the velocity of the point A is known. To construct the velocity diagram for the mechanism, draw $\overrightarrow{oa}$ to represent $\mathbf{v}_A$; then the points b, c, d, e in the

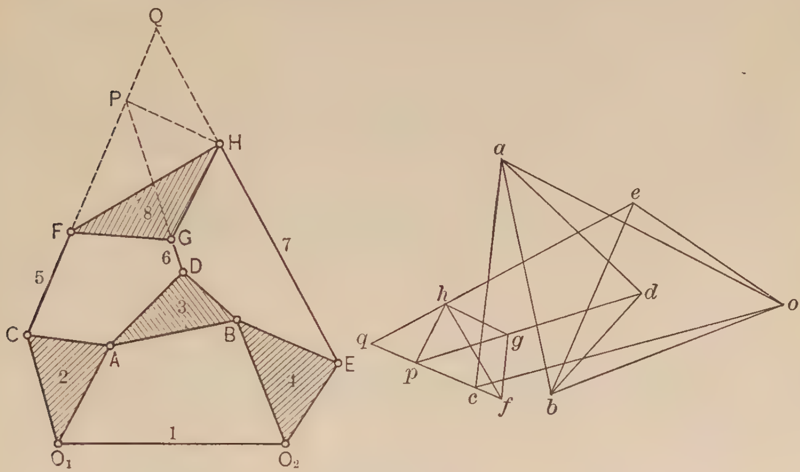


Fig. 144*d*.

velocity diagram are determined by the intersections of the following pairs of lines:

$$\begin{array}{ll} ab \perp AB, & ob \perp O_2B; \\ ac \perp AC, & oc \perp O_1C; \\ ad \perp AD, & bd \perp BD; \\ be \perp BE, & oe \perp O_2E. \end{array}$$

The points f, g, h of the velocity diagram lie respectively on the lines

$$cf \perp CF, \qquad dg \perp DG, \qquad eh \perp EH;$$

but as the directions in which F, G, H are moving are not known, we can not determine the positions of f, g, h directly.

Imagine, now, that the link *8* is enlarged so as to include the point of intersection P of the lines CF and DG. The velocity of P, considered as a point of *8*, is given by

$$\mathbf{v}_P = \mathbf{v}_C + \mathbf{v}_{PC}, \qquad \text{where} \qquad \mathbf{v}_{PC} = \mathbf{v}_{PF} + \mathbf{v}_{FC}$$

is normal to CP since both $\mathbf{v}_{PF}$ and $\mathbf{v}_{FC}$ have this direction. Similarly

$$\mathbf{v}_P = \mathbf{v}_D + \mathbf{v}_{PD}, \qquad \text{where} \qquad \mathbf{v}_{PD} = \mathbf{v}_{PG} + \mathbf{v}_{GD}$$

is normal to DP. These equations show that the point p in the velocity image of *8* is at the intersection of the lines

$$cp \perp CP, \qquad dp \perp DP.$$

Now draw PH; then the point h is determined by the intersection of the lines

$$ph \perp PH, \qquad eh \perp EH.$$

Lastly f and g are determined by the lines

$$hf \perp HF, \qquad cf \perp CF;$$
$$hg \perp HG, \qquad dg \perp DG.$$

The triangle fgh is a polar velocity image of FGH. The velocity of any point in the mechanism is represented by the vector from o its image in the velocity diagram.

The velocity diagram may also be constructed by using the point Q instead of P. The student should carry out this construction.

PROBLEMS

1. Construct the velocity diagram in Example 1 by passing from the links *2* to *6* by means of the instantaneous centers I_{24} and I_{46}.

2. Construct the velocity diagram for the mechanism of Fig. 139*c*. The link a is fixed and ω_b is known.

3. Construct the velocity diagram for the Crosby indicator motion (Fig. 139*d*) assuming that the velocity of the piston d is known.

4. Construct the velocity diagram for the linkage of Fig. 139*g*. The link *1* is fixed and the velocity of A known. Compare the direction of P's motion as obtained from the velocity diagram with that determined from the position of I_{81} (see § 139, Problem 3).

5. Construct the velocity diagram for the linkage of Fig. 139*f*. The link *1* is fixed and ω_2 is known.

[First draw abc, a polar velocity image of ABC. Then locate I_{71} (see § 139, Problem 2); the velocity image of I_{71}, considered as a point of *7*, is the pole o of the velocity diagram. Since the velocities of two points of *7*, namely C and I_{71}, are known, the velocity image of CDE may be constructed and the diagram completed.]

6. Construct the velocity diagram for the Stephenson Link mechanism of Fig. 139*i*. Link *1* is fixed and ω_2 known.

7. Construct the velocity diagram for the mechanism of the Hanna riveter (Fig. 143*b*), assuming that the velocity of the piston *2* is known.

8. Construct the velocity and acceleration diagrams for the beam engine of Fig. 142*a*, assuming that the crank-pin revolves with a known uniform speed.

9. Construct the velocity and acceleration diagrams for the toggle-press of Fig. 142*b*, assuming that A revolves with a known uniform speed.

10. Construct the velocity and acceleration diagrams for Bricard's parallel motion (Fig. 139*h*). Take the dimensions given in § 139, Problem 4, and assume that ω_b is known.

11. Draw the velocity and acceleration diagrams for the mechanism of Fig. 144*c* for seven positions of the crank at intervals of 45° ($\angle EQB = 0°$, 45°, . . .) and also for the two extreme positions of the link OD ($OD \perp QB$).

Take the dimensions of the mechanism as follows:

$$OQ = 1' \, 11'', \quad QB = 8'', \quad OD = DE = 3' \, 8'';$$

and assume that the crank-pin has a uniform speed of 2 ft./sec. Use the scales

$$1\tfrac{1}{2} \text{ in.} = 1 \text{ ft.}, \quad 3 \text{ in.} = 1 \text{ ft./sec.}$$

From the above diagrams construct

(*a*) the curves for v_E and a_E on a time base;

(*b*) the curves for v_E and a_E on a displacement base; and

(*c*) check the acceleration curves by finding a_E from the velocity-space curve (see § 114).

The area between the velocity-time curve and the t-axis lies partly above and partly below the axis. Show that these portions have equal areas.

12. For what positions of the mechanism of Fig. 144*c* is the complementary acceleration $\mathbf{a}_c = 0$ (see Example 1)?

145. Angular Velocity Ratio in Higher Pairing. Figures 145*a* and 145*b* represent *cam trains* consisting of three links, *a*, *b*, *c*, the link *c*, represented by the line I_aI_b, being fixed. The *cam a* is a

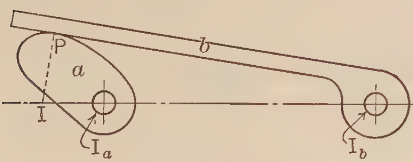


Fig. 145*a*.

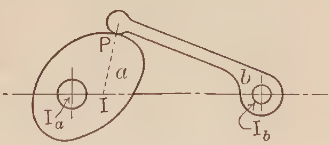


Fig. 145*b*.

flat plate with a curved periphery, which, when driven about its fixed center I_a, transmits motion to the *follower* *b* by line contact (*higher pairing* — see § 136). The follower is held in contact with the cam by gravity or by a spring. When *a* revolves continuously about I_a, *b* oscillates about its fixed center I_b.

Let I denote the relative instantaneous center I_{ab}. Then from (§ 141, 3),

$$\frac{\omega_b}{\omega_a} = \frac{II_a}{II_b}. \tag{1}$$

This angular velocity ratio is therefore known when I is determined. From the Theorem of Three Centers, I (I_{ab}) must lie on the line I_aI_b ($I_{ac}I_{bc}$). Furthermore I must lie on the common normal to the contact surfaces at P; for since a and b remain in continuous contact, the velocity of P_b (P considered as a point of b) *relative to* a must be directed along the common tangent at P. *Thus* I (I_{ab}) *is the point of intersection of the common normal to the contact surfaces with the line of centers.* Cam and follower revolve in the same or opposite directions according as I divides the line of centers externally (Fig. 145a) or internally (Fig. 145b).

The velocity with which the follower slides over the cam is the velocity of P_b relative to a. Since this relative velocity is numerically equal to $\omega_{ba}\cdot IP$, the

$$\text{Sliding Speed} = |\omega_{ba}|\cdot IP = |\omega_b - \omega_a|\cdot IP. \tag{2}$$

In this expression the angular velocities must be given their appropriate signs. From (2) we see that the sliding speed is zero when P coincides with I. *Hence for pure rolling contact, the point of contact must always lie on the line of centers.*

Let us now inquire what condition must be fulfilled in order to obtain a constant ratio of angular velocities in plane contact motions of the above type. From (1) it is clear that ω_b/ω_a will maintain a constant value when, and only when, the position of I remains unaltered during the motion. We therefore have the important result:

If one body transmits plane rotation to another by line contact, the ratio of their angular velocities will remain constant when, and only when, the common normal to the contact surfaces always cuts the line of centers in the same point.

When the position of I is invariable, its path in each of the revolving planes attached to a and b will be a circle. These circles are the centrodes for the relative motion of a and b; and this motion will be exactly reproduced by rolling one circular centrode upon the other. A constant ratio of angular velocities therefore implies the rolling together of circular centrodes.

Finally let us consider a cam train in which the follower has a

rectilinear reciprocating motion (Fig. 145c). Then I_b is at infinity in the direction normal to b's motion, and I is the point where the common normal to the contact surfaces cuts the line I_aI_b. The speed of the follower, v_b, is obtained by noting that I (I_{ab}) has the same speed whether considered as a point of b or as a point of a. Considered as a point of b, $v_I = v_b$; considered as a point of a, $v_I = \omega_a \cdot I_aI$; hence

$$v_b = \omega_a \cdot I_aI. \tag{3}$$

The sliding speed is again given by (2); since $\omega_b = 0$, its value is $|\omega_a| \cdot IP$.

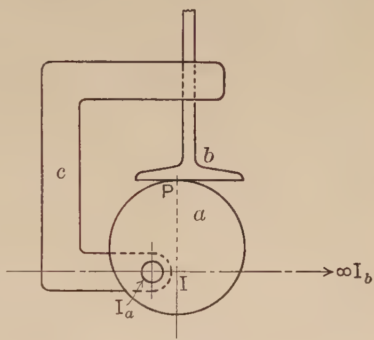


Fig. 145c.

PROBLEMS

1. Let *1* and *2* be two equal ellipses in contact at P; and let F_1, F_1'; F_2, F_2' denote their respective foci. Then if F_2 lies on F_1P prolonged so that $PF_2 = PF_1'$ and F_1,F_2 are chosen as fixed axes of rotation for the ellipses, show that one can drive the other by pure rolling contact.

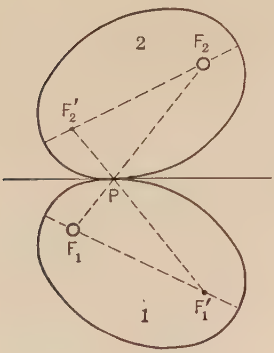


Fig. 145d.

2. Show that ω_2/ω_1 in Problem 1 fluctuates between the extreme values

$$\frac{1-e}{1+e} \quad \text{and} \quad \frac{1+e}{1-e},$$

where e is the eccentricity of the ellipses.

3. Show that with the same fixed centers as in Problem 1, the variable ratio ω_2/ω_1 may be transmitted by means of the four-bar chain $F_1F_1'F_2'F_2$.

Find the space and body centrodes of the coupler of this chain.

146. Spur Gears. In the preceding article we have seen that when motion is transmitted by line contact, the angular velocity ratio will remain constant when the centrodes of the relative motion of driver and follower are circles whose centers lie on the fixed axes of rotation. Denoting the radii of these centrodes by r_a, r_b,

the relation (§ 145, 1), may be written

$$\frac{\omega_b}{\omega_a} = \pm \frac{r_a}{r_b}; \tag{1}$$

and if the bodies a and b are replaced by circular cylinders of radii r_a, r_b (Fig. 146), precisely the same velocity ratio will be transmitted by their pure rolling contact. Such cylinders, or "friction wheels" have found but limited application in practice, since the transmission of power renders them very liable to slip. For this reason the centrode cylinders are commonly provided with teeth formed partly above and partly below their contact surfaces. The gear wheels (*spur gears*) so formed are said to have the circular centrodes as their *pitch circles*. The point of contact I of the pitch circles is called the *pitch point;* the pitch point is thus the instantaneous center I_{ab} of the relative motion of a and b that corresponds to the constant velocity ratio required.

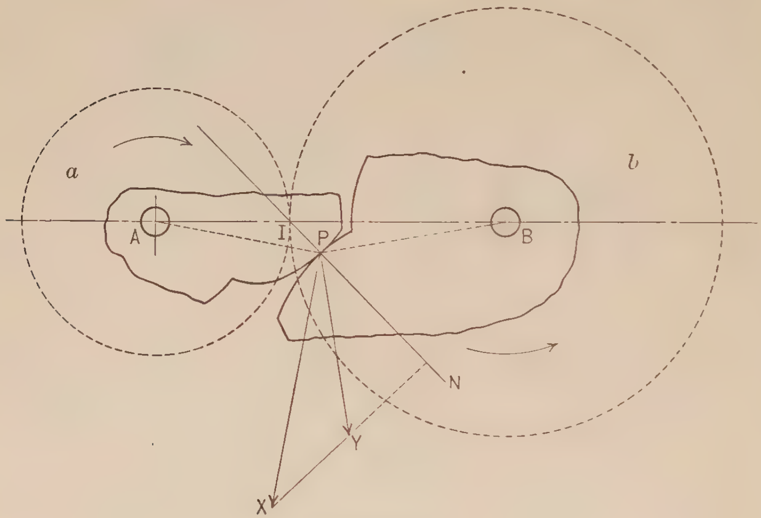


Fig. 146.

The question now arises: What shapes may be given to the gear teeth in order that the ratio of the angular velocities maintain a given constant value? Assuming that the teeth engage by line contact, a general answer may be given at once from the concluding theorem of § 145. *The profiles of the teeth must be so*

formed that their common normal at the point of contact shall always pass through the pitch point.

A clear idea of the action of gear teeth may be obtained from Fig. 146. A pair of teeth are in contact at the point P, and the common normal PN to their profiles passes through the pitch point I. The velocity of P_a is represented by $\overrightarrow{PX}$ ($\perp AP$), the velocity of P_b by $\overrightarrow{PY}$ ($\perp BP$). These velocity vectors must have the same projection along the common normal PN as long as the teeth remain in contact. The velocity with which the tooth of b is sliding over the tooth of a is evidently the velocity of P_b relative to P_a, namely

$$\overrightarrow{PY} - \overrightarrow{PX} = \overrightarrow{XY}.$$

From (2) of the preceding article, the magnitude of this sliding velocity is

$$(2) \qquad |\omega_b - \omega_a| \cdot IP.$$

The angular velocities have the same sign or opposite signs according as the gears mesh internally or externally.

PROBLEMS

1. Suppose that a body a, revolving about a fixed center with the angular velocity ω_a, transmits a translatory motion to a body b by line contact. If v_b denotes the speed of b, show that the ratio v_b/ω_a will remain constant when and only when the common normal to the contact surfaces cuts the line I_aI_b in a fixed point. (See Fig. 145*c*.)

What are the centrodes of the relative motion of a and b in this case?

2. In the case of a spur gear a driving a rack b, what condition must the tooth profiles fulfil in order that the ratio v_b/ω_a remain constant?

147. Involute Teeth. The curve most frequently used for tooth profiles is the *involute of a circle.* This curve is generated by a point of a straight line that rolls without slipping on the circumference of a fixed circle. This definition may also be put in the form: An involute is generated by the end of a taut string that is being unwrapped from the circumference of a fixed circle (the *base circle*).

If we regard the involute in the figure as generated by the point P of the line IP (prolonged) as it rolls upon the circle, the instantaneous center of the line is its point of contact I (§ 126). The

velocity of P is therefore perpendicular to IP and IP is normal to the involute at P (cf. § 87, Example 2). In other words, *a tangent to the base circle is a normal to the involute.*

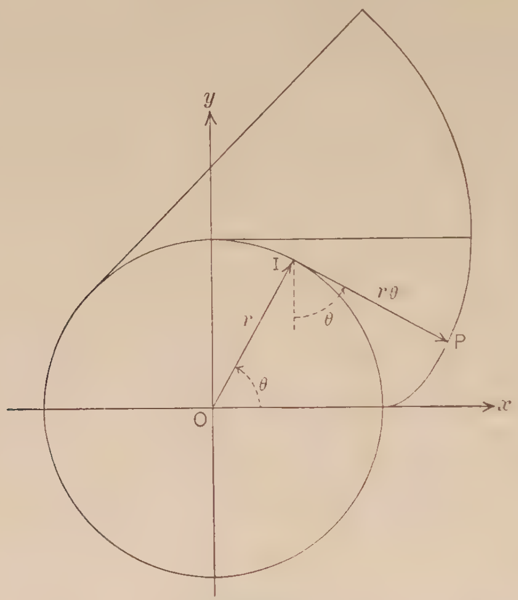


Fig. 147*a*.

Now suppose that the dotted circles a, b in Fig. 147*b* represent the pitch circles of two spur gears having their axes at A, B. Let the full line circles a', b', concentric with a, b, have radii r_a', r_b' proportional to the radii r_a, r_b of the pitch circles:

(1) $$\frac{r_a'}{r_b'} = \frac{r_a}{r_b}.$$

The common internal tangent MN of a', b' divides the line AB in the above ratio and consequently passes through the pitch point I. Let us now regard a', b' as cylindrical disks attached to the planes of the corresponding pitch circles, and the tangent MN as a portion of a crossed belt connecting them. If the disk a' is driven with the peripheral speed v, the belt will communicate this speed to the periphery of b', and the angular velocity ratio of the disks will be

(2) $$\frac{\omega_b}{\omega_a} = -\frac{v/r_b'}{v/r_a'} = -\frac{r_a'}{r_b'} = -\frac{r_a}{r_b},$$

and therefore precisely the same as if the pitch circles were rolling together.

As the disks revolve, a fixed point P of the belt, in its passage from M to N, will describe the curves ST and $S'T'$ on the revolving planes of a and b respectively. The arcs SP and $S'P$ are

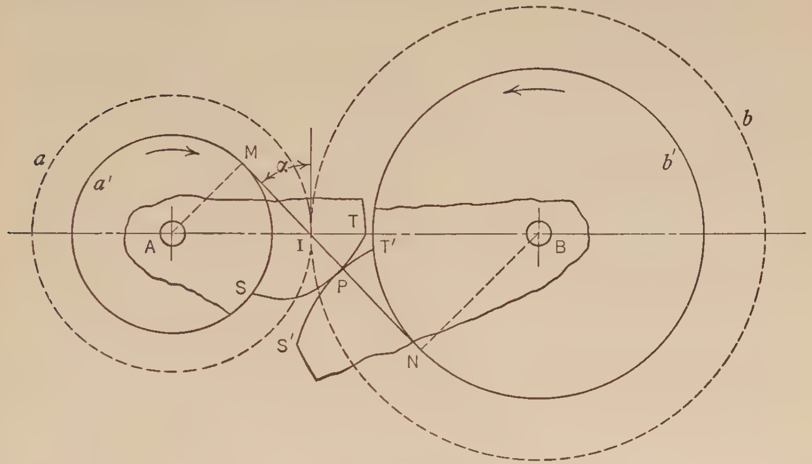


Fig. 147b.

described by P in passing from M to P, the arcs PT and PT' in passing from P to N. The curves SPT and $S'PT'$ are involutes of the circles a' and b' respectively. To see this clearly, imagine the disk a' fixed and an arm c, connecting the shafts A and B, revolved about A. (This arm is not shown in the figure; see Fig. 121b.) This will not change the *relative* motions of a, b and P. If c revolves clockwise from the position shown, P traces on a the arc PS, if counterclockwise, the arc PT. The entire curve SPT is evidently an involute of the base circle a'. Similarly by holding b' fast and revolving c about the shaft B, the curve $S'PT'$ is seen to be an involute of the base circle b'.

The involutes ST and $S'T'$ are suitable tooth profiles on a and b. To prove this we observe the following points:

(1) The curves in all positions have their tracing point P in common.

(2) The point P travels in the line MN while describing the curves; and as MN is tangent to both base circles, it is normal to both involutes at P. The curves thus have a common tangent at P and will work smoothly together.

(3) Since the common normal MN at the point of contact always passes through the pitch point I, the condition for a constant velocity ratio is satisfied.

The angle α between the common internal tangent MN to the base circles and the common tangent to the pitch circles at I is called the *obliquity*. It was formerly standard practice to take $\alpha = 14\frac{1}{2}°$; in recent years, however, values of the obliquity as high as 20° and 22° have been used with excellent results. When the obliquity is known, the radii of the base circles are given by

$$r_a' = r_a \cos\alpha, \quad r_b' = r_b \cos\alpha,$$

as we see at once from the figure.

Gear teeth with involute profiles have important advantages over other types of teeth. One of these is that a slight change in the distance between the centers of the gears does not affect their correct action. For as the base circles are unaltered, their common internal tangent is still normal to the teeth at their contact, and still divides AB in the ratio r_a'/r_b', ensuring the same constant velocity ratio as before.

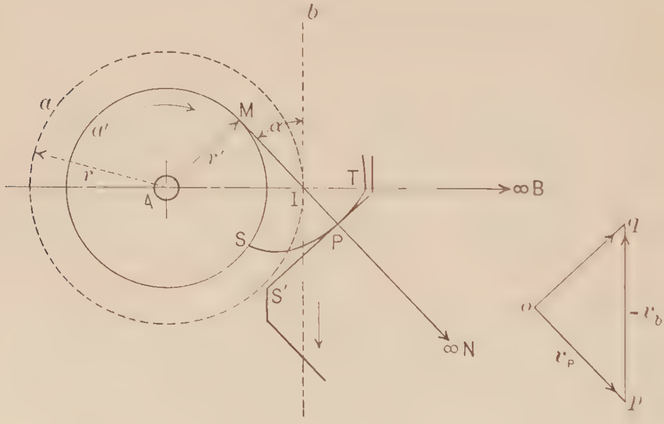


Fig. 147c

Let us now imagine that the circles a and a' are fixed while b and b' grow continuously larger as B recedes to infinity along the line AB. The pitch point I and the *position* of the tangent MIN are not affected by this limiting process. The obliquity therefore remains constant. The pitch circle b approaches a straight line

perpendicular to AI at I, and its interior expands into the "half-plane" to the right of this line. The entire base circle b' moves off to infinity. It is now hardly permissible to regard MP as part of a crossed belt. But if we regard MP as a string being unwrapped from a' and always passing through the point I, the rotation of a will communicate a translatory motion to b by rolling contact. A fixed point P of the string will then trace an involute of the circle a' upon its plane, and a straight line $S'P$, perpendicular to MP, on the plane of b. The latter fact, although fairly obvious from the nature of the limiting process, may be proved directly. For as P is moving with the speed $\omega r'$ in the direction MP, the plane b is moving vertically downward with the speed ωr. Draw

$$\overrightarrow{op} = \mathbf{v}_P, \qquad \overrightarrow{pq} = -\mathbf{v}_b; \qquad \text{then} \qquad \overrightarrow{oq} = \mathbf{v}_P - \mathbf{v}_b$$

is the velocity of P relative to b. Now

$$\frac{op}{pq} = \frac{\omega r'}{\omega r} = \frac{r'}{r} = \frac{MA}{AI} \qquad \text{and} \qquad \angle\, opq = \angle\, MAI;$$

the triangles opq and MAI are therefore similar, and $\angle\, poq = \angle\, AMI = 90°$. Since $\overrightarrow{oq}$, the velocity of P relative to b, is always perpendicular to op (or MP), P must describe on b a straight line perpendicular to MP.

We may now form teeth on b having straight profiles making angles of $90° + \alpha$ with its pitch line and having the same pitch as the involute teeth on the wheel a. When thus provided with teeth, b is called a *rack*. If the involute gear is driven with the constant angular velocity ω, it will communicate a uniform speed of translation ωr to the rack.

PROBLEMS

1. In Fig. 147*d* the circles a and b represent the pitch circles of a pair of spur gears with centers at A and B. The small circle m has its center M on the line AB and is tangent to both a and b at the pitch point I. If all three circles revolve about their fixed centers so as to roll together without slipping, show (1) that a point P on the circumference of m will describe an epicycloid on the plane of a, a hypocycloid on the plane of b, and (2) that these curves may be used as tooth profiles on a and b respectively for the transmission of the constant angular velocity ratio $\omega_b/\omega_a = -r_a/r_b$. [A second describing circle, equal to m and to the left of I, is needed to complete the tooth profiles.]

2. When the radius of the rolling circle is one half that of the fixed circle show that the hypocycloid becomes a diameter of the fixed circle. See § 126, Example.

Show that when the radius of the "describing circle" m in the preceding problem is equal to $\frac{1}{2}r_b$, the teeth of b are thinner at the root than at the pitch circle.

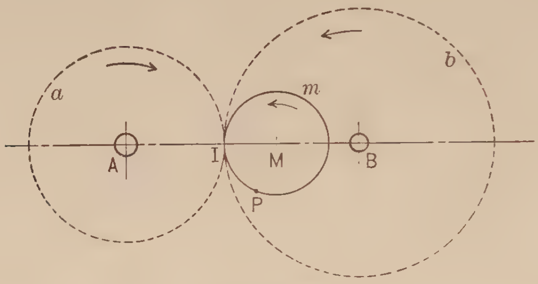


FIG. 147*d*.

3. Apply the method of Problem 1 to obtain the appropriate tooth profiles for a rack b to mesh with the gear a. (Keep the circles a and m fixed while B is allowed to recede to infinity.)

4. Show how the above method of obtaining tooth profiles may be applied to annular wheels. (Consider two pitch circles having internal contact at I.)

148. Summary, Chapter X. The motion of a plane figure in its plane is either an instantaneous translation ($\omega = 0$) or rotation ($\omega \neq 0$). When $\omega \neq 0$, a line through a point of the figure normal to its velocity will pass through the *instantaneous center* I. The intersection of two such lines locates I. The velocities of the figure are the same as if it were revolving about I as a fixed point with the ω for that instant.

The paths described by I in the fixed and moving planes are called the *space* and *body centrodes* respectively. I describes both centrodes with the same velocity; and the motion of the figure is reproduced by rolling the body centrode over the space centrode with this velocity.

If A, B are points of the figure,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA}, \quad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA},$$

where $\mathbf{v}_{BA}$ and $\mathbf{a}_{BA}$ denote the velocity and acceleration that B would have if the figure were revolving about A as a fixed point with the ω and α for that instant.

When $\omega \neq 0$ or $\alpha \neq 0$ there is a single point J of the figure, the *center of acceleration*, whose acceleration is zero. The accelerations of the figure are the same as if it were revolving about J as a fixed point with the ω and α for that instant.

If the vectors $\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C, \ldots$ are drawn from a point i, their end-points form a polygon similar to $ABC \ldots$ in the same sense, and i corresponds to I.

If the vectors $\mathbf{a}_A$, $\mathbf{a}_B$, $\mathbf{a}_C, \ldots$ are drawn from a point j, their end-points form a polygon similar to $ABC \ldots$ in the same sense, and j corresponds to J.

If the motion of a particle P is referred to rigid body in plane motion, its velocity and acceleration are

$$\mathbf{v} = \mathbf{v}_b + \mathbf{v}_r, \quad \mathbf{a} = \mathbf{a}_b + 2\,\boldsymbol{\omega}\times\mathbf{v}_r + \mathbf{a}_r$$

Here $\mathbf{v}_b$, $\mathbf{a}_b$ are the velocity and acceleration of the point of body which coincides with P at the instant and $\mathbf{v}_r$, $\mathbf{a}_r$ are the velocity and acceleration of P relative to the body. The term $2\,\boldsymbol{\omega}\times\mathbf{v}_r$ is called the *complementary acceleration.*

If a and b are plane figures with different angular velocities, they have a single pair of coincident points, I_{ab}, I_{ba} moving with the same velocity. I_{ab} is the instantaneous center of a relative to b. For three plane figures a, b, c, the points I_{ab}, I_{bc}, I_{ca} always lie in straight line (Theorem of Three Centers).

In a *kinematic chain* of a number of *links*, the relative motion of any two of the links determines the relative motions of all the others. By fixing successively the different links of a kinematic chain we obtain its *inversions.*

If the figures a, b have the angular velocities ω_a, ω_b, the angular velocity, ω_{ab}, of a relative to b is $\omega_a - \omega_b$. If c is a third figure

$$\omega_{ab} = \omega_{ac} - \omega_{bc}.$$

If the angular velocities of all the links of a kinematic chain are known relative to one link, the angular velocities relative to any other link may be computed from this equation.

The angular velocities of a and b relative to c are inversely proportional to the directed segments from I_{ab} to I_{ac} and I_{bc}:

$$\frac{\omega_{ac}}{\omega_{bc}} = \frac{I_{ba}I_{bc}}{I_{ab}I_{ac}}.$$

CHAPTER XI

DYNAMICS. FUNDAMENTAL PRINCIPLES

149. Orientation. Dynamics — the study of the motion of bodies as related to the forces acting on them — is divided into two branches, Statics and Kinetics (see Introduction). Statics is concerned with those problems in which the bodies considered remain at rest; Kinetics deals with problems in which the bodies are moving or are set in motion. We have already developed the *statics* of a particle and of rigid bodies from four fundamental principles, labeled A, B, C, D. We propose now to develop the *dynamics* of a particle and of rigid bodies from three fundamental principles which we shall designate by I, II, III. Principle I, the Principle of Force and Acceleration, is new; Principle II is the same as Principle A (Vector Addition of Forces) and Principle III is the same as Principle D (Action and Reaction). Principle B (Transmissibility of a Force) and Principle C (Static Equilibrium) will be deduced from Principles I, II, III. These principles thus form the foundation of Dynamics, that is, of *both* Statics and Kinetics. Principles B and C were adopted provisionally to permit the development of Statics independently of Kinetics.

Before proceeding with the study of Dynamics the student should carefully reread §§ 21, 22, 23, 32 of Chapter II dealing with force, weight, gravity, particles and rigid bodies.

150. Principle I: Force and Acceleration. *If a force acts upon a particle, free to move in any direction, what effect does it produce?* This is the fundamental question in the dynamics of a particle. The answer to this question is based upon a vast amount of experimental evidence, partly direct, but mostly indirect. Before giving it in its general form let us first consider a particular case.

The most familiar force is the weight of a heavy body, that is, the force it exerts on its supports when at rest relative to the earth (§ 22). Suppose now that the local weight of certain body is found by hanging it on a very sensitive spring balance. We shall then find that the weight of the body varies in different locali-

ties; in fact accurate measurements will show that the local weight of the body varies in exactly the same ratio as the local value of falling acceleration g (§ 118). Thus if W and W' denote the weights of the same body in two places where the falling accelerations are g and g', $W/g = W'/g'$. In other words, the ratio W/g for all localities is a constant whose value depends only upon the body weighed and the choice of units for W and g. Denoting this "body constant" by m, we have

$$W = mg.$$

Since the weight and falling acceleration have the same direction, namely downwards along a plumb-line at the locality in question, the above equation may also be written vectorially:

$$\mathbf{W} = m\mathbf{g}. \tag{1}$$

Now the weight $\mathbf{W}$ of the body is equal to the pull of the earth upon it (its *gravity*, § 32). The above equation thus states that the force of gravity on a free body will give it a falling acceleration in its direction and that the magnitude of this acceleration at any place is proportional to the magnitude of the force.

Accumulated experimental evidence now shows that this relation between the force of gravity and the acceleration it produces in a free body may be generalized so as to apply to all forces, whatever their type or supposed origin. We formulate this fundamental principle as follows:

PRINCIPLE I (FORCE AND ACCELERATION). *A free particle acted upon by a force acquires an acceleration in the direction of the force, and the magnitude of the acceleration is proportional to the magnitude of the force; that is*

$$\mathbf{F} = m\mathbf{a},$$

where m is a scalar constant whose value (for given units of length, time and force) depends entirely upon the nature of the body designated as a particle.

The equation $\mathbf{F} = m\mathbf{a}$ is the *fundamental equation of dynamics.* It implies that a frame of reference has been chosen (a system of rectangular axes, for example) relative to which the acceleration is measured. Unless the contrary is explicitly stated the reference frame is regarded as "fixed" although no body absolutely at rest is known to exist. *It is immaterial, however, whether we use a fixed reference frame or one moving with constant velocity*; for the

acceleration of a particle is the same in either case. In dealing with bodies on or near the earth, we usually choose a reference frame at rest relative to the earth and disregard the earth's own motion. But in certain problems a system of axes having directions fixed relative to the stars is required.

The Principle of Force and Acceleration is Newton's *Second Law of Motion.*

151. Mass. If a free particle acquires the accelerations $\mathbf{a}$ and $\mathbf{a}'$ when acted on by the forces $\mathbf{F}$ and $\mathbf{F}'$ respectively,

$$\mathbf{F} = m\mathbf{a} \quad \text{and} \quad \mathbf{F}' = m\mathbf{a}';$$

and as far as magnitudes are concerned

$$\frac{F}{a} = \frac{F'}{a'} = m.$$

Thus when the units of length, time, and force are chosen, there is associated with every body a certain characteristic number, $m = F/a$, where F and a are the magnitudes of any force acting on the body and the acceleration it produces. If the same force acts in turn on two bodies, the one having the larger m will acquire the smaller acceleration. Thus the constant m may be regarded as a measure of the reluctance of the body to change its state of motion or of rest; from this point of view, m is sometimes called the *inertia* of the body. It is customary, however, to call m the *mass* of the body and to regard this number as a measure of the quantity of matter it contains.

If we determine by experiment the acceleration produced in a body by some known force, the quotient *force/acceleration* gives the mass of the body. When the mass m of a body is known, the equation $\mathbf{F} = m\mathbf{a}$ enables us to determine the acceleration acquired by the free body when acted on by any known force, or to find a single force that would produce a given acceleration in the free body.

A body which acquires unit acceleration when acted on by a unit force has a mass of one unit. Thus if a force of 1 pound produces an acceleration of 1 ft./sec.^{2} in a body, this body has unit mass. The name " slug " has been coined for this unit; it is, however, rarely or never used in practice. This *British Gravitational System of Units*, in which the foot, the second, and the pound force are the fundamental units, is the one commonly used by American and British engineers. We recall that the pound force is the weight

of the standard pound body in a locality where $g = 32.1740$ ft./sec.^{2} (§§ 22, 118).

European engineers, other than British, often use the *Metric Gravitational System of Units* in which the meter, the second, and the kilogram force are the fundamental units. The kilogram force is the weight of the standard kilogram body in a locality where $g = 9.80665$ m./sec.^{2} (§§ 22,118).

In electrical engineering and in physics another system of units, called the *C.G.S.* (centimeter-gram-second) *System*, is almost universally used. In this system the unit of mass, the *gram*, is chosen as 1/1000 of the mass of the standard kilogram body. The unit of force, called the *dyne*, is the force that will give a body of 1 gram mass an acceleration of 1 cm./sec.^{2}.

152. Principle II: Vector Addition of Forces. The Principle of Force and Acceleration covers the case in which but one force acts on the particle. In order to apply our fundamental equation $\mathbf{F} = m\mathbf{a}$ when two or more forces act on the particle we apply a second principle, already stated as Principle A of Statics.

PRINCIPLE II (VECTOR ADDITION OF FORCES). *A system of forces acting simultaneously on the same particle may be replaced by a single force, acting on this particle, equal to their vector sum.*

By the aid of this principle we can also deal with particles that are constrained to move on certain curves or surfaces, provided we replace the constraints (supporting surfaces, cords) by appropriate forces (reactions, tensions). This amounts to drawing a *free-body diagram* for the particle as already explained in detail in § 36.

153. Principle III: Action and Reaction. Finally in order to deal with systems of particles and rigid bodies we need a third principle, already stated as Principle D of Statics:

PRINCIPLE III (ACTION AND REACTION). *The interaction between two particles, whether in direct contact or at a distance from each other, may be represented by two forces of equal magnitude and opposite direction acting along their joining line.*

Galileo recognized the truth of this principle; but it was first clearly stated by Newton as his *Third Law of Motion.*

We shall now develop the Dynamics of a particle and of rigid bodies from Principles I, II, and III.

154. Calculation of Mass. A body of local weight W is supported by a string. When at rest the body is in equilibrium under

two forces, the gravity or earth-pull $\mathbf{G}$ and the force $\mathbf{T}$ exerted by the string; hence, from the fundamental equation,

$$\mathbf{G} + \mathbf{T} = 0 \qquad \text{(Prin. I, II).}$$

The weight $\mathbf{W}$ of the body, the "action" it exerts on the string, is balanced by the "reaction" $\mathbf{T}$ of the string in the body:

$$\mathbf{W} + \mathbf{T} = 0 \qquad \text{(Prin. III).}$$

From those equations we have $\mathbf{W} = \mathbf{G}$: *the weight of a body is equal to its gravity.*

Suppose now that the string is cut and that the body begins to fall with the acceleration $\mathbf{g}$ under the force of gravity $\mathbf{G}$; then $G = mg$ (Prin. I) or, since $W = G$,

$$W = mg, \quad m = \frac{W}{g}.$$

The mass of a body is equal to its local weight divided by the local value of the falling acceleration.

When $m = 1$, $W = g$; hence a body of unit mass has a local weight numerically equal to the local value of the falling acceleration. Thus a body whose mass is 1 slug weighs about 32 pounds.

155. The Law of Inertia. When there is no force acting on a particle, or when all the forces combine to a zero resultant, the fundamental equation becomes

$$m\mathbf{a} = 0 \quad \text{or} \quad \frac{d\mathbf{v}}{dt} = 0.$$

Therefore $\mathbf{v}$ is a constant vector. When the particle has the initial velocity $\mathbf{v}_0$, its velocity will have the constant value $\mathbf{v} = \mathbf{v}_0$. If the particle is initially at rest, it will remain at rest (static equilibrium). Otherwise it will travel in a straight line with the velocity $\mathbf{v}_0$, for a constant velocity vector implies a fixed direction of motion. The vector equation of this line, obtained by integrating

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_0, \quad \text{is} \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t;$$

here $\mathbf{r}_0$ gives the initial position of the particle.

Conversely, if a particle is at rest or moving uniformly in a straight line, its acceleration is zero, and, from the fundamental equation, the forces acting on it must combine to a zero resultant.

These results are known as the

LAW OF INERTIA. *A particle will continue in a state of rest or of uniform motion in a straight line unless acted on by an unbalanced force.*

Galileo was the first to realize the truth of this fundamental law. It was stated in the above form by Newton as his *First Law of Motion.*

156. Summary, Chapter XI. Dynamics (Statics and Kinetics) is based on three Fundamental Principles:

I. Force and Acceleration ($\mathbf{F} = m\mathbf{a}$),
II. The Vector Addition of Forces,
III. Action and Reaction.

When the units of length, time and force are given, the constant $m = F/a$ depends only on the nature of the body and is called its *inertia* or *mass.* The mass of a body is equal to its local weight divided by the local value of the falling acceleration.

When the resultant force on a particle is zero it will remain at rest if originally at rest; otherwise it will travel in a straight line with constant velocity (*Law of Inertia*).

CHAPTER XII

DYNAMICS OF A PARTICLE

157. Scalar Equations of Motion. If the forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . . act on a particle we have from Principles I and II,

$$\sum \mathbf{F}_i = m\mathbf{a}.$$

This vector equation is equivalent to three scalar equations obtained by taking the components of both members on the axes of a fixed rectangular system. Thus if $\mathbf{F}_i = [X_i, Y_i, Z_i]$,

$$(1) \qquad \sum X_i = ma_x, \quad \sum Y_i = ma_y, \quad \sum Z_i = ma_z.$$

If the particle moves in a plane we may choose it as the xy-plane. Then $a_z = 0$ and hence $\sum Z_i = 0$; that is, the sum of the components of the forces normal to the plane must vanish. If the forces $\mathbf{F}_i$ lie in the xy-plane this condition is automatically fulfilled; the third equation of (1) is then omitted.

In the solution of problems a free-body diagram of the particle should be drawn showing all the external forces that act on it. The resultant of these forces is equal to $m\mathbf{a}$ where $m = W/g$. The acceleration vector, if shown in the diagram, should be drawn with short dashes to distinguish it from the forces, which are drawn in full. The axes are shown as dotted lines and their positive direction marked with an arrowhead.

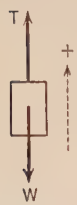


FIG. 157*a*.

Example 1. A body weighing 1 ton is hoisted vertically upward by a cable. It starts from rest and comes to rest with an acceleration numerically equal to 4 ft./sec.². Find the tension of the cable at the beginning and at the end of the motion.

The body is acted on by two forces: its weight, $W = 2000$ lb., and the tension T of the cable (Fig. 157*a*). If the positive direction is upward, we have

$$T - W = \frac{W}{g}a, \quad T = W\left(1 + \frac{a}{g}\right).$$

As the body starts upward, $a = 4$ ft./sec.² and

$$T = 2000\left(1 + \tfrac{4}{32}\right) = 2250 \text{ lb.}$$

As the body comes to rest, $a = -4$ ft./sec.² and

$$T = 2000\left(1 - \tfrac{4}{32}\right) = 1750 \text{ lb.}$$

What is the tension in the cable when the body is hoisted with uniform speed?

Example 2. A box weighing 1 ton rests on the floor of an elevator. Find the reaction of the floor on the box when the elevator is starting upward with an acceleration of 4 ft./sec.^{2}? when it comes to rest with a retardation of 4 ft./sec.^{2}.

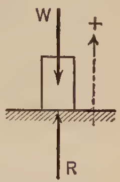


FIG. 157*b*.

The box is acted on by two forces: its weight W and the vertical reaction R of the floor of the elevator (Fig. 157*b*). If the positive direction is upward, we have

$$R - W = \frac{W}{g}a, \quad R = W\left(1 + \frac{a}{g}\right).$$

Thus R has the same values as T in Example 1.

Example 3. How long will it take a box, starting from rest, to slide down a smooth chute 50 ft. long and inclined at an angle $\beta = 30°$ to the horizontal?

The box is acted on by two forces: its weight W and the normal reaction of N of the plane. On taking components parallel and normal to the plane we have (Fig. 157*c*),

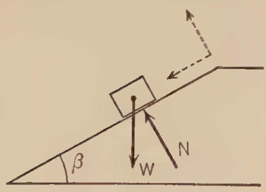


FIG. 157*c*.

$$W \sin \beta = \frac{W}{g}a, \quad N - W \cos \beta = 0;$$

hence $\quad a = g \sin \beta, \qquad N = W \cos \beta.$

If the box travels a distance x in the time t with the acceleration $g \sin \beta$,

$$x = \tfrac{1}{2} g \sin \beta\, t^2, \quad t = \sqrt{\frac{2x}{g \sin \beta}} \qquad (\S\ 116,\ 2).$$

With $x = 50$ ft., $\beta = 30°$, this gives

$$t = \sqrt{\frac{100}{16}} = \frac{10}{4} = 2\tfrac{1}{2} \text{ sec.}$$

At the bottom of the chute the box will have the velocity

$$v = at = 16 \times 2\tfrac{1}{2} = 40 \text{ ft./sec.}$$

Example 4. *Conical Pendulum.* A particle of weight W, attached to a string of length l, revolves uniformly in a horizontal circle of radius r. Find the tension of the string and the time required for a complete revolution (the *period*).

The particle is acted on by two forces: its weight W and the tension T of the string. If the constant angular velocity of the particle is ω rad./sec., its acceleration is $\omega^2 r$ and always directed toward the center of the circle (§ 109). On taking components horizontally and

vertically (Fig. 157d)

$$T \sin \beta = \frac{W}{g} \omega^2 r, \quad T \cos \beta - W = 0.$$

Thus $T = W/\cos \beta$, and from the first equation

$$\tan \beta = \frac{\omega^2 r}{g}, \quad \omega = \sqrt{\frac{g}{h}} \text{ rad./sec.}$$

since $\tan \beta = r/h$. Therefore the

$$\text{Period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}} \text{ sec.}$$

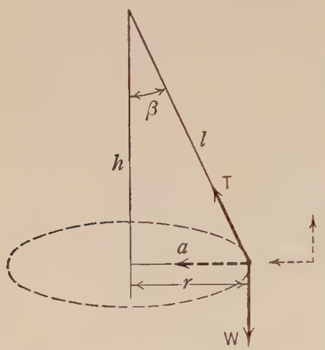


FIG. 157d.

As a numerical example, let

$W = 2$ lb., $l = 2$ ft.,
$r = 1$ ft., $g = 32.16$ ft./sec.²

Then $\sin \beta = \frac{1}{2}$, $\beta = 30°$; the string makes a constant angle of 30° with the vertical as long as ω is constant, and $h = 2 \cos 30° = 1.732$ ft. We now find

$$T = \frac{2}{\cos 30°} = 2.31 \text{ lb.,} \quad \text{Period} = 2\pi\sqrt{\frac{1.732}{32.16}} = 1.458 \text{ sec.}$$

PROBLEMS

1. A weight of 2 tons is raised from the ground to a height of 80 ft. in 5 sec. by a constant tension on the hoisting cable. Find the tension.

2. A sled on reaching the bottom of a hill has a speed of 15 mi. hr. How far will it run on a level against a resistance of 1/10 of the weight of the sled and riders?

3. An engine weighing 120 tons rounds curve of 800-ft. radius at a speed of 30 mi./hr. If the rails are on the same level, find the horizontal thrust on the outer rail.

4. A bicycle rounds a curve of radius r on a track banked at an angle β to the horizontal. If the plane of the bicycle remains normal to the slope of the track, prove that its speed $v = \sqrt{gr \tan \beta}$.

5. At what angle should an automobile speedway be banked on a curve of 400 ft. radius so that there is no side thrust on the tires at a speed of 60 mi./hr.?

6. A 1000-lb. box is pulled up a smooth chute inclined at an angle of 30° to the horizontal by a rope parallel to the chute. If the tension of the rope is constantly 600 lb., find the acceleration of the box.

7. A weight W hangs from two strings inclined at angles α and β to the horizontal. If the second string is cut, show that the tension

in the first is instantaneously reduced from $W \cos\beta/\sin(\alpha+\beta)$ to $W \sin\alpha$.

8. A small pail holding 3 lb. of water is whirled in a vertical circle at 60 r.p.m. by a string held in the hand. If the bottom of the pail is 2 ft. from the hand, find the pressure of the water on it when the pail is at its highest and lowest points. Find the least number of r.p.m. in order that the water may stay in the pail.

9. A stone on a string is whirled in a vertical circle of length r. Show that its angular velocity must be greater than $\sqrt{g/r}$ rad./sec. if the string remains taut.

10. A car weighing W lb. "loops the loop" over a vertical circular track of radius r ft. If its speed is v ft./sec. at the top find the reaction of the track at this point. What is the least value of v if the car does not leave the track?

158. Sliding Friction. When a body A slides over another body B, a force $\mathbf{F}$ is exerted by B upon A along the surface of contact which tends to retard its motion.* This force $\mathbf{F}$ is called the *sliding friction* on A. Experiment has shown that the sliding friction between two dry bodies depends on

(1) the nature of the surfaces of contact,
(2) the normal pressure per unit of area between the bodies,
(3) the relative speed of sliding (F decreases as v increases).

However for moderate values of the speed and unit normal pressure, F may be assumed to be independent of these factors and taken proportional to the total normal pressure N between the bodies. Thus we have a relation

$$F = \mu N$$

of the same form as in the limiting friction of rest (§ 32). Here μ, the *coefficient of sliding friction*, is a numerical constant for a given pair of surfaces. For the same surfaces μ is somewhat less than μ_0, the coefficient of static friction defined in § 32. Experiments on sliding at low speeds indicate that μ increases toward μ_0 as a limit as the speed approaches zero.

The values of μ and μ_0 in a few cases are as follows:

		μ_0	μ
Oak on oak: fibers parallel to motion	dry	0.62	0.48
	soaped	0.44	0.16
Cast iron on cast iron: greased		0.16	0.09
Steel on ice		0.027	0.014

* By the Principle of Action and Reaction, A exerts the force $-\mathbf{F}$ on B.

The decrease of μ with the speed is shown by the following data for steel tires *sliding* on steel rails at different speeds.

Speed in mi./hr.	Start	6.8	13.5	27.3	40.9	60
μ	0.242	0.088	0.072	0.070	0.057	0.027

In the case of a locomotive or car rolling over steel rails the friction between a wheel and rail just prior to sliding may be regarded as limiting static friction and computed from $F = \mu_0 W$, where W is the weight on the wheel considered, and μ_0 a constant depending on the condition of the wheels and rails and known as the *coefficient of adhesion*. It is probable that μ_0 does not vary greatly with the speed. As the tractive effort of a locomotive cannot exceed $\mu_0 D$, where D represents the weight on the driving wheels, the importance of μ_0 in locomotive design is apparent. When both rails and drivers are clean μ_0 reaches its maximum value, about 0.35; but if the surfaces are moist and greasy, μ_0 decreases to about 0.20 or 0.15. In recent locomotive design μ_0 has been assumed between 0.18 and 0.25. If the drivers slip, the coefficient of adhesion gives way to the coefficient of sliding friction μ, which, as seen from the values given above, is much smaller than μ_0, especially at high speeds.

The same considerations apply to all self-propelled vehicles. Thus for automobiles the greatest tractive effort is $\mu_0 D$, where μ_0 is the coefficient of static friction between tires and road, and D is the weight on the rear wheels. Again when the brakes are applied to the rear wheels, the maximum retarding force, $\mu_0 D$, is exerted just before the wheels begin to slip. When skidding occurs, the coefficient of sliding friction μ takes the place of μ_0 and the retarding force is diminished.

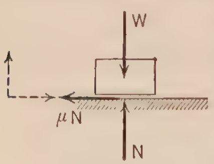


FIG. 158.

Example 1. A flat stone is sent gliding over ice with an initial velocity of 20 ft./sec. If the coefficient of sliding friction is $\mu = 0.025$, how far will it go?

From the free-body diagram (Fig. 158)

$$-\mu N = \frac{W}{g} a, \quad N - W = 0; \quad \text{hence} \quad a = -\mu g.$$

Let the stone travel a distance x before coming to rest; then (§ 116, 3)

$$0 - v_0^2 = 2\,ax = -2\,\mu g x,$$

$$x = \frac{v_0^2}{2\,\mu g} = \frac{20 \times 20}{2 \times 0.025 \times 32} = 250 \text{ ft.}$$

Example 2. A locomotive has a total weight of 50 tons on the drivers; if the coefficient of adhesion is 0.15, find the greatest pull that it can exert. How long will it take the locomotive, exerting this pull, to bring a train of 300 tons gross weight from rest up to a speed of 30 mi./hr., if the train resistance is 10 lb./ton?

The greatest pull is $0.15 \times 50 = 7.5$ tons; the net accelerating force on the train is therefore

$$P = 7.5 - \frac{300 \times 10}{2000} = 6 \text{ tons.}$$

If the gross weight is W tons,

$$P = \frac{W}{g}a = \frac{W}{g}\frac{v}{t};$$

and since 30 mi./hr. = 44 ft./sec.,

$$t = \frac{Wv}{Pg} = \frac{300 \times 44}{6 \times 32} = 69 \text{ sec. (approx.)}$$

Example 3. Solve Example 3, § 157 taking account of the friction between the chute and the box; $\mu = \frac{1}{4}$.

Three forces now act on the box: its weight W, the normal pressure N of the chute, and the friction μN directed up the plane (draw the free-body diagram). On taking components as before,

$$W \sin \beta - \mu N = \frac{W}{g}a, \quad N - W \cos \beta = 0.$$

Putting $N = W \cos \beta$ in the first equation, we find

$$a = g(\sin \beta - \mu \cos \beta) = 32 \sin 30° - 8 \cos 30° = 9.07 \text{ ft./sec.}^2.$$

Since $x = \frac{1}{2}at^2$,

$$t = \sqrt{\frac{2x}{a}} = \sqrt{\frac{100}{9.07}} = \frac{10}{3.01} = 3.32 \text{ sec.}$$

At the bottom of the chute the box will have the velocity $v = at =$ 30.1 ft./sec.

PROBLEMS

1. Solve Problem 6, § 157, when $\mu = 0.1$ between box and chute.

2. A box will slide down a chute inclined at an angle of 14° to the horizontal at a constant speed. In what time will it slide down a 50-ft. chute of 20° inclination if it starts from rest and μ is the same as before?

3. If the inclinations of the chutes in Problem 2 are α and β, show that $\mu = \tan \alpha$ and $a = g \sin(\beta - \alpha)/\cos \alpha$ on the second.

4. A particle slides down a rough plane, inclined at angle β to the horizontal, under the action of gravity. If ϕ is the angle of friction

($\tan \phi = \mu$) show that its acceleration down the plane is $a = g \sin (\beta - \phi)/\cos \phi$.

If the particle is projected up the plane, $a = g \sin (\beta + \phi)/\cos \phi$.

5. Find the least stopping distance on applying brakes to an automobile going 30 mi./hr. when $\mu = 0.6$ (clean, dry city streets) for

(*a*) two-wheel brakes when 60 per cent of the weight is on the rear wheels;

(*b*) four-wheel brakes.

6. Under the conditions of Problem 5 show that the stopping distance x in feet from a speed of V_0 mi./hr. is given by

$$x = 0.0934\ V_0^2 \quad \text{for two-wheel brakes,}$$
$$x = 0.0560\ V_0^2 \quad \text{for four-wheel brakes.}$$

7. On a speedway curve of 400-ft. radius, what is the least banking angle required to keep the wheels from side-skidding at 60 mi. hr. if $\mu = 0.4$?

8. Solve Problem 7 in general terms. Show that least banking angle β to avoid side-skidding on a curve of radius r at speed v is given by

$$\tan (\beta + \phi) = \frac{v^2}{gr} \quad \text{where} \quad \tan \phi = \mu.$$

159. Two Particles. If two bodies move in contact or when connected to each other, a free-body diagram should be drawn for each body before the dynamical equations are applied. When the bodies are connected by a cord, the tension of the cord where it is cut must appear in the diagrams. When the cord is light in comparison with the bodies involved, we shall neglect its weight and assume that its tension is the same at all sections

(1) when it is unsupported or passes over a fixed smooth surface, or

(2) when it passes over a "weightless" pulley turning on a smooth pin.*

Example 1. *Atwood's Machine.* Two bodies of weight W and W' are connected by a cord which passes over a fixed smooth circular cylinder (Fig. 159*a*). Find their common acceleration a and the tension T of the cord.

In the free-body diagrams the tension T acts upward on each body. If $W > W'$ the bodies will move in the directions of the dotted arrows

* These assumptions may be justified from the dynamical equations of a portion of the cord in case (1), from the dynamical equations of the pulley in case (2).

with numerically equal accelerations a. On taking components in these directions we have

$$W - T = \frac{W}{g} a, \quad T - W' = \frac{W'}{g} a.$$

To eliminate T we add these equations; thus

$$W - W' = (W + W')\frac{a}{g}, \quad a = \frac{W - W'}{W + W'} g.$$

The acceleration a is therefore constant and less than g. By choosing $W - W'$ sufficiently small, a can be made so small that it may be conveniently determined by experiment and g then computed from the equation above. The apparatus may therefore be used to determine g.

The tension T of the cord may be found by substituting the value of a in either of the original equations. Thus from the first equation

$$T = W\left(1 - \frac{a}{g}\right) = \frac{2\,WW'}{W + W'}.$$

As a numerical example let $W = 2$ lb., $W' = 1$ lb. Then $a = \frac{1}{3} g$ and $T = 1\frac{1}{3}$ lb.

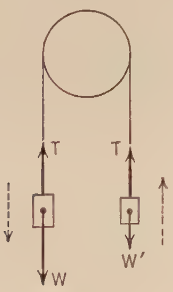


Fig. 159a.

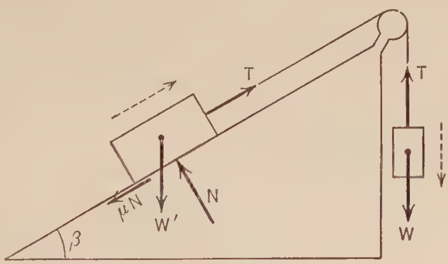


Fig. 159b.

Example 2. In Fig. 159b the body of weight W falls and draws the other body of weight W' up a rough plane. If the cord passes over a fixed smooth disk, find its tension T and the common acceleration a of the bodies.

From the free-body diagrams

(i) $$T - W' \sin\beta - \mu N = \frac{W'}{g} a, \quad N - W' \cos\beta = 0;$$

(ii) $$W - T = \frac{W}{g} a.$$

Since $N = W' \cos \beta$, we have from (i)

$$T - W'(\sin \beta + \mu \cos \beta) = \frac{W'}{g} a.$$

To eliminate T we add this to (ii), obtaining

$$W - W'(\sin \beta + \mu \cos \beta) = (W + W')\frac{a}{g}, \quad \text{whence}$$

$$a = \frac{W - W'(\sin \beta + \mu \cos \beta)}{W + W'} g.$$

We may now compute the tension from (ii)

$$T = W\left(1 - \frac{a}{g}\right).$$

As a numerical example let $W = 80$ lb., $W' = 100$ lb., $\mu = \frac{1}{4}$. Then

$$a = \frac{8.35}{180} 32 = 1.48 \text{ ft./sec.},$$

$$T = 80\left(1 - \frac{8.35}{180}\right) = 76.3 \text{ lb.}$$

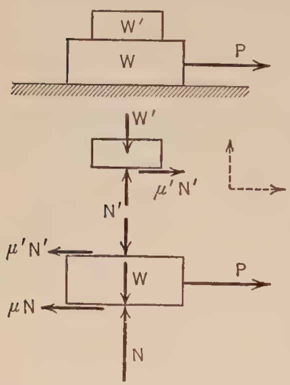


FIG. 159c.

Example 3. A block of weight W, supporting another of weight W', is drawn along a rough plane by a horizontal force P (Fig. 159c). The coefficient of sliding friction between plane and block is μ, the coefficient of static friction between the blocks is μ'. Find the greatest value that P can have if the upper block does not slip.

From the free-body diagrams

(i) $$P - \mu N - \mu' N' = \frac{W}{g} a, \quad N - N' - W = 0;$$

(ii) $$\mu' N' = \frac{W'}{g} a, \quad N' - W' = 0.$$

Hence

$$N' = W', \quad N = W + W',$$

and from (ii), $a = \mu' g$. This is the limiting value of a just before slipping occurs. Substituting the values of N, N' and a in (i) now gives the greatest value of P:

$$P = \mu(W + W') + \mu' W' + \mu' W = (\mu + \mu')(W + W').$$

As a numerical example let $W = 80$ lb., $W' = 40$ lb., $\mu = \frac{1}{4}$, $\mu' = \frac{1}{3}$. Then

$$P = (\tfrac{1}{4} + \tfrac{1}{3})(80 + 40) = 70 \text{ lb.}$$

The above methods may also be applied to three or more connected bodies.

Example 4. Three bodies of weight W_1, W_2, W are connected by a cord sliding over two smooth fixed disks and over the smooth surface of the central body (Fig. 159*d*). Find the accelerations a_1, a_2, a of the bodies and the tension T of the cord.

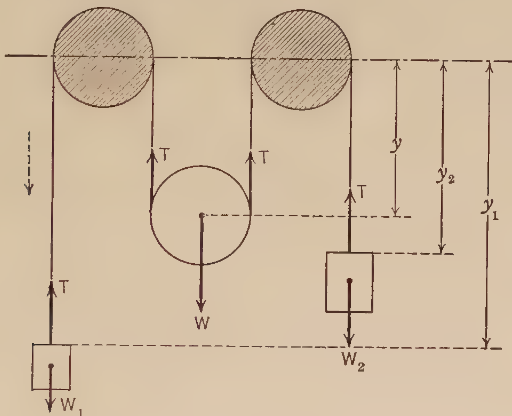


FIG. 159*d*.

Take the positive direction downward; then from the free-body diagrams

$$W_1 - T = \frac{W_1}{g} a_1, \quad W_2 - T = \frac{W_2}{g} a_2, \quad W - 2T = \frac{W}{g} a.$$

To find the relation between a_1, a_2 and a, we make use of the equation

$$y_1 + y_2 + 2y = \text{const.}$$

which states that the length of the connecting cord is constant. On differentiating this twice with respect to the time we get

$$a_1 + a_2 + 2a = 0.$$

If we solve the equations above for a_1, a_2 and a and substitute in this relation we find

$$T\left(\frac{1}{W_1} + \frac{1}{W_2} + \frac{4}{W}\right) = 4.$$

This equation gives T; the first three equations now give a_1, a_2 and a respectively.

For example let $W_1 = 5$, $W_2 = 10$, $W = 20$ lb. Then

$$T(\tfrac{1}{5} + \tfrac{1}{10} + \tfrac{4}{20}) = 4, \quad \text{whence} \quad T = 8 \text{ lb.};$$

$$a_1 = -\tfrac{3}{5} g, \quad a_2 = \tfrac{1}{5} g, \quad a = \tfrac{1}{5} g.$$

PROBLEMS

1. In Fig. 159*e* the weight $W = 30$ lb. falls and pulls the weight $W' = 15$ lb. along a smooth horizontal plane. If the cord passes over a fixed smooth disk, find its tension T and the acceleration of the weights.

2. If the coefficient of sliding friction between the plane and body is $\mu = \frac{1}{3}$ in Problem 1, find T and a. Show, in general, that

$$a = \frac{W - \mu W'}{W + W'} g, \qquad T = \frac{(1 + \mu) W W'}{W + W'}.$$

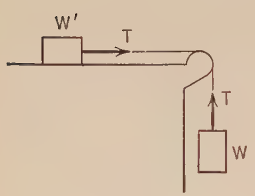


FIG. 159*e*.

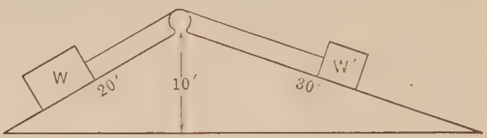


FIG. 159*f*.

3. In Fig. 159*f*, $W = 30$ lb., $W' = 20$ lb. When the planes are smooth, find the acceleration of the bodies and the tension of the cord.

4. In Fig. 159*g* find the acceleration of the bodies and the tensions T, T' when friction is neglected.

5. Solve Problem 4 when $\mu = \frac{1}{4}$ for the 20 lb. and 36 lb. bodies, and the hanging weight is 40 lb.

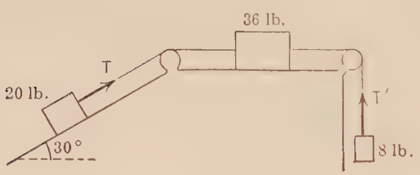


FIG. 159*g*.

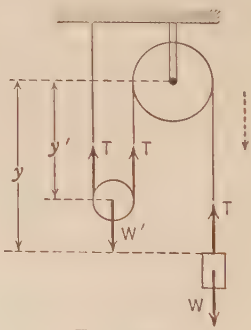


FIG. 159*h*.

6. In Fig. 159*h*, the bodies of weight W and W' are connected by a cord which passes over the smooth rim of the latter and a fixed smooth disk. Show that

$$a' = \frac{W' - 2W}{W' + 4W} g, \quad a = -2a'; \quad T = \frac{3WW'}{W' + 4W}.$$

[From $y + 2y' = \text{const.}$ obtain $a + 2a' = 0$.]

7. Two bodies of weight W and W', connected by a cord, slide down a rough plane inclined $\beta°$ to the horizontal. If the coefficients of friction between the plane and the bodies are μ and μ' ($\mu < \mu'$) and W is the lower body, show that the acceleration and tension of the cord are

$$a = \left(\sin\beta - \frac{\mu W + \mu' W'}{W + W'}\cos\beta\right)g, \quad T = \frac{WW'}{W + W'}(\mu' - \mu)\cos\beta.$$

8. If $\mu > \mu'$ in Problem 7 and the bodies slide down the plane in contact with each other, show that they exert a force

$$R = \frac{WW'}{W + W'}(\mu - \mu')\cos\beta \text{ on each other.}$$

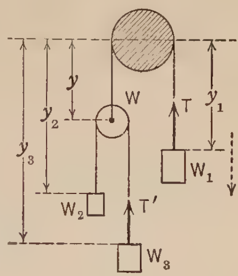


FIG. 159*i*.

9. In Fig. 159*i*, the weights of the four moveable bodies are $W = 1$, $W_1 = 4$, $W_2 = 2$, $W_3 = 1$ lb. The cord connecting W and W_1 passes over a fixed smooth disk (shaded), the cord connecting W_2 and W_3 passes over the smooth rim of W. Taking the positive direction downward, show that the accelerations and tensions are

$$a_1 = \tfrac{1}{23}g, \quad a_2 = \tfrac{7}{23}g, \quad a_3 = -\tfrac{9}{23}g; \quad T = \tfrac{88}{23}\text{ lb.}, \quad T' = \tfrac{32}{23}\text{ lb.}$$
$$[y_1 + y = \text{const. and } y_2 + y_3 - 2y = \text{const.}]$$

10. The Atwood's machine of Example 1 is placed on a platform balance and weighed when the cord is clamped to the cylinder. Show that when the cord is released the machine will decrease in weight by an amount $(W - W')^2/(W + W')$.

160. Differential Equation of Motion. If $\mathbf{F}$ is the resultant of all the forces acting on a particle and we put $\mathbf{a} = d^2\mathbf{r}/dt^2$ in the fundamental equation,

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} \tag{1}$$

where $\mathbf{r}$ is the position vector of the particle referred to a fixed origin. If $\mathbf{F}$ is known as a function of $\mathbf{r}$, this differential equation of the second order determines the motion of the particle, provided that its position and velocity are known at a certain instant, say

$$\mathbf{r} = \mathbf{r}_0, \quad \mathbf{v} = \mathbf{v}_0 \quad \text{when} \quad t = 0.$$

If we refer the motion to fixed rectangular axes and write $\mathbf{r} = [x, y, z]$, $\mathbf{F} = [X, Y, Z]$, the vector equation (1) is equivalent to the

three scalar equations

(2) $$m\frac{d^2x}{dt^2} = X, \quad m\frac{d^2y}{dt^2} = Y, \quad m\frac{d^2z}{dt^2} = Z.$$

If we take the components of (1) along the tangent, principal normal, and binormal to the path, we obtain (see § 107):

(3) $$m\frac{dv}{dt} = F_t, \quad \frac{mv^2}{\rho} = F_n, \quad 0 = F_b.$$

These are called the *intrinsic equations of motion.*

161. Momentum and Impulse. Since $\mathbf{a} = d\mathbf{v}/dt$, we may write (§ 160, 1) as

(1) $$m\frac{d\mathbf{v}}{dt} = \mathbf{F} \quad \text{or} \quad \frac{d}{dt}(m\mathbf{v}) = \mathbf{F}.$$

The vector $m\mathbf{v}$, the product of the mass and velocity, is called the *momentum* of the particle. Equation (1) states that

The time rate of change of the momentum of a particle is equal to the resultant of the forces acting upon it.

As the time changes from t_1 to t_2 let the velocity of the particle change from $\mathbf{v}_1$ to $\mathbf{v}_2$. On integrating (1) between these limits we obtain

(2) $$m\mathbf{v}_2 - m\mathbf{v}_1 = \int_{t_1}^{t_2} \mathbf{F}\, dt.$$

The time integral of the force over any interval is called the *impulse* of the force in this interval. Thus (2) states the

Principle of Impulse and Momentum. *The change in the momentum of a particle in any time interval is equal to the impulse of the resultant force in this interval.*

In the important case when $\mathbf{F}$ is constant the impulse is simply $\mathbf{F}(t_2 - t_1)$, the product of the force and the time interval. Then (2) becomes

(3) $$m\mathbf{v}_2 - m\mathbf{v}_1 = \mathbf{F}\,(t_2 - t_1).$$

By equating the components of both members we obtain three scalar equations.

When the force is measured in pounds, the time in seconds, the unit of impulse is the *pound-second.*

Example 1. A box weighing 20 lb. is given an initial velocity of 8 ft./sec. up a chute inclined 5° to the horizontal. If $\mu = \frac{1}{4}$, in what time will it come to rest?

Since $\mathbf{v}_2 = 0$, (3) gives $-m\mathbf{v}_1 = \mathbf{F}t$. On taking components in the directions shown Fig. 161*a* we obtain

$$-\tfrac{20}{32}\,8 = -(20 \sin 5^\circ + \tfrac{1}{4}N)t, \quad 0 = (N - 20 \cos 5^\circ)t.$$

Hence $N = 20 \cos 5^\circ$ and

$$t = \frac{5}{20 \sin 5^\circ + 5 \cos 5^\circ} = 0.74 \text{ sec.}$$

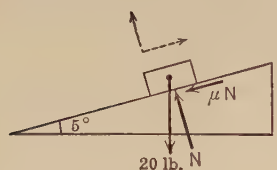


Fig. 161*a*.

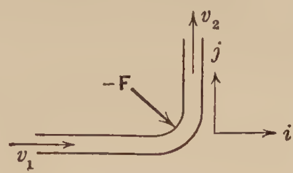


Fig. 161*b*.

Example 2. If 40 cu. ft. of water per minute flow through a 3-inch pipe in which there is a 90° bend, what is the resultant force exerted by the water on the bend if frictional resistances are neglected?

Since $\frac{2}{3}$ cu. ft. of water per second flows through an area of $\pi/16$ sq. ft., its speed v is given by

$$v\frac{\pi}{16} = \frac{2}{3}, \qquad v = 3.4 \text{ ft./sec.}$$

Since 1 cu. ft. of water weighs 62.4 lb., $\frac{2}{3}\,62.4 = 41.6$ lb. of water pass the bend each second. If $\mathbf{F}$ is the force exerted by the bend on the water, the impulse on the water per second is, from (3),

$$\mathbf{F} = \frac{41.6}{32}(3.4\mathbf{j} - 3.4\mathbf{i}) = 4.42\,(\mathbf{j} - \mathbf{i}).$$

The reaction of the water on the bend is $-\mathbf{F} = 4.42\,(\mathbf{i} - \mathbf{j})$, a force numerically equal to $4.42\sqrt{2} = 6.25$ lb. and directed as shown in Fig. 161*b*.

When the force has a constant direction, say $\mathbf{F} = F\mathbf{i}$, its

$$\text{Impulse from } t_1 \text{ to } t_2 = \mathbf{i}\int_{t_1}^{t_2} F\,dt.$$

If F is plotted as ordinate against t as abscissa, we obtain the *force-time curve;* and the integral above is equal to area included between this curve, the t-axis, and the ordinates at t_1 and t_2. The unit to be used in estimating this area is the rectangle formed by the units of the t and F scales.

Example 3. A force F increases at a constant rate from zero to P lb., then decreases at the same rate to zero. If the total time is

t sec., the force-time curve is an isosceles triangle of height P and base t. The impulse of F in the time t is numerically equal to the area of this triangle, $\frac{1}{2}Pt$ lb.-sec.

Example 4. A blow with a hammer gives a 2-lb. body an initial velocity of 1 ft./sec. If the impact lasted 0.001 sec., and the force varied as in Example 3, find its greatest value P.

The impulse of the blow is numerically equal to $\frac{1}{2}Pt = P/2000$ lb.-sec. Equating this to the change of momentum we get

$$\frac{P}{2000} = \frac{2}{32}(1 - 0) = \frac{1}{16}, \qquad P = 125 \text{ lb.}$$

PROBLEMS

1. A stone is thrown vertically upward with a speed of 80 ft./sec. Find its velocity after 2, 3 and 5 sec.

2. A stone is thrown horizontally from a cliff with a speed of 80 ft./sec. Find its velocity after 5 sec.

3. A flat stone is sent gliding over ice with an initial speed of 20 ft./sec. If the coefficient of sliding friction is $\mu = 0.025$, in what time will its speed be 10 ft./sec.?

4. A jet of water $1\frac{1}{2}$ in. in diameter, and with a speed of 60 ft./sec., strikes a vertical wall at right angles. What force F does it exert, assuming that the water flows away along the wall?

If some of the water splashes back from the wall, will the force be greater or less than the foregoing value?

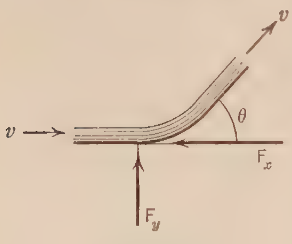


FIG. 161c.

5. A jet of water 2 in. in diameter and with a speed of 40 ft./sec. strikes a fixed curved vane (Fig. 161*c*). If the jet is turned through 30° by the vane, find the force exerted by the water on the vane.

6. If, in the preceding problem, W lb. of water strike the vane per second and are turned through an angle θ, show that the resultant force on the vane is $2(W/g)\,v \sin \frac{1}{2}\theta$ lb.

162. Kinetic Energy and Power. If we multiply both sides of (§ 161, 1) by $\mathbf{v}\cdot$ we obtain the scalar equation

$$m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \mathbf{F}\cdot\mathbf{v},$$

or since

$$\frac{d}{dt}\mathbf{v}^2 = 2\,\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \qquad (\S\ 84,\ 6),$$

$$\frac{d}{dt}(\tfrac{1}{2}m\mathbf{v}^2) = \mathbf{F}\cdot\mathbf{v}.$$

The scalar $\frac{1}{2}mv^2$, one half the product of the mass and the square of the velocity, is called the *kinetic energy* of the particle. The scalar product $\mathbf{F}\cdot\mathbf{v}$ of the force and velocity is called the *power* exerted by the force. In view of these definitions, equation (1) states that:

The time rate of change of the kinetic energy of a particle is equal to the power exerted by the resultant force.

163. Work. The time integral of the power $\mathbf{F}\cdot\mathbf{v}$, between the instant t_1 and any later instant t is called the *work* done by the force $\mathbf{F}$ in this interval. The work W is therefore defined by the equation

$$W = \int_{t_1}^{t} \mathbf{F}\cdot\mathbf{v}\,dt. \tag{1}$$

If the upper limit t is regarded as variable we have

$$\frac{dW}{dt} = \mathbf{F}\cdot\mathbf{v};$$

the power of a force is the time-rate at which it is doing work.

In many cases the force is known when the position of the particle is given. Then the work is more readily computed by changing the variable of integration in (1) from t to s, the arc along the path of the particle. This may be done by writing

$$\mathbf{v} = v\mathbf{T} = \mathbf{T}\frac{ds}{dt} \qquad (§\ 104,\ 2)$$

in (1); we thus obtain

$$W = \int_{s_1}^{s} \mathbf{F}\cdot\mathbf{T}\,ds, \tag{2}$$

the integral being taken over the path of the particle. Therefore *the work done by a force on a particle is equal to the integral of its tangential component taken over the path.*

If we replace $\mathbf{v}$ by $d\mathbf{r}/dt$ in (1), the work done by $\mathbf{F}$ may also be expressed in the form

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} \tag{3}$$

where $\mathbf{r}_1$ denotes the position vector of the particle at the instant t_1.

In the important case when $\mathbf{F}$ is constant the work done is simply

$$W = \mathbf{F}\cdot\int_{\mathbf{r}_1}^{\mathbf{r}} d\mathbf{r} = \mathbf{F}\cdot(\mathbf{r} - \mathbf{r}_1) = \mathbf{F}\cdot\overrightarrow{P_1P}; \tag{4}$$

that is, *the work done by a constant force on a particle as it travels over any path is equal to the scalar product of the force and the displacement vector.* If d is the distance P_1P and θ is the angle between the force and displacement vectors, (4) may also be written

$$W = Fd \cos \theta. \tag{5}$$

Thus W is positive, zero, or negative according as θ is an acute, right or obtuse angle.

A unit force acting on a particle as it moves a unit distance in the direction of the force will do a unit amount of work; if the force is measured in pounds, the distance in feet, the unit of work is called the *foot-pound* (ft.-lb.).

Since power is the time-rate of doing work, the unit of power is one *foot-pound per second* (ft.-lb./sec.). One *horsepower* (hp.) is defined as 550 ft.-lb./sec. or 33,000 ft.-lb./min.

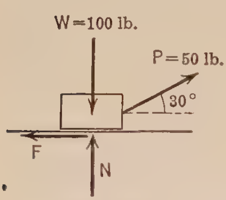


FIG. 163a.

Example 1. A body of weight $W = 100$ lb. is drawn along horizontal plane by a constant force $P = 50$ lb. inclined 30° to the horizontal (Fig. 163a). The work done by P as the body moves 10 ft. in a straight line is, from (5),

$$50 \times 10 \cos 30° = 500 \times 0.866 = 433 \text{ ft.-lb.}$$

The friction on the body is $F = \mu N$, where N is the normal pressure. To find N we have (§ 157)

$$N + 50 \sin 30° - 100 = 0, \qquad N = 100 - 25 = 75 \text{ lb.}$$

If $\mu = \frac{1}{3}$, $F = 75/3 = 25$ lb., and the work done by friction in 10 ft. is

$$25 \times 10 \cos 180° = -250 \text{ ft.-lb.}$$

Both W and N, being perpendicular to the direction of motion, do no work. The total work done on the body is therefore $433 - 250 = 183$ ft.-lb.

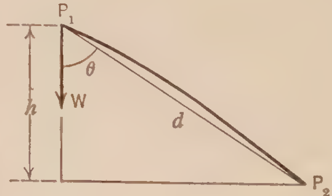


FIG. 163b.

Example 2. The work done by gravity on a body weighing W lb. as it falls from P_1 to P_2 through a height of h ft. (Fig. 163b) is

$$\mathbf{W} \cdot \overrightarrow{P_1P_2} = Wd \cos \theta = Wh \text{ ft.-lb.}$$

This result applies when the body slides along any fixed surface from P_1 to P_2 as well as when it falls freely between these points, as in the case of a projectile.

If a projectile, shot upward, travels from P_2 to P_1, the work done by gravity is

$$\mathbf{W} \cdot \overrightarrow{P_2P_1} = -Wd\cos\theta = -Wh \text{ ft.-lb.}$$

Example 3. A helical spring is compressed 2 in. by an axial load of 120 lb. Within certain limits the force F required to compress a spring an amount x is proportional to x; thus $F = kx$ where k is a constant. In our problem, $120 = 2k$, $k = 60$; hence $F = 60\,x$ lb. when x is measured in inches.

Let us compute the work done by an axial force in compressing this spring 3 in. From (2) we have

$$W = \int_0^3 F\,dx = 60\int_0^3 x\,dx = 30\,x^2\Big|_0^3 = 270 \text{ in.-lb.}$$

Example 4. Find the work done by a gas expanding in a cylinder at constant temperature from the pressure p_1 lb./in.² and volume v_1 in.³ to the volume v_2 in.³.

Let the cylinder be fitted with a weightless piston of cross-section A in.² and let x in. be the distance from the end of the cylinder to the piston (Fig. 163*c*). Then the volume of the gas is $v = Ax$ in.³. The gas exerts a force of pA lb. on the piston and the work done on the piston as it travels from x_1 to x_2 is, from (2),

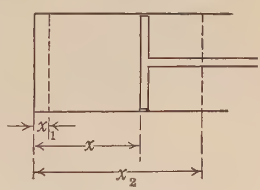


FIG. 163*c*.

$$W = \int_{x_1}^{x_2} pA\,dx = \int_{v_1}^{v_2} p\,dv.$$

Since the temperature is constant we have $pv = p_1v_1$ from Boyle's Law; hence

$$W = p_1v_1\int_{v_1}^{v_2}\frac{dv}{v} = p_1v_1 \ln\frac{v_2}{v_1} \text{ in.-lb.}$$

Example 5. A locomotive pulls a train of 3000 tons up a 1 per cent grade at 6 mi./hr. What horsepower is the engine exerting if the frictional resistance is 5 lb./ton?

The grade resistance on an incline of angle α is

$$3000 \times 2000 \sin\alpha = 6{,}000{,}000 \times 0.01 = 60{,}000 \text{ lb.};$$

for $\sin\alpha$ practically equals $\tan\alpha$ (0.01) for small angles.

Since the frictional resistance is $3000 \times 5 = 15{,}000$ lb., the total force overcome is $F = 75{,}000$ lb. Now 6 mi./hr. = 8.8 ft./sec.; hence the

$$\text{hp.} = \frac{Fv}{550} = \frac{75000 \times 8.8}{550} = 1200.$$

PROBLEMS

1. What work is done against gravity in pulling an ore-car weighing 10,000 lb. up a 3 per cent grade a distance of 500 ft.?

2. A 100-lb. weight slides a distance of 40 ft. down a 30° inclined plane under the action of gravity. If $\mu = \frac{1}{4}$, compute the total work done on the body.

3. A well is dug h ft. deep and A sq. ft. in cross-section. Find the work done in raising the material to the ground level, if one cu. ft. weighs w lb. [The work done in raising a slice of earth y ft. below the ground level and of thickness dy is $wA\,dy{\cdot}y$ ft.-lb.]

Find the work for a well 20 ft. deep and 4 ft. in diameter if $w = 100$ lb./ft.^{3}.

4. A 1500-lb. cable 300 ft. long hangs from a winding-drum. What work is done in winding up 100 ft.?

5. Water is pumped into a tank d ft. above a reservoir, filling it to a height of h ft. If W is the total weight of water moved, show that the work done is $W\,(d + \frac{1}{2}h)$ ft.-lb.

6. A weight of 400 lb. is drawn from a depth of 480 ft. by a cable which weighs 2 lb./ft. How much work is done?

7. The force F needed to give a steel bar, originally l in. long and A in.^{2} in cross-section, a stretch of x in. is $F = AE\,x/l$ where E is the modulus of elasticity of the steel. Find the work done in stretching the bar e in.

Find the work done in stretching a steel bar 10 ft. long, 2 in. in diam., $\frac{1}{4}$ in. $E = 30{,}000{,}000$ lb./in.^{2}.

8. A man weighing 150 lb. walks up a 10° slope at the rate of 4 mi./hr. What horsepower is he exerting?

9. A 100-lb. box is drawn 20 ft. up a rough plane ($\mu = \frac{1}{4}$), inclined at an angle of 30° to the horizontal, by a force P inclined at an angle of 20° to the slope of the plane. Find P if the box moves with constant speed. Show that the work done by P is equal to the work done against gravity and against friction.

If $P = 70$ lb., find the total work done on the box in 20 ft.

10. Air expands adiabatically according to the law $pv^{1.4} =$ const. from $v_1 = 6$ ft.^{3} and $p_1 = 80$ lb./in.^{2} to $v_2 = 20$ ft.^{3}. Find the work done.

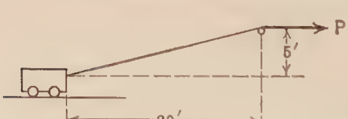


Fig. 163d.

11. A ship is steaming V mi./hr. under a horsepower H. Find the thrust of the screw in tons.

12. A car is pulled along a straight track by means of a rope passing over a pulley as shown in Fig. 163d. If the tension $P = 20$ lb., find the work done while the car moves 10 ft. from the position shown. Solve in two ways.

13. A man walking 6 ft./sec. drags after him 16 ft. of chain weighing 5.5 lb. per ft. If the chain is held 4 ft. above the ground and $\mu = \frac{1}{4}$ between chain and ground, show that the man is exerting 0.15 hp. [If x ft. of chain trail on the ground, the horizontal tension is $H = wc = \mu wx$; hence $c = \mu x$ for the catenary. Now find x from (§ 97, 9).]

164. Graphical Representation of Work. If we represent the tangential component of the force **F** by F_t, the work done by **F** on a particle as it travels the distance $s_2 - s_1$ along a curve is (§ 163, 2)

$$W = \int_{s_1}^{s_2} F_t\, ds. \tag{1}$$

If F_t is plotted as ordinate against s as abscissa we obtain a curve showing the variation of F_t with s. Then (1) shows that the work W may be measured by the area included between this F_t-s curve, the s-axis, and the ordinates at s_1 and s_2 (Fig. 164a). The unit used in evaluating this area is the rectangle formed by the units of the F_t and s scales. Thus if 1 in. horizontal = 5 ft., 1 in. vertical = 10 lb., 1 in.^{2} of area represents 50 ft.-lb. of work.

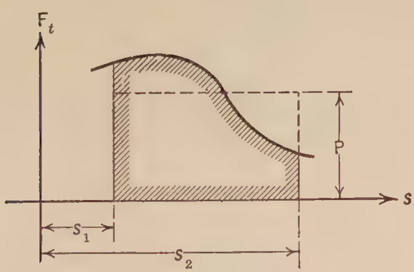


FIG. 164a.

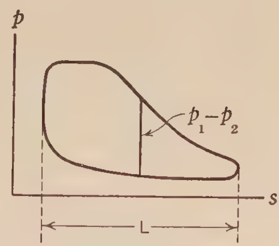


FIG. 164b.

If the constant tangential force P acting over the same distance $s_2 - s_1$ does the same amount of work as the variable F_t, P is called the *space-average* of F_t over this distance. P is represented on the force scale by the height of a rectangle over the base $s_2 - s_1$ whose area is equal to the corresponding area under the F_t-s curve.

Example. Indicated Horsepower of a Steam Engine. To find the horsepower of a single-cylinder steam engine from two indicator diagrams taken at opposite ends of the cylinder, let

A, A' = piston areas (in.²) exposed to steam on the two sides,
L = length of stroke (ft.)
p_1 = driving pressure on piston, forward stroke,
p_1' = back pressure on piston, forward stroke,
p_2' = driving pressure on piston, return stroke,
p_2 = back pressure on piston, return stroke.

These pressures (lb./in.²) are all variable and should be regarded as functions of s, the displacement of the piston from its extreme positions. Then the work done on the piston in the forward and backward strokes is given by

$$\int_0^L (Ap_1 - A'p_1')\,ds \quad \text{and} \quad \int_0^L (A'p_2' - Ap_2)\,ds$$

respectively. The sum of these integrals, which gives the work done in a double stroke, may be written in the form

$$\text{(i)} \quad A\int_0^L (p_1 - p_2)\,ds + A'\int_0^L (p_2' - p_1')\,ds = APL + A'P'L$$

where

$$P = \frac{1}{L}\int_0^L (p_1 - p_2)\,ds, \quad P' = \frac{1}{L}\int_0^L (p_2' - p_1')\,ds$$

denote the space averages of the pressures included by the indicator diagrams taken at opposite ends of the cylinder (Fig. 164b shows one diagram). P and P' are called the *mean effective pressures* for the two sides of the piston. They are readily computed by dividing the areas within the diagrams (measured, for example, with a planimeter) by the length of the diagram. Thus if a diagram has an area of 4.50 in.², a length of 3 in., and the pressure scale is 1 in. = 50 lb./in.², the corresponding mean effective pressure is

$$P = \frac{4.50}{3} \times 50 = 75 \text{ lb./in.}^2$$

If the crank makes N rev./min., the piston makes N double strokes per minute. Hence from (i), the engine has an

$$\text{(ii)} \qquad \text{Indicated hp.} = \frac{(PA + P'A')\,LN}{33{,}000}$$

If the piston-rod extends through the entire cylinder, $A = A'$ = section-area of piston minus section-area of rod.

PROBLEMS

1. The indicator cards from the head and crank ends of a cylinder have areas of 4.72 and 4.60 in.² respectively and both are 4 in. long. If the indicator spring is compressed one inch by a pressure of 60 lb./in.², find the mean effective pressures.

2. The engine in Problem 1 has a 15-in. crank and the cylinder and piston-rod are respectively 12 and 2 in. in diameter. The piston rod extends but one way from the piston. If the engine makes 90 rev./min., find the indicated horsepower.

165. Principle of Work and Energy. In § 162 we found that the time-rate of change of kinetic energy of a particle is equal at any instant to the power exerted by the resultant force **F**:

$$\frac{d}{dt}\left(\tfrac{1}{2}\,mv^2\right) = \mathbf{F}\cdot\mathbf{v}.$$

If we integrate this equation between the instants t_1 and t_2 we obtain

$$\tfrac{1}{2}\,mv_2^2 - \tfrac{1}{2}\,mv_1^2 = \int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}\,dt. \tag{1}$$

The integral on the right is the work done by **F** in this interval (§ 163, 1). If **F** is the resultant of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, . . . acting on the particle, $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots$ and

$$\int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}\,dt = \int_{t_1}^{t_2} \mathbf{F}_1\cdot\mathbf{v}\,dt + \int_{t_1}^{t_2} \mathbf{F}_2\cdot\mathbf{v}\,dt + \ldots;$$

that is, the work done by the resultant of the forces acting on a particle is equal to the sum of the amounts of work (each taken with its proper sign) done by the separate forces. Hence, in view of (1), we have proved the important

PRINCIPLE OF WORK AND ENERGY. *The change in the kinetic energy of a particle in any time interval is equal to the total work done by the forces acting on it in this interval.*

The equation, *Change in kinetic energy = Work done,* is called the *energy equation.*

Example 1. A flat stone is sent gliding over ice with an initial velocity of 20 ft./sec. If $\mu = 0.025$, how far will it go? (§ 158, Example 1.)

Of the forces acting on the stone (Fig. 158*a*) W and N do no work as they are normal to the path, while the friction μW does the work $-\mu Wx$ over the distance x. The energy equation is therefore

$$0 - \frac{1}{2}\,\frac{W}{g}\,v_0^2 = -\,\mu Wx, \text{ whence } x = \frac{v_0^2}{2\,\mu g}.$$

As before we find $x = 250$ ft.

Example 2. An electric car coasts 1000 ft. down a 1 per cent grade and then runs 200 ft. up a 2 per cent grade with its acquired momentum. Find the total resistances (assumed constant) in pounds per ton.

If the car weighs T tons, and the resistances are r lb./ton, the total work done by the forces acting on the car is (Fig. 165*a*):

$$(2000\ T \sin \alpha - rT)\ 1000 - (2000\ T \sin \beta + rT)\ 200 \text{ ft.-lb.}$$

Note that the weight of the car does positive work on the down grade, negative work on the up grade. Since the initial and final speeds are zero the change in kinetic energy is zero. Hence, by the Principle of Work and Energy, the above work is zero, and we find

$$r = 2000 \frac{1000 \sin \alpha - 200 \sin \beta}{1000 + 200}.$$

Since the angles α and β are both small, $\sin \alpha$ and $\sin \beta$ are nearly equal to $\tan \alpha = 0.01$ and $\tan \beta = 0.02$ respectively. Therefore

$$r = 2000 \frac{10 - 4}{1200} = 10 \text{ lb./ton.}$$

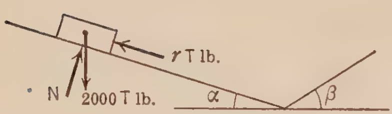


FIG. 165*a*.

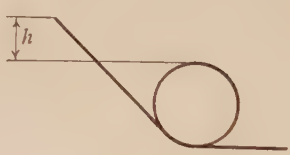


FIG. 165*b*.

Example 3. Let a particle of weight W slide down a smooth surface through a vertical distance h. Then gravity does the work Wh; and as the reaction on the particle, being normal to the motion at every instant, does no work, the energy equation is

$$\tfrac{1}{2} \frac{W}{g} (v^2 - v_0^2) = Wh, \quad \text{and} \quad v^2 = v_0^2 + 2\,gh.$$

If the particle slides upward, the work of gravity is $-Wh$ and $v^2 = v_0^2 - 2\,gh$.

PROBLEMS

1. Solve Problem 2, § 157, by the Principle of Work and Energy.
2. Solve Problem 5, § 158, by the Principle of Work and Energy.
3. A projectile is fired with an initial speed of 500 ft./sec. Find its speed when at a height of 2500 ft. above the point of projection.

4. In "looping a loop" of radius r the car starts from rest at a height h above the top of the loop (Fig. 165*b*). Neglecting friction, show that $h > \frac{1}{2} r$ if the car does not leave the track.

5. A bullet going 700 ft./sec. penetrates a block of wood 4 in. If the block were but 3 in. thick, with what speed would the bullet emerge if the resistance to penetration were constant?

6. A sled, starting from rest, slides 500 ft. down a 10° slope on to a level track. How far will it go on the level if $\mu = 0.10$ for the entire journey?

7. A 1-ton elevator, descending with a speed of 16 ft./sec., is brought to rest with a uniform retardation in 8 ft. What is the tension in the cable?

8. In Fig. 159*e* the floor is 10 ft. below W; $W = W'$ and $\mu = 0.2$ between W' and the plane. If W starts from rest, with what speed will it hit the floor, assuming that W' can travel at least 10 ft. on the horizontal?

9. A particle slides down a smooth circular cylinder, starting from rest. Show that it will leave the cylinder at a point $\frac{2}{3}$ as high above the center as its starting point.

166. Conservation of Energy. Consider a particle subject to a force **F** determined entirely by the position of the particle. Then to every point P in space there corresponds a definite force **F**; such a force distribution is called a *field of force*. If the work done by **F** as the particle moves over any path depends only on the end-points and not on the form of the path, the force **F** (or the field of force) is said to be *conservative*. Let Q be any position of the particle chosen as standard. Then if **F** is conservative,

The potential energy of the particle at the position P is defined as the work done by **F** *on the particle as it moves over any path from P to the standard position Q.* Denoting the potential energy at P by V,

$$V = \int_P^Q \mathbf{F}\cdot d\mathbf{r} \quad \text{over any path joining } P \text{ and } Q.$$

Let us now compute the work done by **F** as the particle moves over any path from P_1 to P_2. We have

$$\int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1}^{Q} \mathbf{F}\cdot d\mathbf{r} + \int_{Q}^{P_2} \mathbf{F}\cdot d\mathbf{r} = \int_{P_1}^{Q} \mathbf{F}\cdot d\mathbf{r} - \int_{P_2}^{Q} \mathbf{F}\cdot d\mathbf{r} = V_1 - V_2,$$

that is, the work done by **F** over any path from P_1 to P_2 is equal to the difference $V - V$ of the potential energies at these points.

Hence by the Principle of Work and Energy,

$$\tfrac{1}{2}\,mv_2^2 - \tfrac{1}{2}\,mv_1^2 = V_1 - V_2, \qquad \text{or}$$
$$\tfrac{1}{2}\,mv_2^2 + V_2 = \tfrac{1}{2}\,mv_1^2 + V_1;$$

that is, for any position of the particle,

$$\tfrac{1}{2}\,mv^2 + V = \text{const.}$$

This is the dynamical aspect of the

Law of Conservation of Energy. *If a particle moves in a conservative field of force, the sum of its kinetic and potential energies is constant.*

It must be remembered, however, that not all forces are conservative. Frictional forces, for example, are not conservative and the total mechanical energy, $\tfrac{1}{2}\,mv^2 + V$, does not remain constant when the motion is retarded by friction. In such cases some of the mechanical energy is converted into heat in overcoming friction. The much wider physical law of the Conservation of Energy applies in this case. Thus if an amount of heat H, expressed in the mechanical units of energy, is produced while the speed changes from v_0 to v, we have

$$\tfrac{1}{2}\,mv^2 + V + H = \tfrac{1}{2}\,mv_0^2 + V_0.$$

We shall now consider three important fields of force that are conservative and find the corresponding potential energy.

1. A constant field of force is conservative. For from (§ 163, 4) we see that the work done over any path from P_1 to P_2 is $\mathbf{F}\cdot\overrightarrow{P_1P_2}$ and therefore independent of the path.

The gravity of a body near any given place on the earth's surface is sensibly constant. Hence the gravitational field of earth in any locality may be regarded as conservative. If we choose the origin of $\mathbf{r}$ as the standard position, the potential energy of a body of weight W is

$$(1) \qquad V = \int_{\mathbf{r}}^{0} \mathbf{W}\cdot d\mathbf{r} = -\,\mathbf{W}\cdot\mathbf{r} = Wh$$

where h is the height of the body *above* a horizontal plane through the origin.

For a free particle subject only to gravity, conservation of energy requires that

$$\tfrac{1}{2}\,mv^2 + mgh = \text{const.} \quad \text{or} \quad v^2 + 2\,gh = \text{const.}$$

2. The particle is attracted towards a fixed point O by a force directly proportional to the distance. Then $\mathbf{F} = k\overrightarrow{PO} = -kr\mathbf{R}$ where k is a positive constant and $\mathbf{R}$ a unit radial vector. Now

$$\mathbf{r} = r\mathbf{R}, \quad d\mathbf{r} = r\,d\mathbf{R} + \mathbf{R}\,dr; \quad \mathbf{F}\cdot d\mathbf{r} = -kr\,dr$$

since $\mathbf{R}\cdot\mathbf{R} = 1$, $\mathbf{R}\cdot d\mathbf{R} = 0$. Hence the work done over any path from P_1 to P_2 is

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = -k\int_{r_1}^{r_2} r\,dr = \tfrac{1}{2}k\,(r_1^2 - r_2^2)$$

and therefore independent of the path. If the origin is chosen as the standard position,

$$V = -k\int_r^0 r\,dr = \tfrac{1}{2}kr^2. \tag{2}$$

3. The particle is attracted toward a fixed point O by a force inversely proportional to the square of the distance. Then

$$\mathbf{F} = -\frac{k}{r^2}\mathbf{R} \quad \text{and} \quad \mathbf{F}\cdot d\mathbf{r} = -\frac{k}{r^2}\,dr.$$

Hence the work done over any path from P_1 to P_2 is

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = -k\int_{r_1}^{r_2} \frac{dr}{r^2} = k\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

and therefore independent of the path. If the standard position is chosen at an infinite distance from O,

$$V = -k\int_r^\infty \frac{dr}{r^2} = -\frac{k}{r}. \tag{3}$$

If the particle is *repelled* from O according to the law of inverse squares, $V = k/r$. This is the case in an electrical field of force due to a charge at O when the particle has a charge of same sign.

PROBLEMS

1. A body is projected from the earth at any angle with an initial speed v_0. Show that it will return to the earth with this same speed

(*a*) when its gravity is regarded as constant;

(*b*) when its gravity varies inversely as the square of the distance from the center.

2. A body weighs W lb. at the earth's surface. Show that its potential energy in the earth's gravitational field is $V = -WR^2/r$ under the law of the inverse square. R is the radius of the earth and r the body's distance from the center.

3. With what speed must a projectile be fired vertically from the earth ($R = 3960$ mi.) in order that it never return? Neglect air resistance.

4. A body is projected vertically from the earth with an initial speed v_0. Show that it would reach a height $R\,v_0^2/(2gR - v_0^2)$ from the surface if the atmosphere were absent.

5. A body of weight W suspended from an elastic string increases its length by an amount e. If the body is set in vertical oscillations, show that its potential energy at a distance x from its equilibrium position is $\frac{1}{2}W\,x^2/e$. [The tension of the string is proportional to its elongation.]

6. In Problem 5 show that the speed v of the body at a distance x from its equilibrium position O is given by

$$v^2 = v_0^2 - gx^2/e,$$

where v_0 is its speed at O.

167. Units and Dimensions. From the definition of any dynamical quantity we may express its units in terms of the fundamental units of length, time and force. Let us denote these units by L, T, F respectively. Then the units of any derived dynamical quantity may be expressed in terms of these units as shown in the third column of the following table. These symbolic formulas for the derived units (or, more properly, the *exponents* of L, T, F in these formulas) are called the *dimensions* of the corresponding dynamical quantity. Velocity, for example, being measured by dividing a length by a time, has the dimensional formula L/T or LT^{-1}; it has the dimensions 1 in length, -1 in time.

The radian measure of an angle is given by the ratio of the lengths of a subtended circular arc to its radius; the dimensional formula for angle is therefore $L/L = L^0$ or simply 1. Thus an angle has no dimensions.

Quantity	Derivation	Dimensional Formula	Unit: British Gravitational	Unit: Metric Gravitational	Unit: C.G.S.
Length	Fundamental	L	Foot (ft.)	Meter (m.)	Centimeter (cm.)
Time		T	Second (sec.)	Second (sec.)	Second (sec.)
Force		F	Pound (lb.)	Kilogram (kg.)	Dyne
Angle	Arc/Radius	1	radian (rad.)	radian	radian
Area	Length × Length	L^2	ft.^{2}	m.^{2}	cm.^{2}
Volume	Length × Area	L^3	ft.^{3}	m.^{3}	cm.^{3}
Velocity	Length/Time	LT^{-1}	ft./sec.	m./sec.	cm./sec.
Acceleration	Velocity/Time	LT^{-2}	ft./sec.^{2}	m./sec.^{2}	cm./sec.^{2}
Ang. Velocity	Angle/Time	T^{-1}	rad./sec.	rad./sec.	rad./sec.
Ang. Acceleration	Ang. Vel./Time	T^{-2}	rad./sec.^{2}	rad./sec.^{2}	rad./sec.^{2}
Mass	Force/Accel.	$FL^{-1}T^2$	"slug"	"metric slug"	gram (gm.)
Density	Mass/Volume	$FL^{-4}T^2$	slug/ft.^{3}	met. slug/m.^{3}	gm./cm.^{3}
Pressure	Force/Area	FL^{-2}	lb./ft.^{2}	kg./m.^{2}	dyne/cm.^{2}
Momentum	Mass × Vel.	FT	lb.-sec.	kg.-sec.	dyne-sec.
Impulse	Force × Time				
Kinetic Energy	$\frac{1}{2}$ Mass × Vel.^{2}	FL	ft.-lb.	kg.-m.	dyne-cm. = 1 erg
Work	Force × Length				
Power	Force × Vel.	FLT^{-1}	ft.-lb./sec.	kg.-m./sec.	erg/sec.
Torque (Moment of Force)	Force × Length	FL	lb.-ft.	m.-kg.	dyne-cm.
Moment of Inertia	Mass × Length^2	FLT^2	slug-ft.^{2}	met. slug-m.^{2}	gm.-cm.^{2}
Angular Momentum	Momentum × Length	FLT	lb.-ft.-sec.	kg.-m.-sec.	dyne-cm.-sec.

In the table the sign / should be read "per." The last two quantities listed are first defined in a later chapter.

The following units, derived from those in the table, are also frequently used.

Length:	1 mile = 5280 ft.
	1 kilometer = 1000 m.
Velocity:	1 mile per hour = 88 ft./sec.
Angular Velocity:	1 revolution per second = 2π rad./sec.
	1 revolution per minute = $\pi/30$ rad./sec.
Force:	1 ton (short) = 2000 lb.
	1 metric ton = 1000 kg.
Work:	1 joule = 10^7 ergs.
	1 kilowatt-hour = 3600×10^{10} ergs.
Power:	1 horsepower = 550 ft.-lb./sec.
	1 watt = 1 joule/sec. = 10^7 ergs/sec.
	1 kilowatt = 1000 watts = 10^{10} ergs/sec.

By an executive order the United States yard is defined as 3600/3937 meter and the avoirdupois pound as 1/2.20462 kilogram. We therefore have the following relations:

1 ft. = 0.304801 m., 1 m. = 3.28083 ft.
1 lb. = 0.453593 kg., 1 kg. = 2.20462 lb.

To find the relation between the dyne and the kilogram we note that one kilogram force will give the standard kilogram body (1000 grams mass) an acceleration of 980.665 cm./sec.^{2} (§ 118); hence

1 kg. force = 1000×980.665 = 980,665 dynes.

These equations enable us to express any unit in one system in terms of the corresponding unit in another. For example, since

1 ft. = 30.4801 cm.,
1 lb. = $0.453593 \times 980{,}665$ = 444,823 dynes,
1 ft.-lb. = $30.4801 \times 444{,}823 = 1.35582 \times 10^7$ ergs.

The dimensional formula for a quantity offers a reliable way of finding its value when the units are changed. Thus to convert an acceleration expressed in miles per hour per second (as in railway practice) to ft./sec.^{2} we may proceed as follows:

$$1(\text{mi./hr.})/\text{sec.} = \frac{1 \text{ mile}}{1 \text{ hr.} \times 1 \text{ sec.}} = \frac{5280 \text{ ft.}}{3600 \text{ sec.} \times 1 \text{ sec.}}$$
$$= 1.467 \text{ ft./sec.}^2$$

If there is essentially but one relation connecting a set of dynamical quantities, and this equation holds good for any choice of fundamental units,* every term of the equation must have the same dimensions in L, T, F. This is the *Principle of Dimensional Homogeneity*.† As an example consider the equations of § 117 for uniformly accelerated motion; in (1) all terms have the dimensions LT^{-1}, in (2) L, in (3) L^2T^{-2}. Again in (§ 161, 2) all terms have the dimensions FT and in (§ 165, 1) the dimensions FL. This principle affords an important check on the accuracy of dynamical formulas. When a problem is solved in general terms, the result should always be tested for dimensional homogeneity. Numerous examples will occur in the following pages.

PROBLEMS

1. Show that

1 kilowatt = 737.56 ft.-lb./sec.; 1 hp. = 0.74570 kilowatt.

2. Test the results given in Problems 2, 4, 5 and 6 of § 166 for dimensional homogeneity.

168. Free Motion under Gravity. When the resistance of the air is neglected, a free body near the surface of the earth will move with the local acceleration of gravity. Its equation of motion is therefore

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{g}$$

where $\mathbf{g}$ is a constant vector directed downwards along the plumb line.

Let the body be projected from a point O, chosen as origin, with the velocity $\mathbf{v}_0$; that is, let

$$\mathbf{r} = 0, \quad \mathbf{v} = \mathbf{v}_0 \quad \text{when} \quad t = 0.$$

Successive integration of the equation of motion gives

$$\frac{d\mathbf{r}}{dt} = \mathbf{g}t + \mathbf{C}_1, \quad \mathbf{r} = \tfrac{1}{2}\,\mathbf{g}t^2 + \mathbf{C}_1 t + \mathbf{C}_2$$

* This is always the case in Dynamics when the units are defined so that a body of unit mass acted on by a unit force will acquire unit acceleration.

† For a proof see Bridgman, *Dimensional Analysis*, Chapter IV. The principle is valid for the equations of general Physics.

and from the initial conditions $\mathbf{C}_1 = \mathbf{v}_0$, $\mathbf{C}_2 = 0$. Hence

(1), (2) $$\mathbf{v} = \mathbf{v}_0 + \mathbf{g}t, \quad \mathbf{r} = \mathbf{v}_0 t + \tfrac{1}{2}\,\mathbf{g}t^2.$$

We may also write (2) in the form

(3) $$\mathbf{r} = \tfrac{1}{2}\,(\mathbf{v}_0 + \mathbf{v}_0 + \mathbf{g}t)t = \frac{\mathbf{v}_0 + \mathbf{v}}{2}\,t.$$

Equations (1) and (2) give the velocity and position of the body in terms of the time. They show that the motion may be regarded as a vector combination of motion with the constant velocity $\mathbf{v}_0$ and free fall from rest under gravity.

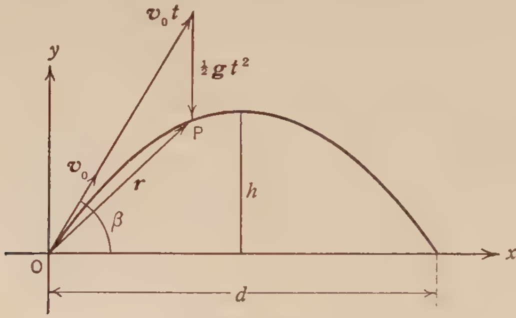


Fig. 168.

On multiplying (1) by m, the mass of the particle, we have

$$m\mathbf{v} - m\mathbf{v}_0 = m\mathbf{g}t,$$

an equation stating that the change of momentum in time t is equal to the impulse of the force $m\mathbf{g}$ (§ 161).

Since the potential energy of the body is mgh (§ 166, 1) where h is the height of the body above O, conservation of energy demands that

$$\tfrac{1}{2}\,mv^2 + mgh = \tfrac{1}{2}\,mv_0^2 + 0, \quad \text{or}$$

(4) $$v^2 = v_0^2 - 2\,gh.$$

Equations (1) to (4) hold for a free particle moving with any *constant* acceleration. These equations, moreover, have the same form as the equations of § 116 (with $x_0 = 0$) for uniformly accelerated motion in a straight line.

If we choose x- and y-axes in the vertical plane through $\mathbf{v}_0$ as in Fig. 168,

$$\mathbf{v}_0 = [v_0 \cos\beta,\, v_0 \sin\beta], \quad \mathbf{g} = [0, -g]$$

where β is the angle of projection. Hence the actual motion is a combination of

(*a*) *Horizontal motion at uniform speed* $v_0 \cos \beta$, *and*

(*b*) *Vertical motion under gravity with the initial speed* $v_0 \sin \beta$.

The equations of § 116 thus apply to the vertical motion.

The coördinates of the body after t seconds are

$$x = v_0 \cos \beta t, \quad y = v_0 \sin \beta t - \tfrac{1}{2} gt^2.$$

When $v_0 \neq 0$ we may eliminate t from these equations and obtain the Cartesian equation of the path:

$$(5) \qquad y = x \tan \beta - \frac{g}{2 v_0^2 \cos^2 \beta} x^2.$$

This represents a parabola with axis vertical. Its vertex, located by putting $dy/dx = 0$ ($v_y = 0$), is the point

$$\left(\frac{v_0^2}{2g} \sin 2\beta, \frac{v_0^2}{2g} \sin^2 \beta\right).$$

The range on a horizontal plane through O is found by putting $y = 0$ in (5): thus the

$$\text{Horizontal range} = \frac{v_0^2}{g} \sin 2\beta.$$

For a given initial speed the range is greatest when $\beta = 45°$.

Example. A projectile is fired from a hill 1024 ft. above a plain with an initial velocity of 960 ft./sec. inclined at an angle of 30° above the horizontal. Find (*a*) its time of flight to the plain, (*b*) the horizontal distance traversed, (*c*) its greatest height above the cannon, (*d*) the striking speed.

Choose horizontal and vertical axes as in Fig. 168. Then $\mathbf{g} = [0, -32]$ ft./sec.² and

$$\mathbf{v}_0 = [960 \cos 30°, 960 \sin 30°] = [831, 480] \text{ ft./sec.}$$

(*a*) Since $y = -1024$ on the plain, the projectile will hit the plain in a time t given by (§ 116, 2):

$$-1024 = 480t - \tfrac{1}{2} 32 t^2 \quad \text{or} \quad t^2 - 30t - 64 = 0.$$

Thus $t = 32$ sec. (the positive root).

(*b*) As the horizontal motion is at the uniform rate of 831 ft./sec., the projectile will traverse the distance

$$831 \times 32 = 26{,}600 \text{ ft.} \quad \text{in} \quad 32 \text{ sec.}$$

(*c*) According to (§ 116, 1) $v_y = 480 - 32t$; hence the projectile will reach its highest point ($v_y = 0$) in 15 sec. Since the mean vertical

velocity in this interval is $\frac{1}{2}(480 + 0) = 240$ ft./sec., the greatest height above O is $240 \times 15 = 3600$ ft. Or we may use (§ 116, 2):

$$y = 480 \times 15 - \tfrac{1}{2}\, 32 \times 15^2 = 3600.$$

(*d*) The speed when $h = -1024$ is given by (4):

$$v = \sqrt{960^2 + 64 \times 1024} = 994 \text{ ft./sec.}$$

PROBLEMS

1. A lump of coal falls from a car going 20 mi./hr. from a height of 9 ft. above the ground. What horizontal distance will it cover before striking? What is its striking velocity?

2. A ball is thrown so that it just clears a 7-ft. fence 100 ft. away. If it left the hand 5 ft. above the ground and at an angle of 45° to the horizontal, what was its initial velocity?

3. A gun is fired at inclination angle β at the foot of an inclined plane of inclination angle α. Show that the bullet will strike the plane in

$$t_1 = \frac{2\, v_0 \cos\beta}{g}(\tan\beta - \tan\alpha) \text{ sec.}$$

and that its range on the plane is $v_0 t_1 \cos\beta/\cos\alpha$.

4. Nuts from an automatic machine are delivered into a smooth slide in the form of a concave elliptic quadrant. They slide down through a vertical distance of 2 ft., pass over its horizontal edge and fall to the floor 10 ft. below their starting point. Where will they hit the floor, assuming that they start from rest?

5. A stone is thrown horizontally from a cliff 900 ft. above the sea with a speed of 80 ft./sec. When will it strike the sea and at what horizontal distance from the cliff?

6. A stone from a quarry blast was thrown 800 ft. away. What is the least possible value of its initial velocity?

169. Central Forces. Consider a particle acted on by a force $\mathbf{F}$ which always passes through a fixed point O. Choose O as the origin of $\mathbf{r}$; then since $\mathbf{F} = m\mathbf{a}$, the acceleration $\mathbf{a}$ also passes through O and

$$\mathbf{r}\times\mathbf{a} = 0 \quad\text{or}\quad \mathbf{r}\times\frac{d\mathbf{v}}{dt} = 0.$$

Hence

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = \frac{d\mathbf{r}}{dt}\times\mathbf{v} + \mathbf{r}\times\frac{d\mathbf{v}}{dt} = 0,$$

so that $\mathbf{r}\times\mathbf{v}$ must be constant vector during the motion, say

(1) $$\mathbf{r}\times\mathbf{v} = \mathbf{h}.$$

If $\mathbf{r} = \mathbf{r}_0, \quad \mathbf{v} = \mathbf{v}_0 \quad \text{when} \quad t = 0, \quad \mathbf{h} = \mathbf{r}_0\times\mathbf{v}_0.$

Case 1. $\mathbf{h} = 0$. Then $\mathbf{r}\times\mathbf{v} = 0$, and if we write $\mathbf{r} = r\mathbf{R}$, where $\mathbf{R}$ is a unit radial vector

$$\mathbf{v} = \frac{dr}{dt}\mathbf{R} + r\frac{d\mathbf{R}}{dt}, \quad \mathbf{r}\times\mathbf{v} = r^2\mathbf{R}\times\frac{d\mathbf{R}}{dt} = 0.$$

Since $d\mathbf{R}/dt$ is perpendicular to $\mathbf{R}$ (§ 84), we conclude that $d\mathbf{R}/dt = 0$ and $\mathbf{R}$ is constant. The motion is therefore *rectilinear.*

Case 2. $\mathbf{h} \neq 0$. On multiplying (1) by $\mathbf{r}\cdot$ we then have $\mathbf{r}\cdot\mathbf{h} = 0$, that is, $\mathbf{r}$ is always perpendicular to $\mathbf{h}$. The motion is therefore *plane.*

Since $|\mathbf{r}\times\mathbf{v}| = h$ is the area of the parallelogram having $\mathbf{r}$ and $\mathbf{v}$ as sides (§ 16), $\frac{1}{2}h$ is the area of the triangle with the sides $\mathbf{r}$ and $\mathbf{v}$. This triangle gives the area that $\mathbf{r}$ would sweep over in the next second if $\mathbf{v}$ were constant. Hence $\frac{1}{2}h$ is the instantaneous rate at which the position vector $\mathbf{r}$ is sweeping over area — the *sectorial speed.* Since h is constant, we have proved the

Law of Areas. *If a force acting on a particle P always passes through a fixed point O, then OP will sweep over equal areas in equal times.*

The vector $\frac{1}{2}\mathbf{h}$, which is normal to the plane of motion, is called the *sectorial velocity.*

If the forces acting on a particle are such that their *projections* on a plane pass through a fixed point, the Law of Areas applies to the *projection* of the motion on this plane.

For let O be the origin of $\mathbf{r}$ and $\mathbf{F}$ the resultant of the forces. Then if P', $\mathbf{r}'$, $\mathbf{v}'$, $\mathbf{a}'$, $\mathbf{F}'$ denote the projections of P, $\mathbf{r}$, . . . on the plane, $\mathbf{F} = m\mathbf{a}$ gives $\mathbf{F}' = m\mathbf{a}'$ on projection and $\mathbf{r}'\times\mathbf{a}' = 0$. On differentiating $\mathbf{r} = \mathbf{r}' + \overrightarrow{P'P}$ with respect to t we find that $\mathbf{v}'$ and $\mathbf{a}'$ are the velocity and acceleration of P'. Hence

$$\mathbf{r}'\times\frac{d\mathbf{v}'}{dt} = 0 \quad \text{and} \quad \mathbf{r}'\times\mathbf{v}' = \text{const.}$$

170. Harmonic Motion. Let a particle P be attracted toward a fixed point O by a force which varies as the distance OP. Then from the fundamental equation, $\mathbf{a} = k\overrightarrow{PO} = -k\overrightarrow{OP}$ where k is a positive constant. If we choose O as origin and write $k = n^2$ (*cf.* § 119), the equation of motion is

$$\frac{d^2\mathbf{r}}{dt^2} = -n^2\mathbf{r}; \tag{1}$$

n^2 is the numerical value of the acceleration at unit distance from O.

The reasoning of § 119 now shows that

$$\mathbf{r} = \mathbf{A} \cos nt + \mathbf{B} \sin nt,$$

where $\mathbf{A}$, $\mathbf{B}$ are any constant vectors, satisfies (1). If the initial conditions are

$$\mathbf{r} = \mathbf{r}_0, \quad \mathbf{v} = \mathbf{v}_0 \quad \text{when} \quad t = 0,$$

we find that $\mathbf{A} = \mathbf{r}_0$, $\mathbf{B} = \mathbf{v}_0/n$. Hence

$$\mathbf{r} = \mathbf{r}_0 \cos nt + \frac{\mathbf{v}_0}{n} \sin nt, \tag{2}$$

$$\mathbf{v} = -n\mathbf{r}_0 \sin nt + \mathbf{v}_0 \cos nt, \tag{3}$$

give the position and velocity of P at time t. At the instant $t + 2\pi/n$, $\mathbf{r}$ and $\mathbf{v}$ are the same as at the instant t: hence the motion repeats itself in the time

$$T = \frac{2\pi}{n}, \quad \text{the } \textit{period}.$$

Since the force on a particle of mass m is $-mn^2\mathbf{r}$, its potential energy is $\frac{1}{2} mn^2r^2$ (§ 166, 2). Hence, from the conservation of energy,

$$\tfrac{1}{2} mv^2 + \tfrac{1}{2} mn^2r^2 = \text{const.} \quad \text{or} \quad v^2 + n^2r^2 = \text{const.}$$

Since the force on the particle always passes through O, $\mathbf{r}\times\mathbf{v}$ is constant and the motion proceeds with constant sectorial speed (§ 169).

If $\mathbf{r}_0\times\mathbf{v}_0 = 0$, the motion given by (2) is a *simple harmonic motion* along a line through O. This has been treated in detail in § 119.

If $\mathbf{r}_0\times\mathbf{v}_0 \neq 0$, the path given by (2) is an ellipse about O as center and the motion is said to be *elliptic harmonic*. If we choose a vertex of the ellipse as the initial position, $\mathbf{v}_0$ will be perpendicular to $\mathbf{r}_0$. On taking the axes of the ellipse as x- and y-axes, (2) is equivalent to the scalar equations

$$x = r_0 \cos nt, \quad y = \frac{v_0}{n} \sin nt.$$

These are the parametric equations of an ellipse of semi-axes r_0, v_0/n in terms of the eccentric angle $\phi = nt$ (Fig. 170a). This angle

increases at the constant rate $d\phi/dt = n$. Hence if a point Q revolves about the auxiliary circle with the constant angular velocity n, its projection P on the ellipse will have elliptic harmonic motion.

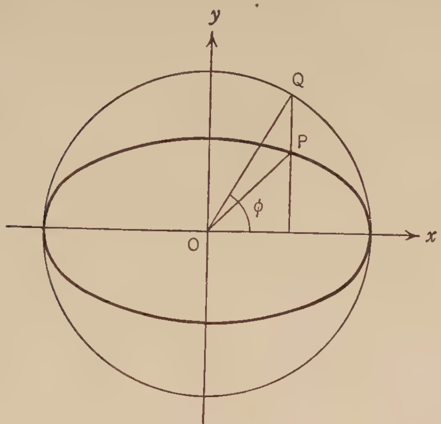


FIG. 170*a*.

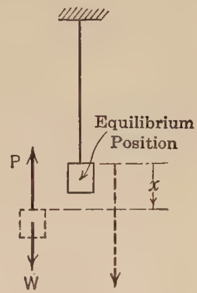


FIG. 170*b*.

Example 1. Let a body of weight W be suspended from an elastic string. When it hangs in equilibrium it will stretch the string an amount e so that the tension developed is just equal to W. Suppose, now that the body is drawn down a further distance h (less than e) and then released. Let P denote the tension in the string when the elongation of the string is $e + x$. Here x is measured from the equilibrium position and the positive direction is downward (Fig. 170*b*). From Hooke's Law the tension of the string is proportional to its elongation:

$$\frac{P}{W} = \frac{e + x}{e}, \quad P = W + \frac{x}{e} W.$$

At any instant the forces on the body are its weight W acting downwards and the tension P acting upwards. Its equation of motion is therefore

$$\frac{W}{g} a = W - P = -\frac{x}{e} W \quad \text{or} \quad a = -\frac{g}{e} x.$$

This is the characteristic equation of a s.h.m. in which $n^2 = g/e$. The period is therefore

$$T = \frac{2\pi}{n} = 2\pi\sqrt{\frac{e}{g}}.$$

Since $v = 0$ when $x = h$, the vibrations will have the amplitude h; and as $h < e$ the string will not become slack during the motion.

Suppose for example that a body, attached to a string of natural length 2 ft., hangs in equilibrium when its elongation is $2\frac{1}{2}$ ft. If it is then displaced and released, it will vibrate in s.h.m. with the period of $2\pi\sqrt{1/64} = \pi/4 = 0.785$ sec. If body is held so that the string has its natural length and then let go, the amplitude of the vibrations will be $\frac{1}{2}$ ft.; when the string has its greatest elongation of 1 ft, its tension will equal twice the weight of the body.

Example 2. Consider a solid immersed wholly or in part in a liquid. To find the resultant pressure P of the liquid on its submerged surface, imagine the solid removed and the space it occupied filled with the liquid. As this portion of liquid is in equilibrium, its weight must be exactly balanced by the total pressure P exerted by the adjacent liquid on its bounding surface, that is, P is a vertical force numerically equal to this weight. We have thus proved the

PRINCIPLE OF ARCHIMEDES. *A body immersed in a liquid experiences an upward force equal to the weight of the liquid displaced.*

In particular, *a floating body in equilibrium must displace its own weight of liquid.*

Suppose now that a body of weight W floats on a liquid of weight w per unit volume; when in equilibrium it will experience an upward thrust from the water equal to W. Now let the body be given a slight downward displacement x. If the section-area A of the body at the water-line is nearly constant for small displacements, the additional weight of liquid displaced is wAx, and the body will experience an additional upward thrust from the liquid of this amount. If the positive direction of x is downward, the equation of motion of the body is

$$\frac{W}{g}a = W - (W + wAx) \quad \text{or} \quad a = -\frac{wAg}{W}x.$$

This is the equation of a s.h.m. in which $n^2 = wAg/W$. The body will therefore vibrate up and down with the period

$$T = 2\pi\sqrt{\frac{W}{wAg}} = 2\pi\sqrt{\frac{V}{Ag}}$$

where V is the volume of liquid displaced when the body is in equilibrium. Note that the right-hand member has the dimensions of time.

Thus a hydrometer which displaces 40 cc. and whose stem is 1.2 cm. in diameter will oscillate with the period

$$T = 2\pi\sqrt{\frac{40}{\pi \times 0.36 \times 981}} = 1.42 \text{ sec.}$$

PROBLEMS

1. A 1-lb. weight, suspended from a helical spring originally 2 ft. long, has a period of 1 sec. when set in vibration. How long is the spring when the weight hangs in equilibrium?

2. If the spring in Problem 1 supports a 2-lb. weight, what is the period of vibration.

3. The springs of a truck deflect 2 in. under the weight of the body and load. What is the period of vibration?

4. A cylinder floats vertically on the water with 2 ft. submerged. If it is given a slight downward displacement, find the period of its vibrations.

5. A particle weighing $\frac{1}{4}$ lb., on a smooth horizontal plane, is attached to two elastic strings, alike in all respects, whose other ends are fastened to two pegs 3 ft. apart. Each string has a natural length of 1 ft. and will support the particle when stretched 3 in. If the particle is displaced toward one of the pegs, find the period of its vibrations.

Solve this problem in general terms: w = weight of particle, d = distance between pegs, l = natural lengths of strings, e = elongation of strings under load w.

6. If the strings in Problem 5 are fastened to two pegs vertically above one another, find the equilibrium position of the particle and its period of vibration. Solve also in general terms.

7. The mercury column in a U-tube is 12 in. long measured on the axial line. Find its period when set in vibration.

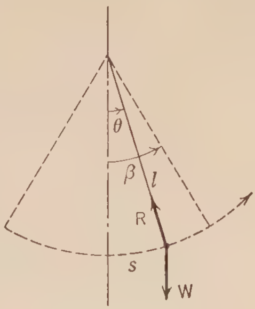


Fig. 171.

171. Simple Pendulum. Consider an ideal pendulum consisting of a weightless string (or rod) of length l supported at one end and attached to a particle of weight W at the other. If the pendulum is displaced a small angle β from the vertical and then released it will swing in a vertical plane. In any position of the pendulum, the forces acting on the particle are its weight W and the tension R of the string (Fig. 171). The equations of motion, obtained by resolving along the tangent and normal to the path, are

$$(1), (2) \qquad \frac{W}{g}\frac{dv}{dt} = -W \sin\theta, \quad \frac{W}{g}\frac{v^2}{l} = R - W\cos\theta.$$

Here both θ and the arc s are measured from the vertical and taken positive in the counterclockwise sense. Then $s = l\theta$, $v = l\, d\theta/dt$ (§ 109, 1) and from (1)

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta. \tag{3}$$

For small oscillations, say $\theta \leqq 5°$, $\sin\theta$ may be replaced by θ (radians) and we have approximately

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta; \quad \theta = \beta, \ \frac{d\theta}{dt} = 0 \quad \text{when} \quad t = 0.$$

This is the differential equation of a s.h.m. (§ 119, 1) with x replaced by θ and $n^2 = g/l$. The period of the pendulum, the time of a complete oscillation to and fro, is therefore

$$T = \frac{2\pi}{n} = 2\pi\sqrt{\frac{l}{g}}, \text{ approximately.} \tag{4}$$

The period is thus independent of the (small) angular amplitude.

The exact value of T for any angle of swing may be obtained from the energy equation. As the angle varies from β to θ, gravity does the work $W(l\cos\theta - l\cos\beta)$ on the bob and hence

$$\begin{aligned}\tfrac{1}{2}\frac{W}{g}v^2 - 0 &= Wl\,(\cos\theta - \cos\beta),\\ v^2 &= 2\,gl\,(\cos\theta - \cos\beta);\\ l^2\left(\frac{d\theta}{dt}\right)^2 &= 4\,gl\,(\sin^2\tfrac{1}{2}\beta - \sin^2\tfrac{1}{2}\theta),\\ \tfrac{1}{4}T &= \tfrac{1}{2}\sqrt{\frac{l}{g}}\int_0^\beta \frac{d\theta}{(\sin^2\tfrac{1}{2}\beta - \sin^2\tfrac{1}{2}\theta)^{\frac{1}{2}}}.\end{aligned} \tag{5}$$

This integral may be reduced to the normal form of an elliptic integral of the first kind and evaluated from tables; or it may be integrated in series.* The first two terms of this series give

$$T = 2\pi\sqrt{\frac{l}{g}}\,(1 + \tfrac{1}{4}\sin^2\tfrac{1}{2}\beta),$$

a result accurate enough for most purposes.

To find the tension R of the string substitute v^2 from (5) in (2); thus

$$R = W\,(3\cos\theta - 2\cos\beta).$$

* See Lamb, *Dynamics*, § 37.

If R assumes negative values during the motion the bob must be attached to a light rod, for a string cannot sustain compression.

If the period of a pendulum of known length is accurately determined in any locality, the local value of g may be computed from (4): $g = 4\pi^2 l/T^2$.

PROBLEMS

1. Compute the length of a seconds pendulum ($\frac{1}{2}T = 1$ sec.) in latitude 40°. [See § 118.]

2. If a pendulum beats seconds in latitude α, show that in latitude β each beat will take $\sqrt{g_\alpha/g_\beta}$ sec.

Show that a seconds pendulum in latitude 35°, sea level, will gain 19 seconds per day in latitude 40°, sea level. [§ 118.]

3. A pendulum consisting of a light rod with a 1-lb. weight at its end is released from its vertical position of instability. Find the stress R in the rod when horizontal and when passing through its lowest position.

At what angle θ (Fig. 171) will $R = 0$? Find R when $\theta = 135°$.

172. Rectilinear Motion in a Resisting Medium. When a body moves through a fluid, such as air or water, it is resisted by a force that depends on its size, shape, and its velocity. For any given body the resistance may therefore be expressed as a function $R(v)$ of the velocity. Experiment shows that R increases indefinitely with v; and that $R \to 0$ as $v \to 0$. For very low velocities R is proportional to v, whereas for higher velocities R is proportional to v^2. To test the quadratic law we may plot R/v^2 against v. Thus for motion in air R/v^2 is constant up to velocities of about 800 ft./sec. Beyond this R/v^2 increases sharply with v, particularly in the neighborhood of the velocity of sound (about 1100 ft./sec.). For still higher values of v, R/v^2 rises to a maximum and then decreases. Thus no simple function $R(v)$ applies for all speeds. In fact no adequate law of air resistance has as yet been formulated.

Consider now the rectilinear motion of a particle subject to a constant accelerating force F as well as to the resistance of the medium. The equation of motion

$$ma = F - R(v)$$

is then readily integrated by the method of § 116, 3.

Let v_0 be the initial velocity. If $F > R(v_0)$, the acceleration is positive and v will increase. Hence $R(v)$ will increase and a decrease; v continues to increase, more and more slowly, to a certain limiting value V which corresponds to $a = 0$ or uniform motion. This *terminal velocity* V is therefore determined by the equation

$$(1) \qquad F - R(V) = 0.$$

If $F < R(v_0)$, the acceleration is negative and v will decrease to a limiting value V determined by (1). In particular if $F = 0$, $V = 0$.

Thus we see that when F is constant the motion tends to become uniform with a certain limiting velocity V. This velocity is theoretically never attained; but in most cases the velocity differs but little from V after a short time and thereafter is practically constant. Thus a falling rain drop attains a terminal velocity of about 25 ft./sec. in about 1 sec.

Example 1. Two bodies of weight W_1, W_2 fall in air under gravity. If their terminal velocities are V_1, V_2, we have from (1)

$$W_1 = R(V_1), \quad W_2 = R(V_2).$$

Hence if $W_1 > W_2$, $R(V_1) > R(V_2)$ and as $R(v)$ increases with v, $V_1 > V_2$. The heavier body will therefore have the greater terminal velocity.

Example 2. A steamer going at full speed V has its engines reversed. How long and how far will it move forward if the resistance of the water varies as v^2?

Let F denote the constant propelling force on the ship. The terminal velocity V of the ship with the engines propelling forward is given by (1):

$$F - kV^2 = 0; \quad \text{hence} \quad k = \frac{F}{V^2}.$$

When the engines are reversed the equation of motion is

$$ma = -F - kv^2 = -F\left(1 + \frac{v^2}{V^2}\right).$$

Putting $a = dv/dt$, we find for the time of forward motion

$$t_1 = -\frac{m}{F}\int_V^0 \frac{dv}{1 + \frac{v^2}{V^2}} = \frac{mV}{F}\tan^{-1}\frac{v}{V}\Bigg|_0^V = \frac{\pi}{4}\frac{mV}{F}.$$

With $a = v\,dv/dx$, we find for the forward distance

$$x_1 = -\frac{m}{F}\int_V^0 \frac{v\,dv}{1+\frac{v^2}{V^2}} = \frac{mV^2}{2F}\ln\left(1+\frac{v^2}{V^2}\right)\Bigg|_0^V = \frac{mV^2}{2F}\ln 2.$$

Do these equations check dimensionally?

Example 3. A man rowing in still water at the rate of v_0 ft./sec. ships his oars. To find his subsequent motion assume that the resistance R varies as v. The equation of motion is then

$$ma = -R \quad \text{or} \quad a = -kv.$$

The velocity and distance passed over may now be found in terms of t as shown in § 115, Example.

PROBLEMS

1. A raindrop has a terminal velocity of 25 ft./sec. If R varies as v, show that the drop, starting from rest, will attain a velocity of 20 ft./sec. in about 1.25 sec.

2. A body falling in air would reach a terminal velocity of 320 ft./sec. If the body is projected vertically upward with this velocity, how high will it rise if R varies as v^2? How high would it rise in the absence of resistance?

[In falling, $a = g - kv^2$ whence $k = g/V^2$.

In rising, $a = -g - kv^2$ or $v\dfrac{dv}{dx} = -g\left(1+\dfrac{v^2}{V^2}\right)$.]

3. A body falls from rest in a medium whose resistance varies as v. The resistance is $\frac{1}{10}$ of the weight when $v = 8$ ft./sec. Find (*a*) the terminal velocity; (*b*) the speed and distance fallen after 5 sec.

173. Damped Oscillations. Let us consider the rectilinear motion of a particle attracted toward a fixed point O of the line with a force proportional to the distance and subject to a resistance proportional to the speed. With O as origin and the line as x-axis, the equation of motion is

$$m\frac{d^2x}{dt^2} = -k_1x - k_2\frac{dx}{dt},^*$$

* In vector form the equation of motion is $m\mathbf{a} = -k_1\mathbf{r} - k_2\mathbf{v}$; putting

$$\mathbf{r} = x\mathbf{i}, \quad \mathbf{v} = \frac{dx}{dt}\mathbf{i}, \quad \mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i}$$

we get the equation above, showing that it is valid in general.

where k_1, k_2 are positive constants; or, if we write $k_1/m = n^2$, $k_2/m = 2f$,

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2x = 0. \tag{1}$$

When $f = 0$, (1) reduces to the differential equation for s.h.m. and we found in § 119 that $x = h \cos (nt + \epsilon)$, where h is the constant amplitude. If the resistance is not too great it is natural to assume that the motion will still be oscillatory, but with a progressively decreasing amplitude. Let us therefore attempt to satisfy (1) with a function of the form

$$x = h \cos (pt + \epsilon) \tag{2}$$

where p and ϵ are constants, but h is now some function of t.

If we substitute this expression for x in (1) we obtain

$$\left[\frac{d^2h}{dt^2} + 2f\frac{dh}{dt} + (n^2 - p^2)h\right] \cos (pt + \epsilon)$$
$$- 2p\left[\frac{dh}{dt} + fh\right] \sin (pt + \epsilon) = 0.$$

This equation will be satisfied when the expressions in brackets vanish:

$$\frac{d^2h}{dt^2} + 2f\frac{dh}{dt} + (n^2 - p^2)\, h = 0, \tag{3}$$

$$\frac{dh}{dt} + fh = 0. \tag{4}$$

The function h is determined by (4); on separating the variables and integrating we obtain

$$\frac{dh}{h} = -f\, dt, \quad \ln h = -ft + \ln c,$$

$$h = ce^{-ft}, \tag{5}$$

where c is an arbitrary constant. The substitution of (5) in (3) now gives

$$ce^{-ft}\,(n^2 - p^2 - f^2) = 0.$$

Since $e^{-ft} \neq 0$ for any finite value of t, this requires that $p^2 = n^2 - f^2$. We now assume that the resistance is sufficiently small so that $f < n$; then p may be chosen as the positive real number

$$p = \sqrt{n^2 - f^2}. \tag{6}$$

We have thus obtained a solution of the form (2), namely

$$x = ce^{-ft} \cos (\sqrt{n^2 - f^2}\, t + \epsilon). \tag{7}$$

Here c and ϵ are arbitrary constants at our disposal to satisfy the initial conditions, say $x = x_0$, $v = v_0$ when $t = 0$.

When $f = 0$, (7) reduces to

$$x = c \cos (nt + \epsilon),$$

the space-time relation for undamped harmonic vibrations. Equation (7) differs from this in the presence of the damping factor e^{-ft} and the replacement of n by $\sqrt{n^2 - f^2}$; these changes affect the amplitude and period respectively. As the time increases, the cosine in (7) oscillates between 1 and -1 while e^{-ft} steadily decreases, approaching zero as t becomes infinite. Equation (7) therefore represents oscillations of continually decreasing amplitude, as shown graphically in the x-t curve of Fig. 173a.

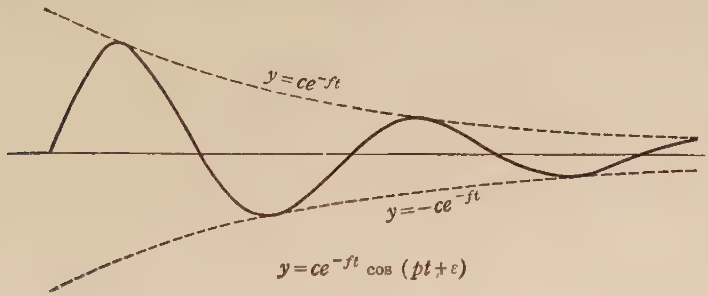


FIG. 173a.

The particle passes through O ($x = 0$) when

$$\cos (\sqrt{n^2 - f^2}\, t + \epsilon) = 0.$$

This occurs at equal intervals of time $\pi/\sqrt{n^2 - f^2}$ with alternating changes of direction. The time between successive passages through O in the same direction is therefore

$$T = \frac{2\pi}{\sqrt{n^2 - f^2}}. \tag{8}$$

This interval is called the *period* of the damped vibration. It is noteworthy that *the period is constant and entirely independent of the initial conditions.* For the same attractive force, the period of the undamped vibration is $2\pi/n$; hence *the period of the damped*

vibration is always greater than the period of the corresponding undamped vibration. With the same central attraction, the period increases without limit as f approaches n. In the next article we shall see that when $f \geqq n$ the motion is so strongly damped that it is no longer oscillatory.

Since the speed $v = dx/dt$, we have from (7)

$$v = -ce^{-ft}[f \cos (pt + \epsilon) + p \sin (pt + \epsilon)].$$

Consequently $v = 0$ whenever

$$\tan (pt + \epsilon) = -f/p$$

and hence at equal intervals $\pi/p = \frac{1}{2}T$. These times mark the greatest excursions of the particle beyond O. As these extreme excursions are alternately positive and negative, the time between successive extremes on the same side of O is precisely the period T.

Consider now two consecutive extremes x_i, x_{i+1} on opposite sides of O. If these occur at the instants t_i and $t_i + \frac{1}{2}T$ we have from (7),*

$$\left|\frac{x_i}{x_{i+1}}\right| = e^{\frac{1}{2}fT}; \text{ hence}$$

the ratio of successive amplitudes is constant. In other words, the successive amplitudes form a decreasing geometric progression. The logarithm of th above ratio,

$$\log |x_i| - \log |x_{i+1}| = \tfrac{1}{2}fT \tag{9}$$

is called the *logarithmic decrement.* Since its value and that of the period may be found by direct observation, (9) may be used to compute the damping coefficient $2f$. Moreover in any actual vibration, the constancy of the logarithmic decrement enables us to test the hypothesis that the damping is proportional to the speed. If the logarithmic decrement is computed from successive amplitudes on the same side, its value is fT.

If the damping is very small, as in the case of a pendulum swinging in air, the period will differ but slightly from that of an undamped vibration. Irrespective of the amount of damping, however, the successive amplitudes, being terms of a decreasing geometric progression, steadily approach zero as a limit, and after a certain time the motion will become imperceptible.

* At the extremes $v = 0$ and $\tan (pt + \epsilon) = -f/p$; $\cos (pt + \epsilon)$ at the extremes has therefore the same *numerical* values.

In § 119 it was shown that an undamped harmonic vibration could be represented as the projection of uniform circular motion on a diameter. To obtain an analogous representation for a damped vibration we note the x in (7) may be regarded as the x-component of a vector $\overrightarrow{OQ}$ of variable length

$$r = ce^{-ft} \quad \text{making an angle} \quad \theta = pt + \epsilon$$

with the x-axis (Fig. 173b). As t increases, the length decreases while OQ revolves at the constant rate of $d\theta/dt = p$. The curve traced by its end-point Q is the equiangular spiral whose equation in polar coördinates is

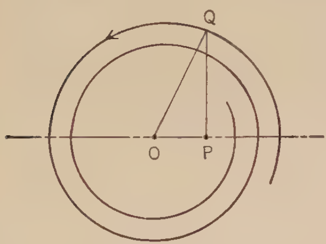


Fig. 173b.

$$r = c'e^{-\frac{f}{p}\theta} \quad \text{where } c' = ce^{\frac{f\epsilon}{p}}.$$

The spiral cuts the radial lines through O at the constant angle α given by

$$\cot \alpha = \frac{1}{r}\frac{dr}{d\theta} = -\frac{f}{p} = -\frac{f}{\sqrt{n^2 - f^2}};$$

when f is small compared with n, α is but slightly greater than a right angle. Thus a damped harmonic vibration may be represented by the projection of uniform angular motion along an equiangular spiral on a line through its pole.

PROBLEMS

1. If the initial conditions for a damped harmonic motion are $x = 0$, $v = v_0$ when $t = 0$, show that

$$x = \frac{v_0}{\sqrt{n^2 - f^2}} e^{-ft} \quad \sin(\sqrt{n^2 - f^2}\, t)$$

2. If the initial conditions for damped harmonic motion are $x = x_0(>0)$, $v = 0$ when $t = 0$, show that

$$c = \frac{nx_0}{\sqrt{n^2 - f^2}}, \qquad \tan \epsilon = -\frac{f}{\sqrt{n^2 - f^2}},$$

where ϵ is chosen between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

3. If three consecutive excursions of a damped galvanometer needle give the readings a, b, c on the scale, show that the equilibrium reading will be $(ac - b)/(a + c - 2b)$. [The successive displacements of

the needle from its equilibrium positive form a geometric progression.]

4. A body, making 30 complete undamped vibrations per min., has its motion damped to $\frac{1}{2}$ amplitude in 10 sec. Find the value of f and the period during damping.

174. Non-periodic Motion under Central Attraction. We have seen that when $f < n$ in

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2x = 0 \tag{1}$$

we have damped oscillations given by (§ 173, 7). When $f = n$, this solution reduces to $c'e^{-ft}$ and contains but one arbitrary constant, and when $f > n$, $\sqrt{n^2 - f^2}$ is imaginary. Let us therefore retain the damping factor e^{-ft} and attempt to satisfy (1) when $f \geqq n$ with an expression of the form.

$$x = e^{-ft}u \tag{2}$$

where u is a function of t involving only real constants. If we substitute (2) in (1) we find that u must satisfy the differential equation

$$\frac{d^2u}{dt^2} - (f^2 - n^2)u = 0.$$

When $f > n$, its general solution is

$$u = Ae^{pt} + Be^{-pt} \quad \text{where} \quad p = \sqrt{f^2 - n^2};$$

and when $f = n$,

$$u = A + Bt.$$

The motion is now given by (2), after the arbitrary constants A, B are determined to satisfy the initial conditions.

In both cases the motion is no longer periodic; and it is easy to show that the particle will pass through O just once or not at all according as A and B differ or agree in sign. In brief, when $f \geqq n$ the damping is so great that the central attraction is unable to produce oscillations.

175. Forced Oscillations. In the last two articles we have studied the motion of a particle attracted by a central force proportional to the distance and resisted by a force proportional to the speed. Let us now consider what effect a disturbing force F *per unit mass of particle* has on the motion. In the notation

of § 173 the equation of motion is now

$$ma = -k_1x - k_2v + mF, \text{ or}$$

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2x = F. \tag{1}$$

In §§ 173, 174 we found the general solution of

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2x = 0 \tag{2}$$

in all cases; let us denote it by X. Then if ξ is any particular solution of (1),

$$x = X + \xi \tag{3}$$

is also a solution of (1). For if we substitute this expression in the left member of (1), the terms in X vanish since X satisfies (2), and the terms in ξ give precisely F since ξ is a solution of (1). Moreover (3) is the *general* solution of (1) since it contains the two arbitrary constants involved in X.

If F is constant, $\xi = F/n^2$ is clearly a particular solution of (1); for since ξ is constant, its derivatives vanish. The general solution of (1) is now given by (3).

The most important case arises when the disturbing force is periodic, say $F = k \cos qt$. Then (1) becomes

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + n^2x = k \cos qt. \tag{1}$$

The total disturbing force $mk \cos qt$ has the greatest value mk; hence k denotes the greatest positive acceleration it can give the particle.

Let us try to find a particular solution of (1) having the same period $2\pi/q$ as the disturbing force, say

$$\xi = b \cos (qt - \alpha); \tag{4}$$

here b and α are constants to be chosen so that (4) will satisfy (1). Putting $x = \xi$ in (1), the left member becomes

$$b(n^2 - q^2) \cos (qt - \alpha) - 2\,bfq \sin (qt - \alpha).$$

The right member may be written

$$k \cos (\alpha + qt - \alpha) = k \cos \alpha \cos (qt - \alpha) - k \sin \alpha \sin (qt - \alpha).$$

These expressions will be identical when

(5) $$b(n^2 - q^2) = k \cos \alpha, \quad 2\,bfq = k \sin \alpha.$$

On solving these equations for b and α we find

(6), (7) $$b = \frac{k}{\sqrt{(n^2 - q^2)^2 + 4\,f^2q^2}}, \quad \tan \alpha = \frac{2\,fq}{n^2 - q^2}.$$

We choose the positive value of b in (6); then the quadrant in which α lies is determined by the signs of $\cos \alpha$ and $\sin \alpha$ in (5). With these values of b and α, (4) gives a particular solution and (3) the general solution of our problem when F is periodic.

The motion represented by (3) may be regarded as a superposition of the *forced vibration* $x = \xi$, due to the disturbing force, upon the *free vibration* $x = X$ that would occur in the absence of this force. When damping is present ($f > 0$), the free vibration dies away in time by reason of the damping factor e^{-ft} in X and the motion ultimately reduces to the forced vibration ξ. We shall therefore examine ξ in some detail.

We note first that ξ has the same period $2\,\pi/q$ as the disturbing force but lags an angle α behind it in phase. From (5) we see that $\sin \alpha \geqq 0$ while

$$\cos \alpha \gtreqless 0 \text{ according as } n \gtreqless q.$$

Hence α may be chosen in the interval from 0 to π and

$$\alpha \lesseqgtr \tfrac{1}{2}\,\pi \text{ according as } \begin{matrix}\text{Free}\\ \text{period}\end{matrix} \lesseqgtr \begin{matrix}\text{Impressed}\\ \text{period}\end{matrix}.$$

When the damping is very slight (f very small) $\sin \alpha$ is small; then α is nearly 0 or π according as the free period is less or greater than the impressed period.

For given values of n and f, the amplitude b of the forced vibration is the function of q given by (6). This equation shows that b has a maximum value when

$$(n^2 - q^2)^2 + 4\,f^2q^2$$

has a minimum; that is, where the derivative

$$-4\,(n^2 - q^2)q + 8\,f^2q = 0 \quad \text{or} \quad q^2 = n^2 - 2\,f^2.$$

We thus find that b reaches its maximum

$$b_m = \frac{k}{2\,f\sqrt{n^2 - f^2}} \quad \text{when} \quad q = \sqrt{n^2 - 2\,f^2}.$$

When f is small compared with n (slight damping), the value of q giving maximum amplitude is nearly equal to n. The impressed period $2\pi/q$ is then nearly equal to the free period $2\pi/n$. The amplitude b_m in this case may be relatively large owing to the small number f in the denominator. Indeed the displacement x may become so large that the restoring force is no longer proportional to x; the basic equation (1) will then cease to apply.

This phenomenon, called *resonance*, may occur in any vibrating system acted on by an external periodic force whose period nearly coincides with the natural period of the system. Thus the heavy rolling of a ship caused by waves having nearly the same period as the ship is a case of resonance. Again, an automobile which gets a series of bumps from the road having nearly the same period as the springs is often set in violent vibration. This display of energy is due to the fact that the velocity of the forced vibration is in phase with the disturbing force, so that both have their greatest values at the same instant.

The term *resonance*, as its derivation suggests, was first used to describe the striking cases that occur in the propagation of sound waves. The electrical analogue of resonance, which occurs in alternating current circuits containing inductance and capacity, is also of great importance.*

* For a circuit of capacity C farads, inductance L henrys, and resistance R ohms, to which a periodic e.m.f. of $E \cos \omega t$ volts is applied, we have the differential equation

$$L\frac{di}{dt} + Ri + \frac{Q}{C} = E \cos \omega t,$$

where Q coulombs is the condenser charge, i amperes the current flowing in the circuit after t seconds, and ω is 2π times the frequency. This states that the impressed voltage at any instant is just sufficient to overcome the counter e.m.f. $L\,di/dt$ of induction, to supply the resistance drop of Ri volts, and to charge the condenser. Since $i = dQ/dt$, the equation may be written

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = \frac{E}{L}\cos \omega t.$$

This is the same equation as (1) if we set

$$Q = x, \quad \frac{R}{L} = 2f, \quad \frac{1}{LC} = n^2, \quad \frac{E}{L} = k, \quad \omega = q.$$

Note that the charge Q and the current $I = dQ/dt$ correspond to the displacement x and the speed $v = dx/dt$ of the vibrating particle, and that L corresponds to its mass. The condition $q = n$ that the free and impressed frequencies coincide becomes $\omega = 1/\sqrt{LC}$. When this is nearly fulfilled, heavy charges and currents may appear in the circuit; this is the electrical analogue of resonance.

In an undamped system ($f = 0$), (6) and (7) give

$$b = \pm \frac{k}{n^2 - q^2}, \quad \text{according as } n \gtrless q.$$

In both cases (4) gives the forced vibration

$$\xi = \frac{k}{n^2 - q^2} \cos qt. \tag{8}$$

This result is approximately true when the damping is slight.

When the free and impressed periods coincide in an undamped system ($n = q, f = 0$), equations (6, 7) lose their meaning and the particular solution (4) no longer applies. In this case (1) becomes

$$\frac{d^2x}{dt^2} + n^2x = k \cos nt. \tag{9}$$

In order to account for resonance in this case, let us try to satisfy (9) with

$$\xi = bt \cos (nt - \alpha)$$

in which the amplitude increases with the time. On substituting this expression for x in (9) we obtain

$$-2\, bn \sin (nt - \alpha) = k \cos nt.$$

This equation is satisfied by taking $b = k/2\,n$, $\alpha = \frac{1}{2}\pi$; hence

$$\xi = \frac{k}{2\,n} t \sin nt. \tag{10}$$

The free vibration, being simple harmonic, is

$$X = A \cos nt + B \sin nt$$

(§ 170); the general solution of (8) is therefore

$$x = A \cos nt + B \sin nt + \frac{k}{2\,n} t \sin nt. \tag{11}$$

The constants A, B must be chosen to fit the initial values of x and v. Thus if $x = 0$, $v = 0$ when $t = 0$, we find $A = B = 0$.

The free vibration now persists since damping is absent; but the amplitude of the forced vibration increases proportionately with the time. This is the phenomenon of resonance in an undamped system.

Finally let us consider the case when the disturbing force

$$F = k \sin qt.$$

We then find that the forced vibration

$$(4)' \qquad \xi = b \sin (qt - \alpha)$$

where b and α are again given by (6), (7). In an undamped system we have

$$(8)' \qquad \xi = \frac{k}{n^2 - q^2} \sin qt \quad \text{when } q \neq n,$$

$$(10)' \qquad \xi = -\frac{k}{2\,n} t \cos nt \quad \text{when } q = n.$$

Example. A body hanging in equilibrium from a light helical spring stretches it 1.5 in. beyond its natural length. If the upper end A of the spring is given a vertical s.h.m. of amplitude 2 in. and of period $\frac{1}{2}$ sec., find the amplitude of the motion of the body when it has reached its steady state.

Use the inch as the unit of length. For the free vibration of the body we have (§ 170, Example 1)

$$n^2 = \frac{g}{e} = \frac{32 \times 12}{1.5} = 256, \quad n = 16.$$

When the free vibration has been damped out, the forced vibration will have the period

$$2\,\pi/q = \tfrac{1}{2}; \quad \text{hence} \quad q = 4\,\pi.$$

Let y denote the displacement of A, taken positive when downward. Then if $y = 0$ when $t = 0$ and A is displaced upward at the start,

$$y = -2 \sin 4\,\pi t,$$

$$F = \frac{d^2y}{dt^2} = 32\,\pi^2 \sin 4\,\pi t \quad \text{and} \quad k = 32\,\pi^2.$$

The amplitude and phase of the forced vibration are now given by (6,) (7). Since $n > q$, $\cos \alpha > 0$ and $\alpha < \frac{1}{2}\,\pi$. Moreover the damping is slight, so that on putting $f = 0$ we have approximately

$$b = \frac{k}{n^2 - q^2} = \frac{32\,\pi^2}{256 - 16\,\pi^2} = 3.21 \text{ in.}, \quad \alpha = 0.$$

The forced vibration is in phase with F and hence opposed in phase to y; in other words, the body is lowest when the upper end of the spring is highest.

PROBLEMS

1. Show that when the free and impressed periods of a damped system coincide, the velocity of the forced vibration is in phase with the disturbing force.

What is the electrical analogue of this property?

2. A pendulum of length l, hanging at rest, is acted on by a constant force F per unit mass. Show that its motion is given by

$$\theta = \frac{2F}{g} \sin^2 \tfrac{1}{2}\sqrt{\frac{g}{l}}\, t$$

where θ is the angle measured from the downward vertical.

3. A plumb-bob hangs vertically from the roof of a railway coach running at constant speed. When the brakes are applied it swings through an angle of 3°. If the retardation is uniform show that it equals $\pi g/120$. [See Problem 2.]

176. Universal Gravitation. The most conspicuous motion in the sky is the monthly revolution of the moon about the earth. The orbit of the moon is nearly a circle about the earth as center and the average period of revolution is about

$$T = 27.3 \text{ days} = 2{,}360{,}000 \text{ sec.}$$

The radius r of its orbit has long been known to be about 60 times the radius of the earth; thus we find that the average distance of the moon is about

$$r = 239{,}000 \text{ miles} = 1.26 \times 10^9 \text{ ft.}$$

The acceleration of the moon towards the earth is therefore

$$a_n = \omega^2 r = \left(\frac{2\pi}{T}\right)^2 r = 0.00893 \text{ ft./sec.}^2.$$

It is now plausible that this " falling " acceleration of the moon is due to the earth's attraction. If we compare a_n with the falling acceleration g at the earth's surface, we find that

$$\frac{g}{a_n} = \frac{32.2}{0.00893} = 3600 = 60^2.$$

Thus at a distance of 60 earth radii the falling acceleration is $1/60^2$ of its value at 1 earth radius. In other words, the falling acceleration due to the earth's attraction is inversely proportional to the square of the distance from the center of the earth. This fact has been verified by comparing the weights of equal masses on a beam balance having scale pans at different heights (Jolly's experiment).

If G is the force of gravity exerted by the earth on a body of mass m at a distance r from its center, it will fall to the earth with an acceleration k/r^2 (k, a constant) and hence $G = mk/r^2$ by the

fundamental equation. Moreover from the Principle of Action and Reaction, G must also equal the force with which the body attracts the earth; hence by symmetry we may write

$$G = \gamma \frac{mM}{r^2} \tag{1}$$

where M is the mass of the earth and γ a constant independent of the attracting bodies. Newton now made the great generalization that this equation represents the attraction between any two bodies of masses m and M at a distance r apart. This is the

LAW OF UNIVERSAL GRAVITATION. *Any two bodies, whose dimensions are negligible in comparison with their distance apart, attract each other with forces directed along their joining line, and whose common magnitude is directly proportional to the product of their masses and inversely proportional to the square of their distance apart.*

In (1) γ is a universal constant known as the *constant of gravitation*; it represents the numerical value of the force of attraction between two particles of unit mass at unit distance apart. The average of the best experimental determinations of γ give for its value in C.G.S. units

$$\gamma = 6.675 \times 10^{-8} \text{ cm}^3/\text{gm.sec.}^2.$$

Note that γ, as defined by (1), does not have the dimensions of a force.

It may be shown that a homogeneous sphere, or a sphere formed of homogeneous concentric layers, exerts an attraction at an outside point as if its entire mass were concentrated at its center. Consequently two homogeneous spheres of 1 gram mass having their centers 1 cm. apart attract each other with a force of 6.675×10^{-8} dynes.

On the assumption that the earth is composed of homogeneous concentric layers of total mass M, the attraction between the earth and a particle of mass m at its surface is $\gamma Mm/R^2$, where R is the radius of the earth. But this force is also equal to mg; hence

$$\gamma \frac{Mm}{R^2} = mg, \quad M = \frac{gR^2}{\gamma}. \tag{2}$$

The mean density of the earth is therefore

$$\delta = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3g}{4\pi\gamma R}.$$

If we use the above value of γ, g and R must be taken in C.G.S. units. Thus on putting

$$g = 981 \text{ cm./sec}^2., \quad R = 6.37 \times 10^8 \text{ cm.},$$

we find that $\delta = 5.5$ Thus the mean density of the earth is about 5.5 times the density of water. As the mean density of the earth's crust is about 2.4, the density at the center must be larger than 5.5. This maximum density has been estimated to be about 10.

Example. The attraction of the earth on a body of mass m is

$$\mathbf{G} = -\frac{\gamma M m}{r^2}\mathbf{R}; \quad \text{hence} \quad V = -\frac{\gamma M m}{r}$$

is the potential energy of the body in the earth's gravitational field (§ 166, 3). Hence if the body falls toward the earth, conservation of energy demands that

$$\tfrac{1}{2} mv^2 - \frac{\gamma M m}{r} = \text{const.},$$

or since $\gamma M = gR^2$ from (2),

$$v^2 - \frac{2\,gR^2}{r} = \text{const.} \tag{3}$$

Thus if the body, starting from rest, falls to the earth from a very great distance ($r = \infty$) the constant in (3) is zero. Hence, if it were not for the retarding effect of the atmosphere, the body would strike the earth ($r = R$) with the velocity $V = \sqrt{2\,gR}$. Putting $g = 32.2$ ft./sec.^{2}, $R = 3960 \times 5280$ ft., we find that V is about 36,900 ft./sec. or 7 mi./sec.

PROBLEMS

1. What concentrated mass, in grams, will attract a like mass at a distance of 1 cm. with a force of 1 dyne?

2. If the diameter of the moon is 2160 mi. and its mass 1/81.5 that of the earth, find the falling acceleration of bodies on the moon.

3. A body is projected vertically from the earth with a velocity of 2000 ft./sec. Neglecting air resistance, how high will it rise

(*a*) taking gravity as constant,

(*b*) taking the variation of gravity into account?

4. The initial velocity of a body, projected vertically from the earth, would carry it to a height h if gravity were constant. Show that it will rise a distance $h^2/(R - h)$ higher owing to the decrease in gravity. Air resistance is neglected. [Apply (3).]

Consider the case $h = R$ (radius of earth). What is wrong with the above result when $h > R$?

5. A body falls from rest through a distance d into a center of attraction of mass M. Show that the time of fall is $\pi\sqrt{d^3/8\,\gamma M}$ sec. [From the equation preceding (3) show that

$$v = -\frac{dr}{dt} = \sqrt{2\,\gamma M}\;\sqrt{\frac{1}{r} - \frac{1}{d}}.$$

Find t by integration.]

6. If the orbital motion of the earth were arrested, show that it would fall into the sun in less than 65 days. Distance from earth to sun $= 14.9 \times 10^{12}$ cm., mass of sun $= 19.6 \times 10^{32}$ gm.

177. The Solar System. The equation of motion of a heavenly body attracted toward the sun is, according to the Newtonian Law,

$$m\frac{d\mathbf{v}}{dt} = -\gamma\frac{Mm}{r^2}\mathbf{R}$$

provided that the sun is regarded as fixed in space. M and m denote the masses of the sun and the body and r is the distance between them. With the sun's center as origin, the position vector of the body is $\mathbf{r} = r\mathbf{R}$.

If we write $k = \gamma M$, the above equation becomes

$$\frac{d\mathbf{v}}{dt} = -\frac{k}{r^2}\mathbf{R}. \tag{1}$$

Multiply (1) by $\mathbf{r}\times$ and integrate; then we find, as in § 169,

$$\mathbf{r}\times\mathbf{v} = \mathbf{h}. \tag{2}$$

If $\mathbf{h} \neq 0$, the orbit lies in a plane through the sun perpendicular to $\mathbf{h}$ and the sectorial speed $\frac{1}{2}\,|\mathbf{r}\times\mathbf{v}|$ is constant.

Since

$$\mathbf{r}\times\mathbf{v} = r\mathbf{R}\times\left(\frac{dr}{dt}\mathbf{R} + r\frac{d\mathbf{R}}{dt}\right) = r^2\,\mathbf{R}\times\frac{d\mathbf{R}}{dt},$$

(2) may be written

$$\mathbf{h} = r^2\mathbf{R}\times\frac{d\mathbf{R}}{dt}$$

and on multiplying (1) vectorially by this equation

$$\frac{d\mathbf{v}}{dt}\times\mathbf{h} = -k\mathbf{R}\times\left(\mathbf{R}\times\frac{d\mathbf{R}}{dt}\right) = k\frac{d\mathbf{R}}{dt}.$$

This may be integrated at once to give

(3) $$\mathbf{v}\times\mathbf{h} = k(\mathbf{R} + \boldsymbol{\epsilon})$$

where $\boldsymbol{\epsilon}$ is a constant vector.

The equation of the orbit is obtained by eliminating $\mathbf{v}$ from (2) and (3). Multiply (2) by $\mathbf{h}\cdot$, (3) by $\mathbf{r}\cdot$; then

$$\mathbf{r}\times\mathbf{v}\cdot\mathbf{h} = h^2, \quad \mathbf{r}\cdot\mathbf{v}\times\mathbf{h} = kr\,(1 + \epsilon\cos\theta)$$

where θ is the angle $(\boldsymbol{\epsilon}, \mathbf{r})$. Since the left members are equal (§ 18, 2) the right members are also; hence

(4) $$r = \frac{h^2/k}{1 + \epsilon\cos\theta}.$$

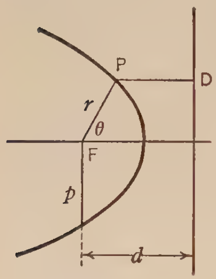


FIG. 177.

This is the equation of a conic section of eccentricity ϵ referred to a focus as pole. For from the focus-directrix definition of conic we have (Fig. 177)

$$\frac{PF}{PD} = \frac{r}{d - r\cos\theta} = \epsilon, \quad r = \frac{\epsilon d}{1 + \epsilon\cos\theta}.$$

This is the same equation as (4) if we put $h^2/k = \epsilon d$.* The orbit is therefore an ellipse, a parabola, or an hyperbola according as the eccentricity ϵ is less than, equal to, or greater than 1. Since the only closed orbits are ellipses, the planets describe ellipses having the sun at one focus. The orbits of comets may be elliptic, parabolic or hyperbolic.

Since the sectorial speed $\frac{1}{2}h$ is constant the period of revolution T in an elliptic orbit is obtained by dividing the area of the ellipse by the sectorial speed. The major axis $2a$ of the orbit is $r_1 + r_2$, where r_1, r_2 are the values of r when $\theta = 0, \pi$. Now from (4)

$$kr_1 = \frac{h^2}{1+\epsilon}, \quad kr_2 = \frac{h^2}{1-\epsilon},$$

and, on addition,

$$2\,ka = \frac{2\,h^2}{1-\epsilon^2}, \quad h = \sqrt{ka(1-\epsilon^2)}.$$

The area of an ellipse of semi-axes a, b is πab or $\pi a^2\sqrt{1-\epsilon^2}$. Hence

$$T = \frac{\pi a^2\sqrt{1-\epsilon^2}}{\frac{1}{2}\sqrt{ka(1-\epsilon^2)}} = \frac{2\,\pi}{\sqrt{k}}a^{\frac{3}{2}}$$

* On putting $\theta = \frac{1}{2}\pi$ we see that h^2/k and ϵd equal p, the ordinate of the ellipse at the focus.

and since $k = \gamma M$,

$$(5) \qquad \frac{T^2}{a^3} = \frac{4\pi^2}{\gamma M}.$$

Thus the ratio T^2/a^3 is the same for all the planets of the solar system. More generally this ratio is constant for all bodies revolving about the same center of attraction, the moons of Jupiter for example (M in (5) is then the mass of Jupiter).

Thus the Newtonian law of gravitation has led to the proof of

KEPLER'S THREE LAWS OF PLANETARY MOTION.

I. *The planets describe ellipses with the sun at one focus.*

II. *The radius vector from the sun to a planet sweeps out equal areas in equal times.*

III. *The squares of the periods of the planets are proportional to the cubes of their mean distances* from the sun.*

These laws were stated by Kepler after a careful study of astronomical observations, the first two in 1609, the third in 1619. The discovery that these laws, so diverse in content, all flow from a single principle, the Law of Gravitation, is one of Newton's greatest achievements and an enduring monument to the mind of man.

Lastly let us consider the equation of energy for a body of mass m in the gravitational field of the sun: $\mathbf{F} = -(\gamma Mm)\mathbf{R}/r^2$. Since its potential energy is $-\gamma Mm/r$ (§ 166, 3), conservation of energy demands that $\frac{1}{2}mv^2 - \gamma Mm/r$ be constant, that is

$$(6) \qquad v^2 - \frac{2\gamma M}{r} = \text{const.}$$

If $v = 0$ when $r = \infty$, the constant in (6) is zero. Hence a body, starting from rest at infinity, will acquire the speed

$$v_c = \sqrt{\frac{2\gamma M}{r}}$$

at a distance r from the sun. This *critical speed* at the distance r has a bearing on the type of orbit that a body will describe when launched into the sun's field. From (3) we have

$$(\mathbf{v}\times\mathbf{h})^2 = k^2(1 + 2\,\mathbf{R}\cdot\boldsymbol{\epsilon} + \epsilon^2),$$

* Since the major semi-axis $a = \frac{1}{2}(r_1 + r_2)$, it is called the planet's *mean distance* from the sun.

or since $|\mathbf{v}\times\mathbf{h}| = vh \sin 90° = vh$,

$$\frac{v^2h^2}{k^2} = 1 + 2\,\epsilon \cos\theta + \epsilon^2.$$

On putting $\epsilon \cos\theta = h^2/kr - 1$ (from (4)) in this result we find that

$$\epsilon^2 - 1 = \frac{h^2}{k^2}\left(v^2 - \frac{2\,k}{r}\right) = \frac{h^2}{k^2}(v^2 - v_c^2).$$

Thus the orbit will be elliptic ($\epsilon < 1$), parabolic ($\epsilon = 1$), or hyperbolic ($\epsilon > 1$) according as the speed at the point of departure is less than, equal to, or greater than the critical speed at this point.

Example. *Halley's Comet* has a period of about 76 years. At perihelion its distance from the sun is 0.58 of an astronomical unit (the mean distance of the earth from the sun). Find its distance at aphelion and the eccentricity of its orbit.

With astronomical units and years as units, Kepler's Third Law, applied to the comet and the earth, gives $T^2/a^3 = 1^2/1^3 = 1$. Hence the mean distance of the comet from the sun is

$$a = T^{\frac{2}{3}} = 76^{\frac{2}{3}} = 17.94 \text{ ast. units.}$$

The distance at aphelion is therefore

$$2 \times 17.94 - 0.58 = 35.30 \text{ ast. units.}$$

Halley's Comet therefore travels beyond the orbit of Neptune ($a = 30.07$) and within the orbit of Venus ($a = 0.615$).

The eccentricity of its orbit may be found from the equations for r_1 and r_2 given above; thus

$$\frac{r_1}{r_2} = \frac{1 - \epsilon}{1 + \epsilon}, \qquad \epsilon = \frac{r_2 - r_1}{r_2 + r_1} = \frac{34.72}{35.88} = 0.97.$$

178. Summary, Chapter XII. If a number of forces $\mathbf{F}_i$ act on a particle

$$\sum \mathbf{F}_i = m\mathbf{a} \qquad \text{(Prin. I, II).}$$

This is equivalent to the three scalar equations $\sum X_i = ma_x$, etc. All external forces $\mathbf{F}_i$ should be shown in the *free-body diagram* for the particle.

At moderate speeds the sliding friction $F = \mu N$, where N is the total normal pressure; μ is less than the coefficient of static friction for the same surfaces.

The differential equation of motion

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}$$

is equivalent to the three scalar equations, $m\,d^2x/dt^2 = X$ etc., or to the *intrinsic equations*

$$m\frac{dv}{dt} = F_t, \quad m\frac{v^2}{\rho} = F_n, \quad 0 = F_b.$$

The product of the mass and velocity of a particle (the vector $m\mathbf{v}$) is called its *momentum*. The time integral of a force over any interval is called the *impulse* of the force in this interval. The *Principle of Impulse and Momentum* states that the change in momentum of a particle is equal to the impulse of the resultant force in the interval.

The scalar $\frac{1}{2}mv^2$ is called the *kinetic energy* of the particle. The scalar product $\mathbf{F}\cdot\mathbf{v}$ is called the *power* of the force **F**. The time rate at which the kinetic energy of a particle is changing is equal to the power of the resultant force.

The time integral of the power $\mathbf{F}\cdot\mathbf{v}$ over any interval is called the *work* done by **F** in this interval. If **F** depends only upon the position of the particle, the work done by **F** is equal to the integral of its tangential component taken over the path:

$$\text{Work} = \int_{s_1}^{s_2} F_t\,ds = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r}.$$

If **F** is constant the work equals the scalar product of the force and displacement, that is, $\mathbf{F}\cdot\overrightarrow{P_1P_2} = Fd\cos\theta$. Work may be represented graphically as the area under the F_t-s curve.

The *Principle of Work and Energy* states that the change in the kinetic energy of a particle is equal to the total work done by the forces acting on it in the interval.

If the work done by a force **F** over any path depends only on the end-points, the force is said to be *conservative*. Then the *potential energy* of a particle in any position P due to the force **F** is the work done by **F** as the particle moves from P to some arbitrary standard position. The potential energy of a body of weight W in the gravitational field of the earth is Wh, where h is its height above some standard horizontal plane.

If a particle moves in a conservative field of force, the sum of its kinetic and potential energies is constant. *Its total energy is therefore conserved.*

In general, every term of a dynamical equation will have the same dimensions in L, T, F.

The motion of a projectile (neglecting air resistance) is a com-

bination of uniform horizontal motion with its initial horizontal velocity and of vertical motion accelerated by gravity with its initial vertical velocity.

When the force on a particle P always passes through a fixed point O, $\mathbf{r} \times \mathbf{v}$ is constant. The motion is plane (rectilinear if $\mathbf{r} \times \mathbf{v} = 0$) and the line OP will sweep over equal areas in equal times (*Law of Areas*).

A particle P has *harmonic motion* when it is attracted to a fixed point O by a force which varies as the distance OP; its equation of motion is $\mathbf{a} = -n^2\mathbf{r}$. When $\mathbf{r}_0 \times \mathbf{v}_0 = 0$ the motion is rectilinear and *simple harmonic*. When $\mathbf{r}_0 \times \mathbf{v}_0 \neq 0$ the motion is *elliptic harmonic*; the orbit is an ellipse about O as center described with constant sectorial speed. In both cases the period is $2\pi/n$.

A simple pendulum of length l has the period $2\pi\sqrt{l/g}$ for small vibrations; its motion is approximately simple harmonic.

For low speeds the resistance of a medium to a moving body varies as v; for higher speeds, as v^2. A body subject to a constant force $\mathbf{F}$ in a resisting medium has a limiting or *terminal velocity* V given by putting $\mathbf{a} = 0$ in its equation of motion.

In a *damped vibration* (resistance proportional to v) the period is constant but less that of the corresponding undamped vibration, and the successive amplitudes decrease in a constant ratio. If the damping exceeds a certain limit the central attraction will no longer produce vibrations and the particle will pass at most once through the center of attraction.

A periodic disturbing force applied to vibrating particle will superimpose on its free vibration a *forced vibration* having the same period as the force but lagging behind it in phase. When the free vibration is damped out, the forced vibration alone remains. If the impressed period nearly coincides with the free period, the forced vibration may involve large amplitudes and speeds when the damping is slight (*resonance*).

Any two particles attract each other according to the Law of Inverse Squares. A particle attracted to a fixed center of force O according to this law will describe a conic section having O as focus with constant sectorial speed. The planets P of the Solar System describe ellipses about the sun O as focus, the focal radius OP sweeps out equal areas in equal times, and the squares of their periods are proportional to the cubes of their mean distances from the sun (*Kepler's Laws*).

PROBLEMS

1. A bicyclist coasts down a 1 per cent grade at a uniform speed of 7.5 mi./hr. What horsepower would he exert in going up this grade at the same speed if the bicycle and rider weigh 175 lb.?

2. A bicyclist, going 15 mi./hr. on a level road, puts on a brake which applies a normal pressure of 30 lb. to the tire. If bicycle and rider weigh 160 lb. and $\mu = \frac{1}{3}$ between brake and tire, find the stopping distance. Neglect rotational inertia of wheels.

3. The bob of a simple pendulum 2 ft. long is moving 8 ft./sec. as it passes its lowest point. To what angle with the downward vertical will it rise?

4. A 50-ton interurban car requires a tractive effort of 12.5 lb./ton to move it along a level track at 30 mi./hr. Between 30 and 60 mi./hr. the wind resistance and other frictional forces are proportional to the square of the car speed. Find the total tractive force necessary to accelerate this car at the rate of 2 mi. per hr. per sec. up a 1 per cent grade at the instant when it reaches 50 mi./hr.

At what limiting speed would this car coast freely down a 2 per cent grade, and what would be its kinetic energy while coasting?

CHAPTER XIII

DYNAMICS OF A SYSTEM OF PARTICLES

179. Two Basic Theorems. The forces acting on any system of particles may be divided into two classes: (1) the *external forces* exerted by particles not in given system, and (2) the *internal forces* consisting of the mutual actions between particles of the system. The internal forces may be grouped in pairs which represent the interaction of two particles of the system. Then, according to the Principle of Action and Reaction, the forces of each pair are equal in magnitude, opposite in direction, and have a common line of action. Hence the vector sum of the forces in each pair is zero; and the sum of their moments about any point is zero (§ 66). Since the internal forces occur only in such pairs we conclude that

In any system of particles the vector sum of the internal forces, and of their moments about any point, is zero.

Consider now any particle P of mass m belonging to the system and acted on by the external forces $\mathbf{F}$ and internal forces $\mathbf{F}'$. Then

$$m\mathbf{a} = \sum \mathbf{F} + \sum \mathbf{F}' \qquad \text{(Prin. I, II)}.$$

Moreover if A is any point, moving or fixed, chosen as the center of moments and $\mathbf{r} = \overrightarrow{AP}$,

$$\mathbf{r}\times m\mathbf{a} = \sum \mathbf{r}\times\mathbf{F} + \sum \mathbf{r}\times\mathbf{F}'.$$

Here $m\mathbf{a}$ is called the *mass-acceleration* of the particle and $\mathbf{r}\times m\mathbf{a}$ the *moment of the mass-acceleration* about A. Now form these equations for each particle of the system and add the equations of each set. Then since both $\sum\sum \mathbf{F}' = 0$ and $\sum\sum \mathbf{r}\times\mathbf{F}' = 0$, we obtain

$$(1) \qquad \sum m\mathbf{a} = \sum\sum \mathbf{F},$$

$$(2) \qquad \sum \mathbf{r}\times m\mathbf{a} = \sum\sum \mathbf{r}\times\mathbf{F}.$$

These fundamental equations in the dynamics of systems may be stated as follows:

THEOREM I. *For any system of particles, the vector sum of the mass-accelerations is equal to the sum of the external forces.*

THEOREM II. *For any system of particles, the vector sum of the moments of the mass-accelerations about any point (fixed or moving) is equal to the sum of the moments of the external forces about the point.*

The double sums above indicate that the forces or moments are first added at each particle, and the sums then added for the entire system. In the future, however, we shall simply write $\sum \mathbf{F}$, $\sum \mathbf{r}\times\mathbf{F}$ for these total sums.

The mass-accelerations of the system are often regarded as a set of fictitious forces localized at the particles, and called the *inertia forces* of the system. If these inertia "forces" are reduced to another set of vectors by shifting them along their lines of action and by vector addition of concurrent "forces" (just as in Statics, § 70, the forces on a rigid body were reduced to an equivalent system by use of Principles A and B), the new set of vectors is said to be *equivalent* to the inertia forces. Now Theorems I and II state that the inertia forces $m\mathbf{a}$ and the external forces $\mathbf{F}$ have the same force-sum and moment-sum about a point; these two sets of vectors are therefore equivalent (§ 74). Thus we have proved the theorem known as

D'ALEMBERT'S PRINCIPLE. *The inertia forces and the external forces acting on any system of particles are equivalent sets of vectors in the sense that either set may be obtained from the other by shifting along lines of action and by vector addition at a point. The system composed of both the reversed inertia forces* $-m\mathbf{a}$ *and the external forces is therefore equivalent to zero.*

180. Conservation of Momentum. The *momentum* of a system is defined as the vector sum of the momenta of its particles. Thus the momentum is $\sum m\mathbf{v}$ and

$$\frac{d}{dt}\sum m\mathbf{v} = \sum m\mathbf{a} = \sum \mathbf{F} \tag{1}$$

from Theorem I. Hence if $\sum \mathbf{F} = 0$, the momentum is constant.

THEOREM ON MOMENTUM. *The time rate of change of the momentum of a system is equal to the sum of the external forces acting on it.*

If the sum of the external forces is zero, the momentum remains constant.

The second part of the theorem is known as the *Conservation of Momentum.*

On integrating (1) between the instants t_1 and t_2 we obtain

$$(2) \qquad (\sum m\mathbf{v})_2 - (\sum m\mathbf{v})_1 = \sum \int_{t_1}^{t_2} \mathbf{F}\, dt:$$

the change in momentum is equal to the sum of the impulses of the external forces.

181. Center of Mass. A system of particles on or near the earth has its *center of gravity* defined by

$$\mathbf{r}^* = \frac{\sum w\mathbf{r}}{\sum w} \qquad (\S\ 77),$$

where w is the weight and $\mathbf{r}$ the position vector of a particle. If we put $w = mg$,

$$(1) \qquad \mathbf{r}^* = \frac{\sum m\mathbf{r}}{\sum m}$$

provided $\mathbf{g}$ *may be taken as constant throughout the system.* The point P^* given by (1) is called the *center of mass* of the system. For a body of mass m the sums in (1) are replaced by integrals as in § 76:

$$(2) \qquad \mathbf{r}^* = \frac{\int \mathbf{r}\, dm}{m}.$$

For ordinary terrestrial bodies the center of mass may be regarded as coincident with the center of gravity. But for heavenly bodies, or systems composed of them, the term center of gravity is not appropriate; their centers of mass, however, defined by (1) or (2), are points of great dynamical importance.

On differentiating (1) twice with respect to the time we obtain

$$(3) \qquad (\sum m)\mathbf{v}^* = \sum m\mathbf{v},$$

$$(4) \qquad (\sum m)\, \mathbf{a}^* = \sum m\mathbf{a},$$

where $\mathbf{v}^*$ and $\mathbf{a}^*$ denote the velocity and acceleration of the center of mass. Since $\sum m\mathbf{v}$ is the momentum of the system, (3) states that

The momentum of the system is the same as that of a particle having the total mass of the system and moving with its center of mass.

If we put $\sum m\mathbf{a} = \sum \mathbf{F}$ (Theorem I) in (4),

$$(5) \qquad (\sum m)\mathbf{a}^* = \sum \mathbf{F}.$$

If (5) is regarded as the equation of motion of a particle of mass $\sum m$ situated at the center of mass P^*, its content may be stated as the

THEOREM ON THE MOTION OF THE CENTER OF MASS. *The center of mass of a system of particles moves like a free particle having the mass of the entire system and acted on by the vector sum of all the external forces.*

If, in particular $\sum \mathbf{F} = 0$, the Law of Inertia (§ 155) applies to the motion of P^*:

If the sum of the external forces acting on a material system is zero, its center of mass will remain at rest or move uniformly in a straight line.

Example 1. *Motion of the Sun.* The external forces on the solar system consist of the attractions of all exterior matter. If these forces may be neglected in view of the vast distances involved, we may conclude that the center of mass of the solar system (a point in the sun itself) is at rest or moves with constant velocity.

Example 2. *Recoil of Firearms.* When a gun is discharged, the system formed by the gun, charge, and bullet, originally at rest, is set in motion through the agency of internal forces. The center of mass of the system therefore remains at rest; and as the bullet and gases move in one direction, the gun recoils in the other.

182. Problem of Two Bodies: To find the motions of two particles P_1, P_2 of masses m_1, m_2, subject only to their mutual Newtonian attractions.

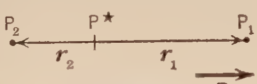


FIG. 182.

As no external forces act on the *system* composed of P_1, P_2, their center of mass P^* will remain at rest or move with constant velocity. Choose a frame of reference having this constant velocity and take P^*, which is at rest in this frame, as origin. If $\mathbf{r}_1 = \overrightarrow{P^*P_1}$, $\mathbf{r}_2 = \overrightarrow{P^*P_2}$, and $\mathbf{R}$ is a unit vector in the direction $\overrightarrow{P_2P_1}$ (Fig. 182), the equations of motion of P_1 and P_2 are

$$(1), (2) \qquad m_1 \frac{d^2\mathbf{r}_1}{dt^2} = -\gamma \frac{m_1 m_2}{(\mathbf{r}_1 - \mathbf{r}_2)^2} \mathbf{R}, \quad m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \gamma \frac{m_1 m_2}{(\mathbf{r}_1 - \mathbf{r}_2)^2} \mathbf{R}.$$

Since $\mathbf{r}^* = 0$, we have, from (§ 181, 1),

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0, \quad \mathbf{r}_2 = -\frac{m_1}{m_2}\mathbf{r}_1.$$

With this value of $\mathbf{r}_2$, (1) becomes

$$\frac{d^2\mathbf{r}_1}{dt^2} = -\gamma \frac{m_2}{\left(1 + \frac{m_1}{m_2}\right)^2} \frac{\mathbf{R}}{r_1^2} \tag{3}$$

This equation states that P_1 is attracted toward P^* according to the law of inverse squares. P_1 therefore describes an ellipse having P^* as one focus (§ 177). In the same way we may show that P_2 describes an ellipse about P^* as focus. The law of areas applies to each motion and P_1 and P_2 always lie in line with P^*. The period is given by (§ 177, 5) on replacing M by $m_2\ (1 + m_1/m_2)^{-2}$.

If we divide (1) by m_1, (2) by m_2 and subtract the resulting equations, we obtain

$$\frac{d^2}{dt^2}(\mathbf{r}_1 - \mathbf{r}_2) = -\gamma \frac{m_1 + m_2}{(\mathbf{r}_1 - \mathbf{r}_2)^2}\mathbf{R}.$$

This equation has the form of (§ 177, 1)* with $\mathbf{r}_1 - \mathbf{r}_2$ in place of $\mathbf{r}$ and $m_1 + m_2$ in place of M. The orbit of P_1 *relative to* P_2 is therefore an ellipse about P_2 as focus; and the period and major semi-axis in this relative orbit are connected by the equation

$$\frac{T^2}{a^3} = \frac{4\,\pi^2}{\gamma(m_1 + m_2)}. \tag{4}$$

Thus in the orbit of a planet P_1 relative to the sun ($m_2 = M$), T^2/a^3 depends upon m_1 as well as M. Hence T^2/a^3 is not strictly the same for all planets, as Kepler's Third Law asserts. But since the mass of the sun is more than 1000 times that of Jupiter, the greatest planet, $m_1 + M$ may be replaced by M in (4) without serious error. The third law (§ 177, 5) is therefore very nearly true.

183. Moment of Momentum. If P is a particle of mass m, the moment of its momentum about a point Q is $\overrightarrow{QP}\times m\mathbf{v}$. The *moment of momentum* about Q of a system of particles is defined as the sum of their moments of momentum about Q. Denoting this vector by $\mathbf{H}_Q$,

$$\mathbf{H}_Q = \sum \mathbf{r}\times m\mathbf{v} \quad \text{where } \mathbf{r} = \overrightarrow{QP},$$

$$\frac{d\mathbf{H}_Q}{dt} = \sum m\left(\mathbf{r}\times\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{r}}{dt}\times\mathbf{v}\right).$$

* Written $\dfrac{d^2\mathbf{r}}{dt^2} = -\gamma\dfrac{M}{\mathbf{r}^2}\mathbf{R}$.

Now $d\mathbf{v}/dt = \mathbf{a}$; and if O is a fixed origin

$$\mathbf{r} = \overrightarrow{OP} - \overrightarrow{OQ}, \quad \frac{d\mathbf{r}}{dt} = \mathbf{v} - \mathbf{v}_Q.$$

Hence $$\frac{d\mathbf{H}_Q}{dt} = \sum m\mathbf{r}\times\mathbf{a} - \mathbf{v}_Q\times \sum m\mathbf{v}.$$

Since $\sum m\mathbf{v} = (\sum m)\mathbf{v}^*$ (§ 181, 3), we may write this equation

(1) $$\sum \mathbf{r}\times m\mathbf{a} = \frac{d\mathbf{H}_Q}{dt} + (\sum m)\mathbf{v}_Q\times\mathbf{v}^*.$$

Also $$\sum m\mathbf{a} = (\sum m)\mathbf{a}^* \qquad (§ 181, 4).$$

Hence from § 70, Theorem 2, we conclude that

When $\mathbf{v}_Q\times\mathbf{v}^* = 0$ *the inertia forces of a system of particles may be reduced to the*

$$\textit{Force } (\textstyle\sum m)\mathbf{a}^* \textit{ at } Q \quad \textit{and} \quad \textit{Couple of moment } \frac{d\mathbf{H}_Q}{dt}.$$

If $\mathbf{M}_Q$ denotes the moment-sum of the external forces about Q, $\sum \mathbf{r}\times m\mathbf{a} = \mathbf{M}_Q$ by the basic Theorem II; hence from (1)

(2) $$\frac{d\mathbf{H}_Q}{dt} = \mathbf{M}_Q \quad \text{only when } \mathbf{v}_Q\times\mathbf{v}^* = 0,$$

that is, when $\mathbf{v}_Q = 0$, $\mathbf{v}^* = 0$, or $\mathbf{v}_Q \parallel \mathbf{v}^*$.

THEOREM OF MOMENT OF MOMENTUM. *The time rate of change of the moment of momentum about any point Q is equal to the sum of the moments of the external forces about this point in three cases only:*

(*a*) *Q is at rest,*
(*b*) *the center of mass P^* is at rest,*
(*c*) *the velocities of Q and P^* are parallel (in particular when Q is P^*).*

If $\mathbf{M}_Q = 0$ in (2), $\mathbf{H}_Q$ is constant. This result is known as the *Conservation of Moment of Momentum.*

The moment of momentum is frequently called *angular momentum.*

184. Moment of Relative Momentum. The *moment of relative momentum* of a system about a point Q (fixed or moving) is defined as

$$\mathbf{H}_Q' = \sum \mathbf{r}\times m\mathbf{v}' \quad \text{where} \quad \mathbf{r} = \overrightarrow{QP}$$

and $\mathbf{v}'$ is the velocity of P relative to Q. Now

$$\frac{d\mathbf{H}_Q'}{dt} = \sum m\left(\mathbf{r}\times\frac{d\mathbf{v}'}{dt} + \frac{d\mathbf{r}}{dt}\times\mathbf{v}'\right).$$

If O is a fixed origin, $\mathbf{r} = \overrightarrow{OP} - \overrightarrow{OQ}$ and

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} - \mathbf{v}_Q = \mathbf{v}', \qquad \frac{d\mathbf{v}'}{dt} = \mathbf{a} - \mathbf{a}_Q;$$

hence

$$\frac{d\mathbf{H}_Q'}{dt} = \sum m\mathbf{r}\times(\mathbf{a} - \mathbf{a}_Q) = \sum \mathbf{r}\times m\mathbf{a} - \left(\sum m\mathbf{r}\right)\times\mathbf{a}_Q.$$

Since $\sum m\mathbf{r} = (\sum m)\mathbf{r}^*$ (§ 181, 1), we may write this equation

$$\sum \mathbf{r}\times m\mathbf{a} = \frac{d\mathbf{H}_Q'}{dt} + \left(\sum m\right)\mathbf{r}^*\times\mathbf{a}_Q. \tag{1}$$

Also
$$\sum m\mathbf{a} = \left(\sum m\right)\mathbf{a}^* \qquad (\S\ 181,\ 4).$$

From these equations we conclude that

When $\mathbf{r}^*\times\mathbf{a}_Q = 0$ *the inertia forces of a system of particles may be reduced to the*

$$\textit{Force } \left(\sum m\right)\mathbf{a}^* \textit{ at } Q \quad \textit{and} \quad \textit{Couple of moment } \frac{d\mathbf{H}_Q'}{dt}.$$

If $\mathbf{M}_Q$ denotes the moment-sum of the external forces about Q, $\sum \mathbf{r}\times m\mathbf{a} = \mathbf{M}_Q$ by the basic Theorem II; hence from (1)

$$\frac{d\mathbf{H}_Q'}{dt} = \mathbf{M}_Q \quad \text{only when } \mathbf{r}^*\times\mathbf{a}_Q = 0, \tag{2}$$

that is, when $\mathbf{r}^* = 0$, $\mathbf{a}_Q = 0$, or $\mathbf{a}_Q \parallel \mathbf{r}^*$ $(\overrightarrow{QP^*})$.

THEOREM ON MOMENT OF RELATIVE MOMENTUM. *The time rate of change of the moment of relative momentum about any point Q is equal to the sum of the moments of the external forces about this point in three cases only:*

(*a*) *Q is the center of mass P^*,*
(*b*) *Q has constant velocity (or is at rest),*
(*c*) *the acceleration vector of Q passes through P^*.*

185. Kinetic Energy. The sum $\sum \frac{1}{2} m\mathbf{v}^2$ of the kinetic energies of the particles of a system is called the kinetic energy of the system.

THEOREM. *The kinetic energy of a system of particles is equal to the kinetic energy of a particle having the total mass of the system and moving with its center of mass, plus the kinetic energy of the particles in their motion relative to the center of mass.*

Proof. The velocity of any particle P equals $\mathbf{v} = \mathbf{v}^* + \mathbf{v}'$, where $\mathbf{v}^*$ is the velocity of P^*, and $\mathbf{v}'$ the velocity of P relative to P^*. Hence

$$\begin{aligned}\sum \tfrac{1}{2} m\mathbf{v}^2 &= \sum \tfrac{1}{2} m(\mathbf{v}^{*2} + 2\,\mathbf{v}^*\cdot\mathbf{v}' + \mathbf{v}'^2)\\ &= \tfrac{1}{2}\left(\sum m\right)\mathbf{v}^{*2} + \mathbf{v}^*\cdot\sum m\mathbf{v}' + \sum \tfrac{1}{2} m\mathbf{v}'^2.\end{aligned}$$

Also

$$\sum m\mathbf{v} = \sum m(\mathbf{v}^* + \mathbf{v}') = \left(\sum m\right)\mathbf{v}^* + \sum m\mathbf{v}',$$

so that $\sum m\mathbf{v}' = 0$ in view of (§ 181, 3). The above expression for kinetic energy thus reduces to

$$\sum \tfrac{1}{2} m\mathbf{v}^2 = \tfrac{1}{2}\left(\sum m\right)\mathbf{v}^{*2} + \sum \tfrac{1}{2} m\mathbf{v}'^2.$$

186. Work and Energy. The equation of motion of any particle of a system is

$$m\frac{d\mathbf{v}}{dt} = \sum \mathbf{F} + \sum \mathbf{F}' \qquad (\S\ 179).$$

Multiply this equation by $\mathbf{v}\cdot$; then since

$$m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\tfrac{1}{2} m\mathbf{v}^2\right),$$

$$\frac{d}{dt}\left(\tfrac{1}{2} m\mathbf{v}^2\right) = \sum \mathbf{F}\cdot\mathbf{v} + \sum \mathbf{F}'\cdot\mathbf{v}.$$

On adding the set of such equations for all the particles of the system we obtain

$$\frac{d}{dt}\sum \tfrac{1}{2} m\mathbf{v}^2 = \sum\sum \mathbf{F}\cdot\mathbf{v} + \sum\sum \mathbf{F}'\cdot\mathbf{v}.$$

Integrate this from $t = t_1$ to $t = t_2$; then

$$\left(\sum \tfrac{1}{2} m\mathbf{v}^2\right)_2 - \left(\sum \tfrac{1}{2} m\mathbf{v}^2\right)_1 = \sum\sum \int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}\,dt + \sum\sum \int_{t_1}^{t_2} \mathbf{F}'\cdot\mathbf{v}\,dt.$$

This expresses the

PRINCIPLE OF WORK AND ENERGY. *The change in the kinetic energy of a system of particles in any time interval is equal to the total work done by both external and internal forces in this interval.*

In applying this principle it is often necessary to find the work done by gravity on a system of particles of total weight W as it moves from one position to another. The work of gravity on a *particle* of weight w as it moves from $\mathbf{r}_1$ to $\mathbf{r}_2$ is

$$\mathbf{w}\cdot(\mathbf{r}_2 - \mathbf{r}_1) = \mathbf{k}\cdot(w\mathbf{r}_2 - w\mathbf{r}_1) \qquad (\S\ 163,\ 4),$$

where $\mathbf{k}$ is a unit vector in the direction of gravity. Hence the work of gravity on the *system* is

$$\mathbf{k}\cdot\left(\sum w\mathbf{r}_2 - \sum w\mathbf{r}_1\right) = \mathbf{k}\cdot(W\mathbf{r}_2^* - W\mathbf{r}_1^*) = W\mathbf{k}\cdot(\mathbf{r}_2^* - \mathbf{r}_1^*) \qquad (\S\ 181,\ 1).$$

Here $\mathbf{r}_1^*$ and $\mathbf{r}_2^*$ give the initial and final positions of the center of gravity. Since $\mathbf{r}_2^* - \mathbf{r}_1^*$ is the displacement of the center of gravity, $\mathbf{k}\cdot(\mathbf{r}_2^* - \mathbf{r}_1^*)$ is the distance h^* through which it *falls*. Therefore the

(1) $$\textit{Work of Gravity} = Wh^*,$$

the weight of the system times the vertical distance through which its center of gravity falls.

Example 1. A flexible rope, weighing w lb./ft. hangs from a smooth peg with a ft. on side, b ft. on the other, when released. Find the work done by gravity up to the point when one end clears the peg.

The center of gravity of the rope, originally

$$\frac{wa\cdot\frac{1}{2}a + wb\cdot\frac{1}{2}b}{w(a+b)} = \frac{1}{2}\frac{a^2+b^2}{a+b} \text{ ft. below the peg,}$$

falls to $\frac{1}{2}(a+b)$ ft. below the peg as one end passes over. Hence the work done by gravity is

$$w(a+b)\left\{\frac{1}{2}(a+b) - \frac{1}{2}\frac{a^2+b^2}{a+b}\right\} = wab \text{ ft.-lb.}$$

If the short end of the rope, weighing wa lb., were cut off, dropped b ft., and attached to the long end, the work done would be wab ft.-lb. — a brief method of obtaining the result above.

Example 2. A rope passing over a smooth pulley supports a stone and a monkey of the same weight W at its ends. If the monkey climbs up to the pulley, how much work has he done?

At the outset the tension of the rope is W. As the monkey pulls on the rope to climb upward he increases its tension to $W + F$. This downward force on the rope is paired with a numerically equal force acting upward on the monkey. His acceleration a_m is therefore given by

(i) $$\frac{W}{g}a_m = W + F - W = F.$$

At the same time the rope transmits the upward force $W + F$ to the stone; its acceleration a_s is thus given by

$$\text{(ii)} \qquad \frac{W}{g} a_s = W + F - W = F.$$

From (i) and (ii) we have $a_m = a_s$. Now monkey and stone start from rest on the same distance h below the pulley and both have the same upward acceleration at each instant. They must therefore arrive at the pulley at the same instant. (Let the student give a detailed proof.)

Consider now the system composed of monkey, stone, and rope. Neglecting the weight of the rope, the work done on the system by gravity as the monkey climbs up is $-2\,Wh$. As the system is at rest at the beginning and end, the change in its kinetic energy is zero. Hence, by the principle of work and energy, the total work done by both external and internal forces is zero. Since the former work is $-2\,Wh$, the work done by the internal forces (the muscular forces of the monkey) is precisely $2\,Wh$.

PROBLEMS

1. An 18-ft. flexible rope hangs from a smooth peg, 12 ft. on one side, 6 ft. on the other, when released from rest. What is the speed of the rope as one end just passes over the peg?

2. In the Atwood's machine of Fig. 159*a*, $W = 3$ lb., $W' = 1$ lb. If the weights start from rest, what will be their speed when W has fallen 2 ft?

3. A cable, weighing w lb./ft., passes over a drum and hangs with free ends a ft. and b ft. long. If the cable is wound up so that the free ends become $a - x$ ft. and $b + x$ ft., show that the work done against gravity is $wx(a - b - x)$ ft.-lb. How is this result to be interpreted when $b > a$? when $x > a - b$?

187. Summary, Chapter XIII. *For any system of particles the sum of the mass-accelerations and of their moments about any point are equal respectively to the sum of the external forces and of their moments about that point.* These theorems give the basic dynamical equations:

$$\text{(1), (2)} \qquad \sum m\mathbf{a} = \sum \mathbf{F}, \quad \sum \mathbf{r}\times m\mathbf{a} = \sum \mathbf{r}\times\mathbf{F}.$$

The mass-accelerations (" inertia forces ") and external forces are equivalent sets of vectors (*D'Alembert's Principle*).

The center of mass P^* of a system of particles has the position vector $\mathbf{r}^*$ defined by

$$(\textstyle\sum m)\mathbf{r}^* = \sum m\mathbf{r}; \quad \text{and}$$

$$(\textstyle\sum m)\mathbf{v}^* = \sum m\mathbf{v}, \quad (\sum m)\mathbf{a}^* = \sum m\mathbf{a}.$$

Equation (1) may be replaced by

$$(\textstyle\sum m)\mathbf{a}^* = \sum \mathbf{F}; \tag{3}$$

hence P^* moves like a free particle having the total mass of the system and acted on by the sum of the external forces. If this sum is zero, P^* remains at rest or moves with constant velocity.

Definitions: For a system of particles P

$$\text{Momentum} = \textstyle\sum m\mathbf{v} = (\sum m)\mathbf{v}^*,$$

$$\text{Moment of Momentum } \mathbf{H}_Q = \textstyle\sum \mathbf{r}\times m\mathbf{v},$$

$$\text{Moment of Relative Momentum } \mathbf{H}_Q' = \textstyle\sum \mathbf{r}\times m\mathbf{v}'$$

where $\mathbf{r} = \overrightarrow{QP}$ and $\mathbf{v}'$ is the velocity of P relative to Q.

The moment of the mass-accelerations about Q

$$\textstyle\sum \mathbf{r}\times m\mathbf{a} = \dfrac{d\mathbf{H}_Q}{dt} \quad \text{when} \quad \mathbf{v}_Q\times\mathbf{v}^* = 0,$$

$$\textstyle\sum \mathbf{r}\times m\mathbf{a} = \dfrac{d\mathbf{H}_Q'}{dt} \quad \text{when} \quad \mathbf{a}_Q\times\overrightarrow{QP^*} = 0.$$

In these cases the inertia forces are equivalent to

(*a*) a single force $(\sum m)\mathbf{a}^*$ at Q, and

(*b*) a couple of moment $d\mathbf{H}_Q/dt$ or $d\mathbf{H}_Q'/dt$ respectively.

If $\mathbf{M}_Q$ denotes the moment-sum of the external forces about Q, (2) may be replaced by

$$\frac{d\mathbf{H}_Q}{dt} = \mathbf{M}_Q \quad \text{when} \quad \mathbf{v}_Q = 0, \quad \mathbf{v}^* = 0, \quad \text{or} \quad \mathbf{v}_Q \parallel \mathbf{v}^*; \tag{4}$$

$$\frac{d\mathbf{H}_Q'}{dt} = \mathbf{M}_Q \quad \text{when} \quad \mathbf{a}_Q = 0, \quad \overrightarrow{QP^*} = 0, \quad \text{or} \quad \mathbf{a}_Q \parallel \overrightarrow{QP^*}. \tag{5}$$

In particular both (4) and (5) hold when Q is at rest or at the center of mass.

The kinetic energy of the system is

$$\sum \tfrac{1}{2} m\mathbf{v}^2 = \tfrac{1}{2}\left(\sum m\right)\mathbf{v}^{*2} + \sum \tfrac{1}{2} m\mathbf{v}'^2$$

where $\mathbf{v}'$ is the velocity of P relative to P^*. The change in kinetic energy in any interval is equal to the total work done by both external and internal forces in this interval.

CHAPTER XIV

DYNAMICS OF RIGID BODIES

188. Bodies as Continuous Mass Distributions. A body is any definite portion of matter. Bodies, according to modern physics, consist of an aggregate of atoms, which in turn are miniature solar systems consisting of a central nucleus or *proton* about which the planetary *electrons* revolve. Thus in a very real sense bodies are aggregates of particles — electrons and protons. The number of such particles in ordinary bodies is enormous, but nevertheless finite. In order to deal with such distributions of matter we treat the matter as if it were continuously distributed and *replace the sums of the last chapter by integrals.* Thus the basic Theorems I and II become

$$\int \mathbf{a}\, dm = \sum \mathbf{F}, \quad \int \mathbf{r}\times\mathbf{a}\, dm = \sum \mathbf{r}\times\mathbf{F}.$$

For a body of mass m, the

$$\text{Momentum} = \int \mathbf{v}\, dm = m\mathbf{v}^*,$$

$$\text{Moment of Momentum } \mathbf{H} = \int \mathbf{r}\times\mathbf{v}\, dm,$$

$$\text{Kinetic Energy} = \tfrac{1}{2}\int v^2\, dm.$$

The ideal *rigid body* has been defined as one that suffers no deformation from the forces acting on it. We shall now also assume that *the distances between the particles of a rigid body remain constant.* This assumption is at variance with the state of affairs described above; but this is to be expected since perfectly rigid bodies do not exist.

189. Principle of Work and Energy for a Rigid Body. Let the particles A, B of a rigid body have the velocities $\mathbf{v}_A$, $\mathbf{v}_B$; and let $\mathbf{F}'$ at A and $-\mathbf{F}'$ at B be the pair of internal forces between them. Then

$$\mathbf{F}'\cdot\mathbf{v}_A + (-\mathbf{F}')\cdot\mathbf{v}_B = \mathbf{F}'\cdot(\mathbf{v}_A - \mathbf{v}_B) = 0;$$

for if O is a fixed point,

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}, \qquad \frac{d}{dt}\overrightarrow{BA} = \mathbf{v}_A - \mathbf{v}_B,$$

and since $\overrightarrow{BA}$ has constant length, $d\overrightarrow{BA}/dt$ or $\mathbf{v}_A - \mathbf{v}_B$ is perpendicular to AB (§ 84) and hence to $\mathbf{F}'$. Since all the internal forces occur in pairs $\mathbf{F}'$, $-\mathbf{F}'$, we see that $\sum \mathbf{F}' \cdot \mathbf{v} = 0$. *The total work done by all the internal forces in any displacement of the body is therefore zero*:

$$\sum \int_{t_1}^{t_2} \mathbf{F}' \cdot \mathbf{v}\, dt = \int_{t_1}^{t_2} \sum \mathbf{F}' \cdot \mathbf{v}\, dt = 0.$$

Hence from § 186 we have the following

Principle of Work and Energy. *The change in the kinetic energy of a rigid body in any interval is equal to the work done by the external forces acting on it.*

As a body, rigid or deformable, may be regarded as a system of particles, the result (§ 186, 1) shows that

The work done by gravity during the motion of any body, is equal to the product of its weight and the vertical distance through which its center of gravity falls.

190. Kinetics of Translation. In a *translation* of a rigid body the velocities of all of its particles are the same at any instant (§ 121). Consequently we may speak of the velocity $\mathbf{v}$ of the body, and also of its acceleration $d\mathbf{v}/dt = \mathbf{a}$. Since all the particles have the same acceleration $\mathbf{a}$ at any instant,

$$\int \mathbf{a}\, dm = m\mathbf{a}, \quad \int \mathbf{r} \times \mathbf{a}\, dm = \left(\int \mathbf{r}\, dm\right) \times \mathbf{a} = m\mathbf{r}^* \times \mathbf{a},$$

where $\mathbf{r}^*$ is the position vector of the center of mass referred to any origin O, fixed or moving. Thus the inertia forces are equivalent to a force $m\mathbf{a}$ at the center of mass; for their sum is $m\mathbf{a}$ and their moment-sum about O is $\mathbf{r}^* \times m\mathbf{a}$. In brief:

The inertia forces of a body in translation have the resultant $m\mathbf{a}$ *at center of mass.*

Hence, by D'Alembert's Principle,

The external forces on a body in translation are equivalent to the vector $m\mathbf{a}$ *at the center of mass.*

If $\mathbf{F}$ and $\mathbf{M}_O$ represent the force-sum and moment-sum of all the external forces acting on the body, we have the dynamical

equations

(1), (2) $$m\mathbf{a} = \mathbf{F}, \quad \mathbf{r}^* \times m\mathbf{a} = \mathbf{M}_O.$$

These equations state the basic Theorems I and II applied to a translation.

Problems in the kinetics of translation should be solved as follows:

Draw a free-body diagram showing all the external forces and the resultant $m\mathbf{a}$ *of the inertia forces acting at the center of mass. Then express the equivalence of* $m\mathbf{a}$ *to the external forces by taking components in any direction or moments about any axis so that the resulting equations are as simple as possible.*

In the cases of translation that occur in practice the motion is usually plane. Such problems are simply solved by writing three scalar equations, two by equating components along perpendicular axes in the plane, the third by equating moments about an axis normal to the plane. In particular, the external forces have zero moment about an axis through the center of mass. In the examples that follow we shall refer to moments about an axis normal to the plane as moments about the *point* where the axis cuts the plane.

The momentum and kinetic energy of a body in translation are $m\mathbf{v}$ and $\frac{1}{2}mv^2$. In any interval the change in $m\mathbf{v}$ is equal to the sum of the impulses of the external forces (§ 180); and the change in $\frac{1}{2}mv^2$ is equal to the work done by the external forces (§ 189).

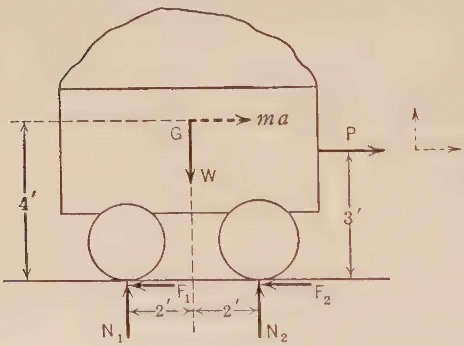


FIG. 190*a*.

Example 1. A loaded ore-truck weighing 1 ton is drawn uniformly along its track by a horizontal pull P of 80 lb. Find the reactions on its wheels if its center of gravity is at G in Fig. 190*a*.

When the horizontal pull P is 160 lb., find the acceleration of the truck and the reactions.

Uniform motion. Resolve the reactions into horizontal and vertical components. Since $P = 80$ lb., $a = 0$, we have from (1):

$$80 - F_1 - F_2 = 0, \quad N_1 + N_2 - 2000 = 0;$$

and from (2), on taking moments about G,

$$1 \times 80 - 4\,(F_1 + F_2) + 2\,N_2 - 2\,N_1 = 0.$$

Hence
$$N_1 + N_2 = 2000, \quad N_2 - N_1 = 120;$$
$$N_2 = 1060 \text{ lb.}, \qquad N_1 = 940 \text{ lb.}$$

The separate values of F_1 and F_2 cannot be found; but their sum is 80 lb.

Accelerated Motion. With $P = 160$ lb.,

$$160 - F_1 - F_2 = \frac{2000}{32}a, \quad N_1 + N_2 - 2000 = 0;$$

and on taking moments about G,

$$1 \times 160 - 4\,(F_1 + F_2) + 2\,N_2 - 2\,N_1 = 0.$$

If the frictional resistances are the same as before $F_1 + F_2 = 80$ lb., and

$$a = \frac{80 \times 32}{2000} = 1.28 \text{ ft./sec.}^2.$$

Also
$$N_1 + N_2 = 2000, \quad N_2 - N_1 = 80;$$
$$N_2 = 1040 \text{ lb.}, \qquad N_1 = 960 \text{ lb.}$$

Example 2. The side rod of a locomotive of weight W has its end pins at a distance r from the center of the drivers of radius R (Fig. 190*b*). When the locomotive is running at uniform speed V find the reactions exerted by the pins on the rod.

Each point of the side rod describes a circle of radius r with the constant angular velocity of $\omega = V/R$ of the drivers. The acceleration of the center of mass G is purely normal and equals $r\omega^2$. Hence if we resolve the reactions into horizontal and vertical components, we have

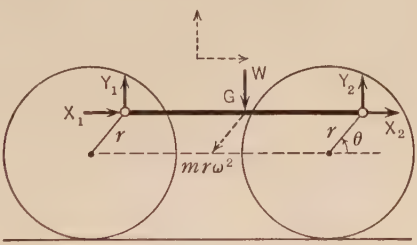


Fig. 190*b*.

$$X_1 + X_2 = -\frac{W}{g}r\omega^2\cos\theta, \quad Y_1 + Y_2 - W = -\frac{W}{g}r\omega^2\sin\theta;$$

and on taking moments about G we find that $Y_1 = Y_2$. Hence

$$Y_1 = Y_2 = \tfrac{1}{2}\,W\left(1 - \frac{r\omega^2}{g}\sin\theta\right).$$

X_1 and X_2 cannot be found separately, but their sum is given above.

Thus when the rod is in its lowest position $\theta = 270°$, and

$$X_1 + X_2 = 0, \quad Y_1 = Y_2 = \tfrac{1}{2} W\left(1 + \frac{r\omega^2}{g}\right).$$

If $\quad V = 60$ mi./hr. $= 88$ ft./sec., $r = 14$ in., $R = 3$ ft.,

$$Y_1 = Y_2 = \tfrac{1}{2} W(1 + \tfrac{14}{12}(\tfrac{88}{3})^2 \tfrac{1}{32}) = 16.2\ W.$$

Thus the vertical reactions, which are $\frac{1}{2} W$ when the locomotive is at rest, become 32 times as great at 60 mi./hr.

Example 3. A block of weight W rests on a truck. Assuming that the block does not slip, what is the greatest acceleration that may be given to the truck before the block tips over (Fig. 190*c*).

The reaction **R** on the block and its weight **W** have the resultant m**a** passing through G. Since **W** acts through G, **R** does likewise. On resolving horizontally and vertically we have

$$R\cos\theta = \frac{W}{g}a, \quad R\sin\theta - W = 0;$$

hence $\tan\theta = g/a$. Just before the block tips over R will act at its left edge. In this position $\tan\theta = h/d$. Hence if

$$\frac{h}{d} \leqq \frac{g}{a} \quad \text{or} \quad a \leqq \frac{d}{h}g$$

the block will not tip. Thus if $d = 3$ ft., $h = 6$ ft., a must not exceed $\frac{1}{2}g$.

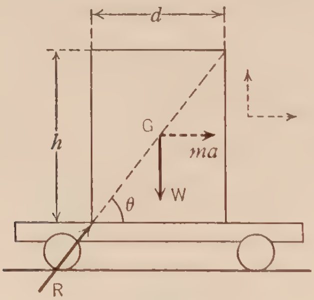


Fig. 190*c*.

Fig. 190*d*.

Example 4. What horizontal acceleration applied to a rod juggled at the end of a finger will maintain it at a constant angle β to the vertical (Fig. 190*d*).

The pressure **R** of the finger on the rod must be in the direction of the rod in order that the resultant of **R** and **W** pass through G. Hence

$$R\sin\beta = \frac{W}{g}a, \quad R\cos\beta - W = 0;$$

$$\tan\beta = \frac{a}{g}, \quad a = g\tan\beta.$$

Example 5. Car Driven by Torque on Rear Axle. If the engine exerts a turning moment or *torque* M on the rear axle, the friction F of the road acts forward to prevent the rear wheels from slipping backwards (Fig. 190*e*). Consider the rear wheels and axle as a free body acted on by the torque M and the friction F. On taking moments about the axle, $Fr - M = 0$ if we neglect the moment of the mass-accelerations (Theorem II) and $F = M/r$.

If the car is moving straight ahead with the acceleration a we have from (1)

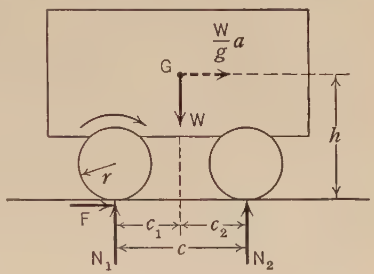


FIG. 190*e*.

$$F = \frac{W}{g}a, \quad N_1 + N_2 - W = 0;$$

and from (2), on taking moments about G,

$$Fh + N_2c_2 - N_1c_1 = 0.$$

Hence from the second and third equations

$$N_1 = \frac{Wc_2 + Fh}{c}, \quad N_2 = \frac{Wc_1 - Fh}{c}$$

where $c = c_1 + c_2$. Since $N_1 = Wc_2/c$, $N_2 = Wc_1/c$ when $F = 0$, the torque M on the rear axle increases the pressure on the rear wheels and decreases that on the front wheels by Fh/c.

If the coefficient of friction (coef. of adhesion; cf. §158) between road and wheels is μ, the rear wheels will not slip as long as F does not exceed μN_1. The condition for no slip is therefore

$$F \leqq \mu \frac{Wc_2 + Fh}{c} \quad \text{or} \quad F \leqq \frac{\mu W c_2}{c - \mu h},$$

and the greatest acceleration is

$$a_{\max} = \frac{F}{W/g} = \frac{\mu c_2}{c - \mu h} g.$$

When the power is off and brakes are applied to the rear wheels, the car will suffer a retardation a'. The friction F' now acts backward to prevent the rear wheels from slipping forward and $F' = Wa'/g$. The moment equation now becomes

$$-F'h + N_2c_2 - N_1c_1 = 0;$$

and the above results all hold if we replace h by $-h$. The braking thus decreases the pressure on the rear wheels and increases that on the front wheels by $F'h/c$; and

$$a'_{\max} = \frac{\mu c_2}{c + \mu h} g.$$

As an example consider a touring car which has a 9-ft. wheel-base and which, at rest, has two-thirds of its weight on the rear wheels. Then $c_1 = 3$ ft., $c_2 = 6$ ft.; and if $h = 3$ ft., $\mu = 0.6$,

$$a_{\max} = \tfrac{1}{2}g, \quad a'_{\max} = \tfrac{1}{3}g.$$

Example 6. *Car Turning a Curve.* Consider first a car on a level track; and let its center of gravity G describe a circle of radius r with uniform speed v. We treat the motion as a circular translation. Then in order that G have the normal acceleration v^2/r, the road must give a transverse thrust P to the wheels (Fig. 190*f*) such that

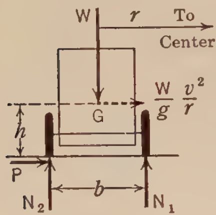


FIG. 190*f*.

$$P = \frac{W}{g}\frac{v^2}{r}.$$

Let N_1 and N_2 denote the upward pressures on the inner and outer pairs of wheels and b their *tread*. On resolving vertically and taking moments about an axis joining the points of contact of the outer wheels, we have

$$N_1 + N_2 - W = 0, \quad N_1 b - W\frac{b}{2} = -\frac{W}{g}\frac{v^2}{r}h;$$

hence

$$N_1 = W\left(\tfrac{1}{2} - \frac{v^2}{gr}\frac{h}{b}\right), \quad N_2 = W\left(\tfrac{1}{2} + \frac{v^2}{gr}\frac{h}{b}\right).$$

Thus the normal acceleration v^2/r of the car decreases the pressure on the inside wheels and increases that on the outside wheels by Wv^2h/grb; and the moment of these forces about a longitudinal axis through G, namely Wv^2h/gr, just balances the moment Ph of P about this axis tending to tip the car outwards. When $N_1 = 0$, that is when

$$v^2 = \frac{grb}{2h},$$

the inner wheels no longer bear on the road and the car is just on the point of upsetting. Thus for a motor-car turning on a curve of 20 ft. radius and for which $b = 56$ in., $h = 30$ in., the overturning velocity is

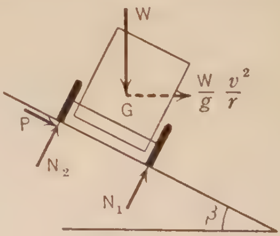


FIG. 190*g*.

$$v = \sqrt{\frac{32 \times 20 \times 56}{2 \times 30}} = 24.5 \text{ ft./sec.} = 16.7 \text{ mi./hr.}$$

Consider now a car on a track sloping toward the center at an angle β to the horizontal (Fig. 190*g*). The angle β can be chosen so

that the thrust $P = 0$ at a given speed v; for on resolving parallel to the slope of the track we have

$$P + W \sin \beta = \frac{W}{g}\frac{v^2}{r} \cos \beta, \quad \text{and}$$

$$\tan \beta = \frac{v^2}{gr} \quad \text{when} \quad P = 0.$$

In this case $N_1 = N_2$ as we see on taking moments about a longitudinal axis through G.

The outer rail of a curved railway track is always given a *superelevation* e above the inner rail so that the flanges of the inner wheels will not exert a side thrust on the rail at a certain speed v at which trains are supposed to take the curve. If the gage of the track is b (standard gage = 4 ft. $8\frac{1}{2}$ in.), the superelevation is $b \sin \beta$ or, since β is small,

$$e = b \tan \beta = \frac{bv^2}{gr}.$$

Thus for a curve of 800-ft. radius on a standard gage track, the superelevation for 30 mi./hr. or 44 ft./sec. is

$$e = \frac{56.5 \times 44^2}{32.2 \times 800} = 4.25 \text{ in.}$$

PROBLEMS

1. In the ore truck of Fig. 190a, $W = 2$ tons, $P = 100$ lb. Neglecting friction, find the reactions N_1, N_2 on each pair of wheels.

2. A 1000-lb. box, 4 ft. high, slides down a plane inclined 30° to the horizontal. If $\mu = \frac{1}{4}$, find its acceleration and the magnitude and position of the normal reaction.

3. A man is running on a horizontal floor at the rate of 20 ft./sec. If $\mu = \frac{1}{3}$ between his shoes and the floor, find his shortest turning radius and his inclination to the vertical on this curve.

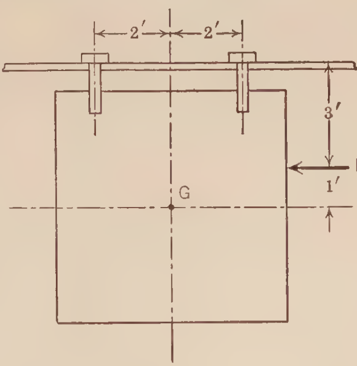


FIG. 190h.

4. What force P applied to the 160-lb. hanging door of Fig. 190h will give it an acceleration of 2 ft./sec. if $\mu = \frac{1}{4}$ between shoes and track. Find the normal reactions of the track on the shoes.

5. An iron pipe 4 ft. in diameter, lying across the floor of a truck, is held in place by two blocks of height h in front and back. What is the least value of h in order to keep the pipe from rolling when the truck is stopped as quickly as possible with rear-wheel brakes? Assume $\mu = 0.6$ between tires and street and that 60 per cent of the weight of the loaded truck is on the rear wheels.

6. A motor-car is running on a level road around a curve of 50-ft. radius. Its wheel-base is 56 in. in width and its center of gravity 28 in. above the ground. Assuming that no skidding occurs, find the greatest speed the car may have without tipping.

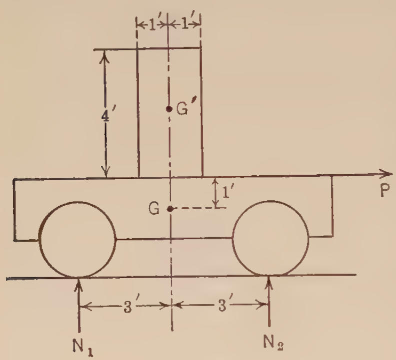


FIG. 190*i*.

7. If the car in Fig. 190*e* is pulled to the right by a force P acting in a horizontal line a distance d below G, show that the front wheels will leave the ground when $P > c_1W/d$.

8. In Fig. 190*i* the car weighs 1 ton, the block 700 lb. and their centers of gravity are at G and G'. If the block does not slide, find the greatest force P that can be applied to the car without tipping the block. What are the reactions N_1, N_2 on each pair of wheels?

9. A particle of weight W' slides down the smooth inclined face of a wedge of weight W resting on a smooth horizontal plane (Fig. 190*j*). Find the acceleration a of the wedge, the acceleration a_r of the particle relative to the wedge, and the reactions R, N on the particle and wedge. [The acceleration of the particle is $\mathbf{a} + \mathbf{a}_r$ (§ 134, 2).]

Compute a and a_r when $W = 10$, $W' = 4$ lb., $\beta = 30°$.

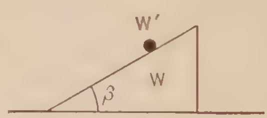


FIG. 190*j*.

191. Rotation about a Fixed Axis. A rigid body of mass m rotates about a fixed axis (z-axis) under the action of certain external forces **F**. If the angle of rotation is θ, its angular velocity and acceleration are

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

the positive directions of θ, ω, α being given by the positive direction on the axis (§ 108).

If p is the perpendicular distance of any particle from the axis, its acceleration

$$\mathbf{a} = a_t\mathbf{T} + a_n\mathbf{N} = p\alpha\mathbf{T} + p\omega^2\mathbf{N}$$

provided the unit of angle is the radian (§ 109). The moment of **a** about the axis (Fig. 191) is therefore the sum of

$$\text{Moment } p\alpha\mathbf{T} = p^2\alpha \quad \text{and} \quad \text{Moment } p\omega^2\mathbf{N} = 0.$$

The moment of the mass-accelerations $\mathbf{a}\,dm$ of the entire body about the axis is therefore

$$\alpha \int p^2\,dm = \alpha I.$$

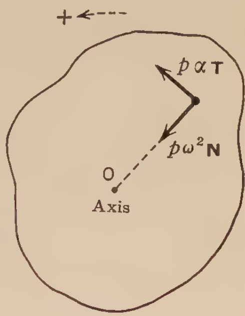


Fig. 191.

The positive number $I = \int p^2\,dm$ is called the *moment of inertia* of the body about the axis; it represents the sum of the products obtained by multiplying the mass of each particle by the square of its distance from the axis. The moment of inertia I may be determined by experiment (§§ 197, 200) or, if the body has a simple geometrical form, calculated as $\int p^2\,dm$. This calculation in certain important cases is given in §§ 193–196.

To determine the angular motion of the body we now apply D'Alembert's Principle. This states that the mass-accelerations of the body form a system of vectors equivalent to the external forces. The sum of the moments of these systems of vectors about the axis are therefore equal. If M_z denotes the moment-sum of the external forces about the z-axis, we have the equation

$$(1) \qquad I\alpha = M_z \quad (\alpha \text{ in rad./sec.}^2).$$

Given the initial conditions

$$\theta = \theta_0, \quad \omega = \omega_0 \quad \text{when} \quad t = 0,$$

equation (1) determines the motion. In particular if M_z is constant, α is also and the rotation is uniformly accelerated; we may then apply equations (1) to (4) of § 117.

Since the speed of a particle at a distance p from the axis is

$v = \omega p$, the body has the kinetic energy

$$\tfrac{1}{2}\int v^2\,dm = \tfrac{1}{2}\,\omega^2\int p^2\,dm = \tfrac{1}{2}\,\omega^2 I; \qquad \text{hence}$$

(2) $$\text{Kinetic Energy} = \tfrac{1}{2}\,I\omega^2 \quad (\omega \text{ in rad./sec.}).$$

To obtain the energy equation from (1) multiply it by ω, obtaining

$$I\omega\frac{d\omega}{dt} = M_z\omega,$$

and integrate from t_1 to t_2:

$$\tfrac{1}{2}\,I\omega_2{}^2 - \tfrac{1}{2}\,I\omega_1{}^2 = \int_{t_1}^{t_2} M_z\omega\,dt$$

If M_z is a function of θ alone we may write $\omega\,dt = d\theta$ and integrate from θ_1 to θ_2; then

$$\tfrac{1}{2}\,I\omega_2{}^2 - \tfrac{1}{2}\,I\omega_1{}^2 = \int_{\theta_1}^{\theta_2} M_z\,d\theta.$$

Since the left member is the change in kinetic energy, the right member must represent the work done by the external forces (§ 189).

This expression for the work may be deduced directly. Let the force $\mathbf{F}$ act at the point P of a body rotating with the angular velocity ω about the z-axis. The vector angular velocity is $\boldsymbol{\omega} = \omega\mathbf{k}$ and the velocity of P is

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} \quad \text{where} \quad \mathbf{r} = \overrightarrow{OP} \qquad (\S\ 109).$$

In the interval from t_1 to t_2, $\mathbf{F}$ does the work

$$\int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}\,dt = \int_{t_1}^{t_2} \mathbf{F}\cdot\boldsymbol{\omega}\times\mathbf{r}\,dt = \int_{t_1}^{t_2} \mathbf{r}\times\mathbf{F}\cdot\boldsymbol{\omega}\,dt.$$

If $\mathbf{F}$ depends only on the angle of rotation θ, we may put

$$\boldsymbol{\omega}\,dt = \mathbf{k}\frac{d\theta}{dt}\,dt = \mathbf{k}\,d\theta$$

and integrate with respect to θ; then

(3) $$\text{Work} = \int_{\theta_1}^{\theta_2} \mathbf{r}\times\mathbf{F}\cdot\mathbf{k}\,d\theta = \int_{\theta_1}^{\theta_2} M_z\,d\theta \qquad (\S\ 66,\ 1).$$

The work of *all* the external forces is given by the same expression provided M_z denotes the *sum* of their moments about the z-axis.

The moment M_z is often called a *torque* about the z-axis, and the integral in (3) the work done by the torque. When the torque M_z is constant

$$\text{Work} = M_z(\theta_2 - \theta_1): \tag{4}$$

the work done by a constant torque is the product of the torque and the angle of rotation expressed in radians. The work is positive when the turning effect of the torque is in the direction of rotation, negative in the contrary case.

192. Comparison of Translation and Rotation. Before proceeding to the solution of problems in rotation let us compare a rectilinear translation with a rotation about a fixed axis. For a translation along the x-axis let F_x denote the sum of the x-components of the external forces. For a rotation about the z-axis let M_z denote the sum of the moments of the external forces about this axis. We then have the following correspondences.

TRANSLATION		ROTATION
	KINEMATICS	
x		θ
$v = \dfrac{dx}{dt}$		$\omega = \dfrac{d\theta}{dt}$
$a = \dfrac{dv}{dt} = v\dfrac{dv}{dx}$		$\alpha = \dfrac{d\omega}{dt} = \omega\dfrac{d\omega}{d\theta}$
	UNIFORMLY ACCELERATED MOTION	
a constant		α constant
$v = v_0 + at$		$\omega = \omega_0 + \alpha t$
$x = v_0 t + \frac{1}{2}at^2$	$\left\{\begin{array}{l} x_0 = 0 \\ \theta_0 = 0 \end{array}\right\}$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2\,ax$		$\omega^2 = \omega_0^2 + 2\,\alpha\theta$
	DYNAMICS	
$F_x = ma$		$M_z = I\alpha$
F_x		M_z
m		I
$\frac{1}{2}mv^2$	Kinetic Energy	$\frac{1}{2}I\omega^2$
$\int_{x_1}^{x_2} F_x\,dx$	Work	$\int_{\theta_1}^{\theta_2} M_z\,d\theta$

Thus when we replace the quantities

$$x, v, a, m, F_x \quad \text{by} \quad \theta, \omega, \alpha, I, M_z$$

the formulas for translation go over into those for rotation. In the dynamical formulas for rotation, *the radian must be chosen as the unit of angle.*

In rotation the moment of inertia of the body about the axis plays the part of mass in translation. From the definition $I = \int p^2\,dm$, I has the dimensions of mass $\times$ (length)^2. If the entire mass of the body were concentrated at a distance k from the axis, the moment of inertia would be mk^2; and if k is chosen so that

$$mk^2 = I \text{ (the actual moment of inertia)}$$

k is said to be the *radius of gyration* of the body about the axis. The moment of inertia of a body is often specified by giving its mass and radius of gyration.

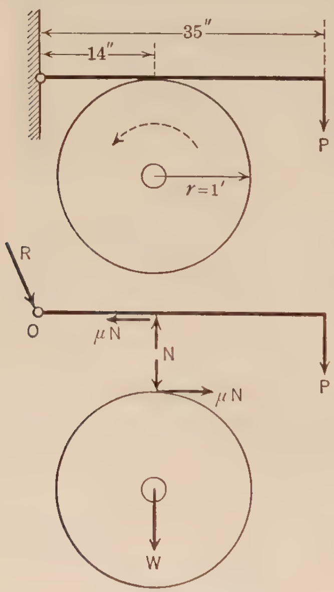


Fig. 192*a*.

Example 1. A force P, applied to the brake of a cylindrical drum making 150 r.p.m. brings it to rest in 10 sec. (Fig. 192*a*). Find the number of revolutions made by the drum in coming to rest. If the drum weighs $W = 800$ lb., its radius $r = 1$ ft., its radius of gyration $k = 0.9$ ft. and $\mu = 0.2$ is the coefficient of brake friction, find the force P.

First Solution. Consider first the kinematic problem. In 10 sec. the angular velocity changes from

$$\omega_0 = \frac{150 \times 2\pi}{60} = 5\pi \text{ rad./sec.} \quad \text{to} \quad 0;$$

hence

$$\alpha = \frac{0 - 5\pi}{10} = -\tfrac{1}{2}\pi \text{ rad./sec.}^2.$$

While coming to rest the wheel will turn through an angle θ given by $0 = \omega_0^2 + 2\,\alpha\theta$ (see table above); hence

$$\theta = -\frac{\omega_0^2}{2\,\alpha} = \frac{25\,\pi^2}{\pi} = 25\,\pi \text{ rad.} = 12.5 \text{ rev.}$$

In the free-body diagrams N and μN are the normal and tangential forces between brake and drum. R is the reaction at O.

The brake is in equilibrium; hence on taking moments about O

$$14\ N - 35\ P = 0, \quad P = 0.4\ N.$$

The forces N, μN and W acting on the drum have the moments 0, $-\mu Nr$, 0 about its axis.

Hence from (1)

$$I\alpha = -\mu Nr, \quad N = -\frac{I\alpha}{\mu r}.$$

Since
$$I = \frac{800}{32}(0.9)^2 = 25 \times 0.81,$$

$$N = \frac{25 \times 0.81 \times \frac{1}{2}\pi}{0.2 \times 1} = 159 \text{ lb.}, \quad P = 0.4 \times 159 = 63.6 \text{ lb.}$$

Second Solution. The work done by the frictional moment $-\mu Nr$ as the wheel comes to rest is $-\mu Nr\theta$. Hence from the Principle of Work and Energy

$$0 - \tfrac{1}{2} I\omega_0^2 = -\mu Nr\theta, \quad N = \frac{I\omega_0^2}{2\ \mu r\theta},$$

from which N may be computed. Since $0 = \omega_0^2 + 2\ \alpha\theta$, this expression for N may be reduced to the one given above.

Example 2. A flywheel, keyed to a shaft free to turn in smooth horizontal bearings, is set in motion by a falling weight w attached to a cord wrapped around the shaft (Fig. 192*b*). If the combined wheel and shaft, of weight W, have a radius of gyration k, and the radius of the shaft is r, find the falling acceleration a of the weight.

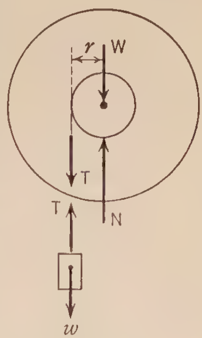


Fig. 192*b*.

The acceleration a is equal to the tangential acceleration $r\alpha$ of a point on the surface of the shaft. The angular acceleration of wheel and shaft is therefore $\alpha = a/r$.

Cut the cord and introduce the tension T at the cut ends. Then the weight w is a free body having translatory motion and

$$w - T = \frac{w}{g}a.$$

The wheel and shaft form a free body rotating about a horizontal axis and acted on by the forces T, N, W (neglecting the journal friction). Their moments about the axis are Tr, 0, 0; hence from (1)

$$Tr = I\alpha = \frac{W}{g}k^2\frac{a}{r}.$$

From these equations we have

$$T = w\left(1 - \frac{a}{g}\right), \quad T = W\frac{k^2}{r^2}\frac{a}{g}.$$

Equate these values of T and solve for a; then

$$a = \frac{g}{1 + \dfrac{W}{w}\dfrac{k^2}{r^2}}.$$

If, for example, $W = 100$ lb., $w = 10$ lb., $k = 6$ in., $r = 1$ in.,

$$a = \frac{32}{1 + 10 \times 36} = 0.089 \text{ ft./sec.}^2.$$

Example 3. A thin uniform rod of length l is free to turn about a horizontal axis at one end. If it is released from a horizontal position, find its angular velocity when vertical (Fig. 192c).

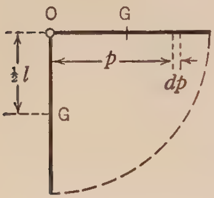


Fig. 192c.

We first find the moment of inertia of the rod. If its weight is W, an element of the rod of length dp has the mass $\frac{dp}{l}\frac{W}{g}$ and its moment of inertia about a transverse axis through O is

$$I = \int p^2\,dm = \frac{W}{gl}\int_0^l p^2\,dp = \frac{W}{g}\frac{l^2}{3}.$$

The square of the radius of gyration is $\frac{1}{3}l^2$.

As the rod falls from rest to its lowest position its center of gravity G drops a vertical distance $\frac{1}{2}l$ and the work done by gravity is $\frac{1}{2}Wl$ (§ 189). On the other hand, its kinetic energy changes from 0 to $\frac{1}{2}I\omega^2$. Hence, from the Principle of Work and Energy,

$$\frac{1}{2}\frac{W}{g}\frac{l^2}{3}\omega^2 - 0 = \tfrac{1}{2}Wl, \quad \omega = \sqrt{\frac{3g}{l}}.$$

Example 4. The rod in Example 3 is displaced slightly from its lowest position and released (Fig. 192d). Find the character of its oscillations.

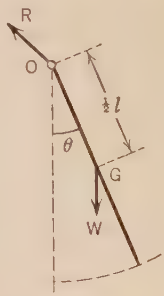


Fig. 192d.

If we neglect bearing friction and air resistance, the only forces acting on the rod are the reaction R at O and the weights of its particles. The resultant of the latter is the total weight **W** acting through G and their moment about O is $-\frac{1}{2}Wl\sin\theta$ when the positive sense of rotation is counterclockwise. The equation of motion (1) thus becomes

$$\frac{W}{g}\frac{l^2}{3}\alpha = -\tfrac{1}{2}Wl\sin\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{g}{\frac{2}{3}l}\sin\theta.$$

This is precisely the equation of motion of a simple pendulum of length $\frac{2}{3}l$ (§ 171, 3). The rod will therefore swing to and fro with the constant period $2\pi\sqrt{\frac{2}{3}l/g}$ (§ 171, 4).

Example 5. A cord passing over a pulley of radius r and weight P has the weights W_1, W_2 attached to its ends (Fig. 192*e*). Find the acceleration of the weights and the tensions in the two parts of the cord, neglecting axle friction.

If a is the acceleration of the weights, $\alpha = a/r$ is the angular acceleration of the pulley. From the free-body diagrams for the weights and pulley we have

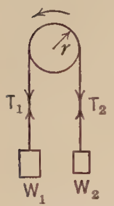


FIG. 192*e*.

$$\text{(i)} \qquad W_1 - T_1 = \frac{W_1}{g}a \qquad \text{(for } W_1\text{)},$$

$$\text{(ii)} \qquad T_2 - W_2 = \frac{W_2}{g}a \qquad \text{(for } W_2\text{)},$$

$$\text{(iii)} \qquad (T_1 - T_2)r = \frac{P}{g}k^2\frac{a}{r} \qquad \text{(for pulley)},$$

where k is the radius of gyration. To eliminate T_1, T_2, divide (iii) by r and add to (i) and (ii); thus

$$W_1 - W_2 = \left(W_1 + W_2 + P\frac{k^2}{r^2}\right)\frac{a}{g}$$

from which we can find a. Equations (i) and (ii) now give T_1 and T_2.

PROBLEMS

1. Find the dimensions of moment of inertia in F, L, T. Apply the check of dimensions to formulas (1), (2), (3) of § 191.

2. A thin uniform rod 4 ft. long and weighing 12 lb. is suspended from a smooth horizontal axis at one end. Find its instantaneous angular acceleration when acted on by a horizontal force of 10 lb. at its mid-point. [$k^2 = \frac{1}{3}l^2$ from Example 3.]

3. A 2-lb. gyroscope is spun by pulling on a string 2 ft. long, wrapped around its axle, with a tension of 9 lb. If $k = 2$ in., find the angular velocity generated in rev./sec.

4. A grindstone with $k = 1$ ft. is making 120 rev./min. Left to itself, it comes to rest under axle friction after 80 rev. If the axle is 1.5 in. in diameter, find the coefficient of axle friction.

5. A flywheel, keyed to a shaft free to turn in smooth horizontal bearings, is set in motion by a falling 10-lb. weight attached to a cord wrapped around the shaft (Fig. 192*b*). Wheel and shaft weigh 200 lb. and the axle is 3 in. in diameter. If the weight falls 5 ft. from rest in 10 sec., find the radius of gyration of wheel and shaft.

6. Solve Example 5 if $r = 6$ in., $P = 3.6$, $W_1 = 7$, $W_2 = 6.5$ lb., and the pulley is dynamically equivalent to a weight of 3.6 lb. concentrated 5 in. from its center.

7. A flywheel and shaft slow down from 200 to 180 rev./min. in 42 sec. under axle friction. Under a braking torque of 18 ft.-lb. it slows down from 200 to 180 rev./min. in 18 sec. Find the moment of inertia.

8. If in Fig. 159*e* the string passes over a pulley of radius r, radius of gyration k, and weight P, show that the acceleration and tensions are

$$a = \frac{W - \mu W'}{W + W' + Pk^2/r^2}g, \quad T = W\left(1 - \frac{a}{g}\right), \quad T' = W'\left(\mu + \frac{a}{g}\right),$$

where μ is the coefficient of sliding friction.

9. A 1-ton cylindrical drum, mounted on smooth horizontal bearings, has 160 ft. of cable weighing 2 lb./ft. wound upon it. The drum is 8 ft. in diameter and $k = 3.75$ ft. The free end of the cable carries a weight of $\frac{1}{2}$ ton hanging next to the drum. Find the angular velocity of the drum in rev./sec. when the weight has fallen 100 ft. from rest.

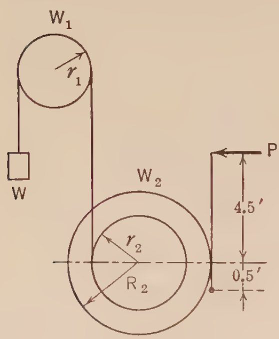


FIG. 192*f*.

10. In Example 5 (Fig. 192*e*) $W_1 = 100$, $W_2 = 75$, $P = 36$ lb.; $r = 6$ in., $k = 5$ in. By means of the equation of energy find the speed of W_1 after it has fallen 8 ft. from rest. Neglect axle friction.

Also solve this problem neglecting the inertia of the pulley.

11. In the hoist shown in Fig. 192*f* the car W, weighing 1000 lb., has a downward velocity of 8 ft./sec. Find the braking pressure P that will stop the car in 20 ft. if $\mu = \frac{1}{4}$ is the coefficient of brake friction and

$$W_1 = 360 \text{ lb.}, \quad k_1 = 15 \text{ in.}, \quad r_1 = 1.5 \text{ ft.};$$
$$W_2 = 720 \text{ lb.}, \quad k_2 = 22 \text{ in.}, \quad r_2 = 2 \text{ ft.}, \quad R_2 = 3 \text{ ft.}$$

Neglect axle friction and the weight of cable

193. Moment of Inertia of Solids of Revolution. The moment of inertia of a body about an axis has been defined as $\int p^2\,dm$ where p is the distance of the element of mass dm from the axis. The moment of inertia of a body is essentially positive. If a body consists of several parts, the moment of inertia of the whole about

any axis is equal to the sum of the moments of inertia of its parts.

If a body of mass m has the moment of inertia I about an axis, its radius of gyration k about this axis has been defined by the equation

$$I = mk^2.$$

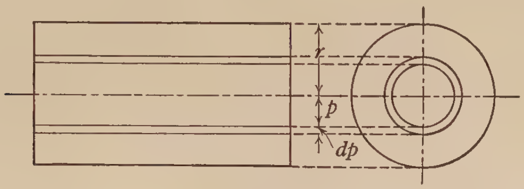


FIG. 193a.

Consider now a right circular cylinder of radius r, length l, and of uniform density δ. To compute its moment of inertia about the axis we take a cylindrical shell of radii p and $p + dp$ as element of mass; then $dm = \delta \cdot 2\,\pi pl\,dp$ and

$$I = \int p^2\,dm = \delta \cdot 2\,\pi l \int_0^r p^3\,dp = 2\,\delta\pi l \frac{r^4}{4} = m\frac{r^2}{2}$$

where $m = \delta\pi r^2 l$ is the mass of the cylinder. Hence $k^2 = \frac{1}{2}\,r^2$: *the square of the radius of gyration of a circular cylinder about its axis equals one half the square of its radius.*

This result may be used to compute the moment of inertia of any homogeneous solid of revolution about its axis. Thus in Fig. 193b the solid is divided into thin slices by planes perpendicular to its axis (x-axis). Regard the slices as circular cylinders of variable radius y; then the moment of inertia of a slice is,

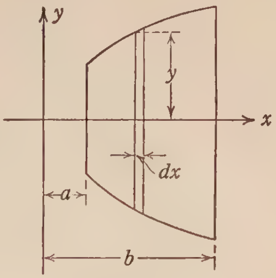


FIG. 193b.

$$\text{mass} \times (\text{rad. gyr.})^2 = \delta\pi y^2\,dx \cdot \tfrac{1}{2}\,y^2$$

and the moment of inertia of the solid is

$$I = \tfrac{1}{2}\,\delta\pi \int_a^b y^4\,dx.$$

When a meridian of the solid is given by an equation $y = f(x)$, this integral may be computed.

Example 1. *Hollow Circular Cylinder.* Let r_1, r_2 be the inner and outer radii of a hollow cylinder of length l. Its moment of inertia about the axis is equal to the difference of the moments of inertia of solid cylinders of radii r_1, r_2. Hence

$$\begin{aligned} I &= \delta\pi r_1^2 l\cdot\tfrac{1}{2}\,r_1^2 - \delta\pi r_2^2 l\cdot\tfrac{1}{2}\,r_2^2 = \tfrac{1}{2}\,\delta\pi l(r_1^4 - r_2^4),\\ m &= \delta\pi r_1^2 l - \delta\pi r_2^2 l = \delta\pi l(r_1^2 - r_2^2),\\ k^2 &= \frac{I}{m} = \frac{r_1^2 + r_2^2}{2}. \end{aligned}$$

The square of the radius of gyration of a hollow cylinder is equal to the mean of the squares of its radii.

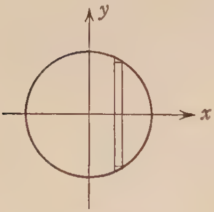


FIG. 193c.

Example 2. *Sphere.* For a sphere of radius r the meridian is a circle $x^2 + y^2 = r^2$ (origin at center); hence

$$\begin{aligned} I &= 2\int_0^r \delta\pi y^2\,dx\cdot\tfrac{1}{2}\,y^2 = \delta\pi\int_0^r (r^2 - x^2)^2\,dx\\ &= \delta\pi\int_0^r (r^4 - 2\,r^2x^2 + x^4)\,dx = \tfrac{8}{15}\,\delta\pi r^5;\\ k^2 &= \frac{I}{m} = \frac{\frac{8}{15}\,\delta\pi r^5}{\frac{4}{3}\,\delta\pi r^3} = \tfrac{2}{5}\,r^2. \end{aligned}$$

Example 3. C_1 and C_2 are solid cylinders of the same material; their lengths are the same and their radii r_1, r_2 (Fig. 193*d*). C_2, originally at rest, is lowered and held in contact with C_1 which is turning with the angular velocity ω_0. Find the angular velocities ω_1, ω_2 of the cylinders when they are rolling over each other without slip.

If F is the friction between the cylinders, their equations of motion are

$$I_1\alpha_1 = Fr_1, \quad I_2\alpha_2 = Fr_2.$$

Hence $$\frac{\alpha_1}{\alpha_2} = \frac{r_1\,I_2}{r_2\,I_1} = \frac{r_1\,r_2^4}{r_2\,r_1^4} = \frac{r_2^3}{r_1^3}.$$

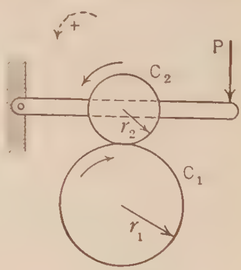


FIG. 193*d*.

If ω_1, ω_2 are the angular velocities of C_1, C_2 after t seconds,

$$\omega_1 = \omega_0 + \alpha_1 t, \quad \omega_2 = \alpha_2 t.$$

Hence

(i) $$\frac{\omega_1 - \omega_0}{\omega_2} = \frac{\alpha_1}{\alpha_2} = \frac{r_2^3}{r_1^3}.$$

When slipping ceases the peripheral speeds must be equal, that is

$$\omega_2 r_2 = -\omega_1 r_1 \quad \text{or} \quad \omega_2 = -\frac{r_1}{r_2}\omega_1.$$

If we multiply (i) by this equation we find

$$\omega_1 - \omega_0 = -\frac{r_2^2}{r_1^2}\omega_1 \quad \text{or} \quad \omega_0 = \left(1 + \frac{r_2^2}{r_1^2}\right)\omega_1.$$

For example if $r_1 = 2\,r_2$ and $\omega_0 = -100$ r.p.m. Then when slipping ceases

$$-100 = \tfrac{5}{4}\,\omega_1, \quad \omega_1 = -80 \text{ r.p.m.}, \quad \omega_2 = 160 \text{ r.p.m.}$$

PROBLEMS

1. Show that $k^2 = 3\,r^2/10$ for a right circular cone about its axis; r is the base radius.

2. An anchor ring is generated by revolving a circle of radius b about an axis at a distance a from the center. Prove that $k^2 = a^2 + \frac{3}{4}\,b^2$ about this axis.

3. A top has the form of a solid generated by revolving a sector of a circle about one of its sides, that is, a cone capped with a spherical segment. The cone has a height of 4 in., slant height 5 in. Show that $k^2 = 2.8$ in.^{2} about the axis of revolution.

4. A 250-lb. iron sphere, 1 ft. in diameter, is making 300 rev./min. about a fixed central axis. What force tangential to its equator will stop it in 4 sec.? Neglect axle friction.

5. A 320-lb. solid disk, 2 ft. in diameter, is mounted on an axle turning in smooth bearings. Find the torque which would generate 360 rev./min. in 30 sec. Neglect inertia of axle.

6. A hollow cylindrical iron drum, weighing 2400 lb., is 2 in. thick and 48 in. in diameter. When making 120 rev./min. a brake applies a normal pressure of 200 lb. to its surface. If $\mu = 0.2$, how many revolutions will it make before stopping? Neglect inertia of spokes and hub.

7. A weight of 96 lb. is hung from a solid homogeneous cylinder by a light cord wrapped around it. The cylinder weighs 384 lb., is 4 ft. in diameter, and revolves on a shaft 6 in. in diameter. Bearing friction is 48 lb. If the weight has an initial velocity of 10 ft./sec. downward, how far will it have fallen when its velocity is 30 ft./sec.?

194. Transfer Theorem. *The moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through the center of mass plus the product of its mass and the square of the distance between the axes.*

Proof. Let I and I^* denote the moments of inertia about an axis s and a parallel axis s^* through the center of mass P^*. Then $I = \int p^2\,dm$, $I^* = \int q^2\,dm$, where p and q are perpendiculars from P on s and s^* (Fig. 194a). Since

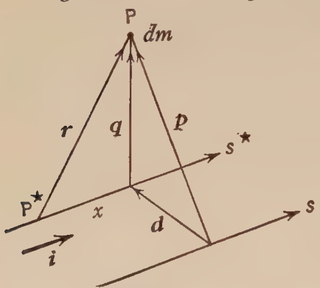


FIG. 194a.

$$p^2 = \mathbf{p}^2 = (\mathbf{d} + \mathbf{q})^2 = d^2 + 2\,\mathbf{d}\cdot\mathbf{q} + q^2,$$

$$I = md^2 + 2\,\mathbf{d}\cdot\int \mathbf{q}\,dm + I^*.$$

With P^* as origin of position vectors and s^* as x-axis, we have from (§ 181, 2)

$$\int \mathbf{r}\,dm = m\mathbf{r}^* = 0, \qquad \int x\,dm = mx^* = 0, \qquad \text{and}$$

$$\int \mathbf{q}\,dm = \int (\mathbf{r} - x\mathbf{i})\,dm = \int \mathbf{r}\,dm - \mathbf{i}\int x\,dm = 0.$$

Hence the above expression for I becomes

$$(1) \qquad I = I^* + md^2.$$

The theorem shows that the moment of inertia of a body about an axis through its center of mass is less than that about any parallel axis. If we introduce the radii of gyration, $I = mk^2$, $I^* = mk^{*2}$, and (1) becomes

$$(2) \qquad k^2 = k^{*2} + d^2.$$

Example 1. The radius of gyration of a sphere of radius r about an axis at a distance d from its center, is

$$k^2 = k^{*2} + d^2 = \tfrac{2}{5}r^2 + d^2 \qquad (\S\ 193,\ \text{Ex. } 2).$$

Example 2. The radius of gyration of a right circular cylinder about a generator is

$$k^2 = \tfrac{1}{2}r^2 + r^2 = \tfrac{3}{2}r^2.$$

Example 3. For a thin uniform rod of length l, $k^2 = \frac{1}{3}l^2$ about a normal axis at one end (§ 192, Ex. 3). For a parallel axis through the center we have

$$k^{*2} = k^2 - (\tfrac{1}{2}l)^2 = \tfrac{1}{3}l^2 - \tfrac{1}{4}l^2 = \tfrac{1}{12}l^2.$$

Knowing k^*, we may find k for any normal axis at a distance d from the center by the Transfer Theorem:

$$(\text{i}) \qquad k^2 = \tfrac{1}{12}l^2 + d^2.$$

Example 4. *Moment of Inertia of a Flywheel.* Let W_1, W_2, W be the respective weights of the rim, hub and spokes (all) for the flywheel shown in section in Fig. 194*b*. Since the rim and hub are hollow cylinders, their radii of gyration are given by § 193, Example 1. For the spokes, regarded as thin rods, k^2 is given by (i) above. Since moments of inertia are additive, we have for the entire wheel

$$I = \frac{W_1}{g}\frac{r_1^2 + R_1^2}{2} + \frac{W_2}{g}\frac{r_2^2 + R_2^2}{2} + \frac{W}{g}\left(\tfrac{1}{12}\, l^2 + d^2\right).$$

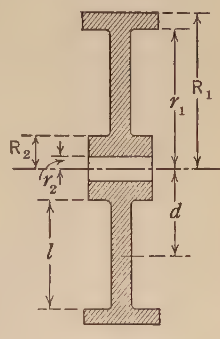


FIG. 194*b*.

PROBLEMS

1. A steel disk, 2 ft. in diameter and weighing 200 lb., has six 2-in. holes whose centers are 9 in. from the axis and a central 4-in. hole. Find its moment of inertia about the axis.

2. Two spheres of radius r, connected by a horizontal rod of length l, revolve about a vertical axis through the rod's center. Each sphere weighs W lb., the rod w lb. Find their combined moment of inertia.

3. A sphere of radius r, at the end of a rod of length l, revolves about a vertical axis through the end of the rod. The weights of sphere and rod are W and w respectively. If the rod makes an angle β with the axis, find their combined moment of inertia.

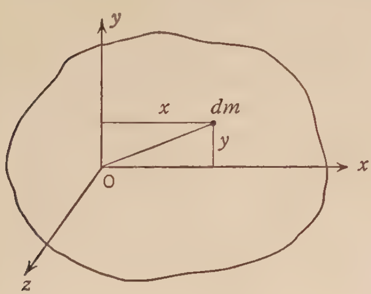


FIG. 195*a*.

195. Moment of Inertia of Thin Flat Plates. We shall treat the mass of a thin, flat plate as if it were concentrated in a plane and had a uniform surface density σ (mass per unit area). Let x and y be perpendicular axes in this plane and z an axis perpendicular to both through their point of intersection O (Fig. 195*a*). Denoting the moments of inertia about these axes by I_x, I_y, I_z,

$$I_x = \int y^2\, dm, \quad I_y = \int x^2\, dm, \quad I_z = \int (x^2 + y^2)\, dm.$$

Hence we have the important relation

$$I_x^2 + I_y^2 = I_z^2, \tag{1}$$

or on introducing radii of gyration

(2) $$k_x^2 + k_y^2 = k_z^2.$$

Example 1. *Circle about a Diameter.* If r is the radius we have $k_z^2 = \frac{1}{2} r^2$ (§ 193). By symmetry $k_x = k_y$; hence

$$2\,k_x^2 = k_z^2 = \tfrac{1}{2} r^2, \quad k_x^2 = \tfrac{1}{4} r^2.$$

Example 2. *Rectangle about a Base.* For a rectangle of base b and height h (Fig. 195b) we have $dm = \sigma b\,dy$ for the shaded strip, and

$$I = \int_0^h y^2 \cdot \sigma b\,dy = \tfrac{1}{3}\sigma b h^3 = m\frac{h^2}{3}; \quad k^2 = \tfrac{1}{3} h^2.$$

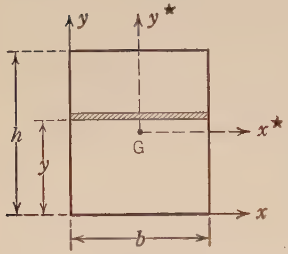


FIG. 195b.

For a parallel axis x^* through its center G the Transfer Theorem gives

$$k_x^2 = k^2_{x^*} + (\tfrac{1}{2} h)^2, \quad k^2_{x^*} = \tfrac{1}{12} h^2.$$

Similarly for an axis y^* through G, $k^2_{y^*} = \frac{1}{12} b^2$.

Example 3. *Rectangle about its Center.* For an axis z^* through G (perpendicular to the plane)

$$k^2_{z^*} = k^2_{x^*} + k^2_{y^*} = \tfrac{1}{12}(h^2 + b^2).$$

Example 4. *Triangle about a Base.* For a triangle of base b and height h (Fig. 195c) we have $dm = \sigma l\,dy$ for the shaded strip of length l. From similar triangles

$$\frac{l}{b} = \frac{h - y}{h}, \quad l = \frac{b}{h}(h - y).$$

Hence

$$I_x = \int_0^h y^2 \cdot \sigma l\,dy = \sigma\frac{b}{h}\int_0^h y^2(h - y)\,dy = \tfrac{1}{12} b h^3 \sigma = m\frac{h^2}{6}$$

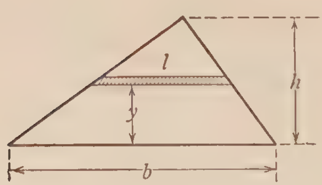


FIG. 195c.

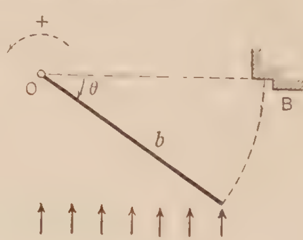


FIG. 195d.

Example 5. An open door on vertical hinges is struck by a gust of wind which exerts a pressure of p lb./ft.^{2} (Fig. 195d). If the door is b ft. wide, h ft. high and is standing nearly at right angles to OB, find the speed with which it will strike B.

For a door weighing w lb./ft.² the moment of inertia about the hinge line is

$$I = \frac{wbh}{g}\frac{b^2}{3} = \frac{wb^3h}{3g}.$$

When the door makes an angle θ with OB, the resultant wind pressure $phb \cos\theta$ acts at a distance $\frac{1}{2} b \cos\theta$ from the hinge; hence its moment about the hinge-line is

$$\tfrac{1}{2} b \cos\theta \cdot phb \cos\theta = \tfrac{1}{2} pb^2h \cos^2\theta.$$

The equation in motion is therefore

$$\frac{wb^3h}{3g}\alpha = \tfrac{1}{2} pb^2h \cos^2\theta \quad \text{or} \quad \alpha = \frac{3}{2}\frac{pg}{wb}\cos^2\theta.$$

Put $\alpha = \omega\, d\omega/d\theta$ in this equation and integrate from $\theta = -\frac{1}{2}\pi$ to 0; then

$$\int_0^\omega 2\,\omega\, d\omega = 3\frac{pg}{wb}\int_{-\frac{1}{2}\pi}^0 \cos^2\theta\, d\theta,$$

$$\omega^2 = 3\frac{pg}{wb}\Big[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\Big]_{-\frac{1}{2}\pi}^0 = \frac{3\pi}{4}\frac{pg}{wb}.$$

We may take $p = 0.0032\, v^2$ where v is speed of the wind in miles per hour.* If $v = 5$ mi./hr., a door 3 ft. wide and weighing 5 lb./ft. will strike B with the speed of

$$\omega b = \sqrt{\frac{3\pi}{4}\frac{pgb}{w}} = \sqrt{\frac{3\pi \times 0.0032 \times 25 \times 32 \times 3}{4 \times 5}} = 1.9 \text{ ft./sec.}$$

PROBLEMS

1. A trapezoid of height h has parallel bases of length b_1, b_2. Show that

$$k^2 = \tfrac{1}{6} h^2 \frac{b_1 + 3b_2}{b_1 + b_2} \text{ about the base } b_1.$$

Consider the cases $b_1 = 0$, $b_2 = 0$, $b_1 = b_2$.

2. The vertices of a regular polygon of n sides are at a distance r from the center O. Show that

$$k^2 = \tfrac{1}{6}\left(2 + \cos\frac{2\pi}{n}\right) r^2$$

about a normal axis through O. What does k^2 approach as $n \to \infty$?

3. Find k_x^2, k_y^2, k_z^2 for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

4. An I-section is composed of two rectangular flanges 10 in. × 1 in. (base) and 8 in. × 1 in. (top) connected by a rectangular web 8 in. × 1 in. Find its moment of inertia about an axis parallel to the base and 3 in. above it.

* Smithsonian Physical Tables (1920), p. 151.

5. An I-section is composed of two rectangular flanges 10 in. $\times$ 2 in. (base) and 8 in. $\times$ 1 in. (top) connected by a rectangular web 10 in. $\times$ 1 in. Find its moment of inertia about a gravity axis parallel to the base.

6. Find k^2 for a rectangular prism of dimensions $a \times b \times c$ about a central axis parallel to the edges c.

7. Find k^2 for a right pyramid with a square base of side a about the axis.

8. A thin rectangular vane b ft. wide and h ft. high, weighing w lb./ft.², revolves about a vertical axis through its center with an initial angular speed of ω_0 rad./sec. The resisting air pressure on any element of the vane is $p = cv^2$ lb./ft.², where $c = 0.0015$ when v is expressed in ft./sec. As the vane slows down under air resistance, find (*a*) how ω depends on the time t, and (*b*) how ω depends on the angle θ through which the vane has turned.

If $b = 4$ ft., $h = 6$ ft., $w = 2$ lb./ft.², in what time will ω be reduced from 30 to 15 rev./min.? How many revolutions will the vane make in this period?

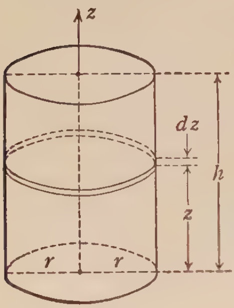


FIG. 196.

196. Application of Transfer Theorem. The use of the Transfer Theorem in computing moments of inertia is illustrated in the following

Example. Moment of Inertia of a Right Circular Cylinder about a Diameter of its Base. Divide the cylinder into thin slices by planes parallel to the base. The radius of gyration of each slice about its own diameter is $\frac{1}{4} r^2$ (§ 195, Ex. 1) and about the parallel diameter of the base $\frac{1}{4} r^2 + z^2$. The moment of inertia of the slice about the base diameter is therefore $\delta\pi r^2\, dz \cdot (\frac{1}{4} r^2 + z^2)$, and hence for the entire cylinder

$$I_x = \delta\pi r^2 \int_0^h (\tfrac{1}{4} r^2 + z^2)dz = \delta\pi r^2(\tfrac{1}{4} r^2 h + \tfrac{1}{3} h^3),$$

$$k_x^2 = \frac{I_x}{\delta\pi r^2 h} = \tfrac{1}{4} r^2 + \tfrac{1}{3} h^2$$

PROBLEM

1. Find k^2 for a right circular cone of height h and base radius r about a diameter of the base.

197. Physical Pendulum. A rigid body free to turn about a horizontal axis and performing oscillations under the influence of

gravity is called a *physical pendulum*. Fig. 197 represents a section of the body by a plane normal to the axis at O and passing through the center of gravity G. The point O is called the *center of suspension*. Let k be the radius of gyration of the pendulum about the axis and $b = GO$. If we neglect friction at the axis, the weight W is the only external force having a moment about O; the equation of motion is therefore

$$\frac{W}{g} k^2 \alpha = - Wb \sin \theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} = - \frac{gb}{k^2} \sin \theta.$$

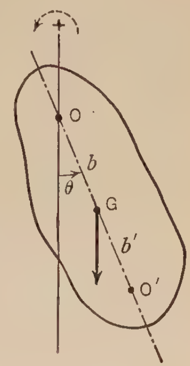


FIG. 197.

This is the same as the equation (§ 171, 3) for a simple pendulum of length $l = k^2/b$. For this reason l is called the *reduced length* of the pendulum. For small vibrations the period of the pendulum is very nearly

$$(1) \qquad T = 2\pi\sqrt{\frac{l}{g}} \quad \text{where} \quad l = \frac{k^2}{b},$$

and is independent of the amplitude.

Let k^* be the radius of gyration about a parallel axis through G; then from the Transfer Theorem $k^2 = k^{*2} + b^2$ and hence

$$(2) \qquad l = \frac{k^{*2}}{b} + b.$$

Thus for a given body the period is the same for all parallel axes at the same distance from the center of gravity.

The point O' on the line OG at a distance l from O is called the *center of oscillation*. Since $b = GO$, $l - b = GO'$ we have from (2)

$$(3) \qquad GO \cdot GO' = k^{*2}.$$

Suppose, now, that the pendulum is suspended from an axis through O' parallel to the first; then if O'' is the new center of oscillation

$$GO' \cdot GO'' = k^{*2} \quad \text{and hence} \quad O'' \equiv O.$$

Thus if O' becomes the center of suspension, O becomes the center of oscillation; more briefly, *the centers of suspension and oscillation are interchangeable.* This theorem is due to Huygens.

If a physical pendulum has the same period about two parallel axes of suspension lying in a plane through G, the axes being on

opposite sides of G and at *different* distances from it, the reduced length of the pendulum is precisely the distance between the axes. For if the distances are b, b',

$$l = \frac{k^{*2}}{b} + b, \qquad l = \frac{k^{*2}}{b'} + b'$$

since the periods are equal. Hence, on eliminating k^*,

$$bl - b^2 = b'l - b'^2 \quad \text{or} \quad (b - b')l = b^2 - b'^2;$$

and on dividing this equation by $b - b'$ ($\neq 0$) we have $l = b + b'$. This is the principle of the *reversible pendulum of Kater*, which has two-knife edges facing one another, one being adjustable so that the periods can be equalized by trial.

When the reduced length of a physical pendulum is accurately known, its period in any locality will determine the local value of g ($g = 4\,\pi^2 l/T^2$).

The reduced length l of a physical pendulum may be obtained by observing its period T; for from (1) $l = gT^2/4\,\pi^2$. If we now determine the distance b of the center of gravity from the axis by balancing the body on a knife edge parallel to the axis, we may compute

$$k^2 = bl = \frac{bgT^2}{4\,\pi^2} \quad \text{and} \quad I = \frac{W}{g}k^2.$$

This gives an experimental method of finding the moment of inertia of a body which can be swung as a pendulum. Thus if a connecting-rod is swung from a knife-edge passing through the hole for the crank-pin or wrist-pin, its moment of inertia about this axis may be found as above. The moment of inertia about any parallel axis may then be computed from the Transfer Theorem.

PROBLEMS

1. Show that, for all axes parallel to a fixed direction, the period of a physical pendulum is least when $b = k^*$.

What is the least period for a thin uniform rod 3 ft. long? Compare with its period when hung from one end.

2. A pendulum consists of a sphere of radius r and mass m attached to a thin wire of length l' and mass m'. Prove that its reduced length is

$$l = \frac{m[\frac{2}{5}\,r^2 + (l' + r)^2] + \frac{1}{3}\,m'l'^2}{m(l' + r) + \frac{1}{2}\,m'l'}.$$

3. A pendulum consists of a light rod with two heavy weights w_1, w_2 at the same distance d from a central knife-edge. If the weights are regarded as particles and $w_1 > w_2$, show that the reduced length is

$$l = \frac{w_1 + w_2}{w_1 - w_2} d.$$

Find the period if $w_2 = 2$ lb., $w_1 = 2.1$ lb., $d = 1$ ft.

198. Kinetics of Rotation. Let O be a fixed point on the axis of rotation Oz and $\mathbf{H}_O$ the moment of momentum of the revolving body about O. Then, according to § 183, the inertia forces of the body may be reduced to a

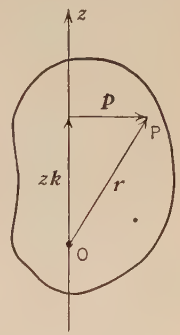


FIG. 198a.

$$\text{Force } m\mathbf{a}^* \text{ at } O \quad \text{and} \quad \text{Couple of moment } \frac{d\mathbf{H}_O}{dt}.$$

We proceed to compute $\mathbf{H}_O \equiv \int \mathbf{r}\times\mathbf{v}\, dm$. Let $\boldsymbol{\omega} = \omega\mathbf{k}$ be the angular velocity at a certain instant. Then if $\mathbf{p}$ is a normal vector from the axis to the particle P (Fig. 198a), its position and velocity are

$$\mathbf{r} = z\mathbf{k} + \mathbf{p}, \quad \mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} = \boldsymbol{\omega}\times\mathbf{p} \qquad (\S\ 109)$$

and

$$\begin{aligned} \mathbf{r}\times\mathbf{v} &= (z\mathbf{k} + \mathbf{p})\times(\boldsymbol{\omega}\times\mathbf{p}) \\ &= (z\mathbf{k} + \mathbf{p})\cdot\mathbf{p}\,\boldsymbol{\omega} - (z\mathbf{k} + \mathbf{p})\cdot\boldsymbol{\omega}\,\mathbf{p} \qquad (\S\ 19) \\ &= p^2\boldsymbol{\omega} - z\omega\mathbf{p} \end{aligned}$$

since $\mathbf{p}\cdot\mathbf{k} = 0$. Hence

$$\mathbf{H}_O = \int \mathbf{r}\times\mathbf{v}\, dm = \boldsymbol{\omega}\int p^2\, dm - \omega\int \mathbf{p}z\, dm,$$

or since $\int p^2\, dm = I$, the moment of inertia of the body about the axis,

$$(1) \qquad \mathbf{H}_O = I\boldsymbol{\omega} - \omega\int \mathbf{p}z\, dm.$$

If the integral $\int \mathbf{p}z\, dm = 0$, the axis is said to be a *principal axis of inertia at O*. Then

$$(2) \qquad \mathbf{H}_O = I\boldsymbol{\omega}, \quad \frac{d\mathbf{H}_O}{dt} = I\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha}$ is the vector angular acceleration $\alpha\mathbf{k}$. We thus have the

THEOREM. *If the axis of revolution is a principal axis of inertia at O, the inertia forces may be reduced to a*

$$\textit{Force } m\mathbf{a}^* \textit{ at } O \quad \textit{and} \quad \textit{Couple of moment } I\boldsymbol{\alpha}.$$

In particular, the inertia forces reduce to

(*a*) *the couple $I\boldsymbol{\alpha}$ when P^* lies on the axis* ($\mathbf{a}^* = 0$),
(*b*) *the force $m\mathbf{a}^*$ at O when ω is constant* ($\alpha = 0$),
(*c*) *zero when P^* lies on the axis and ω is constant.*

Let the plane through P^* normal to the axis cut the axis at O. Then the axis is a principal axis of inertia at O in two cases of great practical importance.

CASE 1. *The mass distribution is symmetric with respect to the plane through P^* normal to the axis of rotation.* Then for each symmetric pair of particles on opposite sides of the plane (Fig. 198*b*).

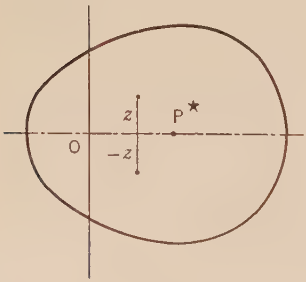


FIG. 198*b*.

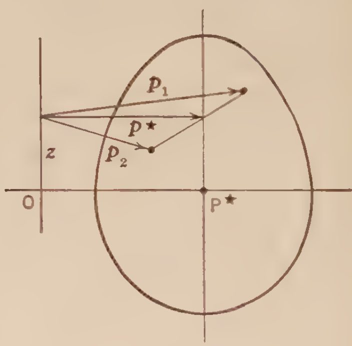


FIG. 198*c*.

$$\mathbf{p}z\,dm + \mathbf{p}(-z)\,dm = 0 \quad \text{and hence} \quad \int \mathbf{p}z\,dm = 0.$$

CASE 2. *The mass distribution is symmetric with respect to a line parallel to the axis of rotation.* Then P^* must lie on the line of symmetry (§ 77). Let $\overrightarrow{OP^*} = \mathbf{p}^*$. Then for each symmetric pair of particles (Fig. 198*c*) we have

$$\mathbf{p}_1 z\,dm + \mathbf{p}_2 z\,dm = 2\,\mathbf{p}^* z\,dm.$$

Hence we may compute $\int \mathbf{p}z\,dm$ as if each particle contributed $\mathbf{p}^*z\,dm$:

$$\int \mathbf{p}z\,dm = \mathbf{p}^* \int z\,dm = \mathbf{p}^*\,mz^* = 0.$$

In Cases 1 *and* 2 *the axis is a principal axis of inertia at the point where the normal plane through the center of mass cuts it.*

Finally consider

CASE 3. *The mass distribution is symmetric with respect to the axis of rotation.* Choose O at pleasure on the axis. Then for each symmetric pair of particles (Fig. 198*d*)

$$\mathbf{p}z\,dm + (-\mathbf{p})z\,dm = 0 \quad \text{and hence} \quad \int \mathbf{p}z\,dm = 0.$$

If the axis of rotation is a line of symmetry, it is a principal axis at ALL *of its points.*

By D'Alembert's Principle the inertia forces are equivalent to the external forces. Hence *if the axis is a principal axis of inertia at O*, and $\mathbf{F}$ and $\mathbf{M}_O$ denote the force-sum and moment-sum about O of all the external forces acting on the body, we have the dynamical equations:

$$(3),\ (4) \qquad m\mathbf{a}^* = \mathbf{F}, \quad I\boldsymbol{\alpha} = \mathbf{M}_O.$$

On equating the z-components of (4) we again obtain the equation of motion of § 191, *valid for any axis, whether principal or not*:

$$(5) \qquad I\alpha = M_z.$$

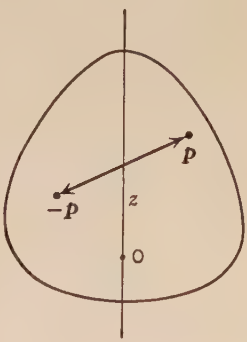


FIG. 198*d*.

In Cases 1 and 2, problems in the kinetics of rotation should be solved as follows:

Draw a free body diagram showing all the external forces, the inertia force $m\mathbf{a}^*$ *acting at O (where the axis is a principal axis) and the inertia couple* $I\boldsymbol{\alpha}$. *Then express the equivalence of the external forces to the inertia force and couple by taking components in any direction or moments about any axis so that the resulting equations are as simple as possible.*

Example. Bearing Reactions. Consider a homogeneous body mounted eccentrically on a vertical shaft and free to turn in *frictionless* bearings at A and B (Fig. 198*e*). If the body has the symmetry of Case 1 or Case 2, the axis will be a principal axis of inertia at the point O. Suppose now that the body is set in rotation and left to the action of its own weight and the bearing reactions. Since these forces have no moment about the axis of rotation, $M_z = 0$ and hence $\alpha = 0$ from (5). The angular velocity ω thus remains constant and the inertia forces reduce to a single "force" $-m\omega^2\mathbf{p}^*$ at O. (Of course in any actual case the bearing friction, however small, exerts a retarding moment about the axis.)

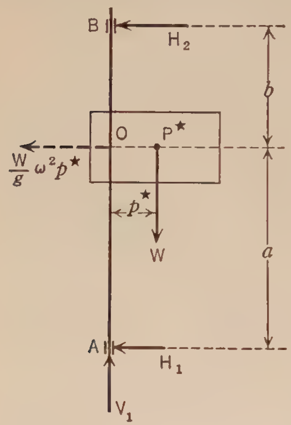


Fig. 198*e*.

By taking moments about B and A in turn, we find

$$H_1(a + b) + Wp^* = \frac{W}{g}\omega^2 p^* b, \quad H_2(a + b) - Wp^* = \frac{W}{g}\omega^2 p^* a.$$

These equations give H_1 and H_2; these horizontal reactions revolve with the body. Finally on taking vertical components, $V_1 - W = 0$.

PROBLEMS

1. A uniform bar of length l is pivoted about a horizontal axis at one end. If it falls from rest when horizontal, show that

$$\alpha = 3\,g \cos\theta/2\,l, \quad \omega^2 = 3\,g \sin\theta/l$$

after falling through an angle θ. If the bar weighs 8 lb., find the reaction $[H, V]$ at the pivot when $\theta = 45°$.

2. If the bar in Problem 1 weighs W lb., show that the radial and transverse components of the reaction are $\frac{5}{2}\,W \sin\theta$, $\frac{1}{4}\,W \cos\theta$ respectively.

3. A thin uniform rod 6 ft. long and weighing 30 lb. hangs at rest from a horizontal axis at one end. If a horizontal force of 20 lb. is applied at its mid-point, find the horizontal reaction at the axis. Where must the force be applied so as to produce no horizontal reaction?

4. A thin rod 6 ft. long is held perpendicular to the edge of a rough

table ($\mu = \frac{1}{4}$) so that 4 ft. projects beyond. If it is released, find the angle through which it turns before slipping.

5. A uniform rod of length a and weight W is supported by two equal wires of length l at its ends, the upper ends of the wires being attached to a point A on a vertical spindle. If the spindle is revolving uniformly at ω rad/sec., find the tension T in the wires and the angle θ between the axis and the line from A to the middle of the rod (Fig. 198f). At what value of ω will the rod begin to rise?

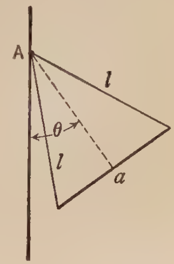


FIG. 198f.

6. A thin, uniform rod of mass m and length $2\,l$ revolves about an axis passing through its center O and making an angle θ with the rod. Show that the axis is *not* a principal axis of inertia at O.

7. When the angular speed ω is constant in Prob. 6, show that the inertia forces of the rod reduce to a couple whose moment is numerically equal to $\frac{1}{3}\,m\,\omega^2\,l \sin\theta\cos\theta$.

199. Center of Percussion. Consider now a body revolving about an axis not passing through the center of mass P^* and having an angular acceleration $\alpha \neq 0$. Then in Cases 1 and 2 of § 198 the inertia forces are equivalent to a

$$\text{Force } m\mathbf{a}^* \text{ at } O \quad \text{and} \quad \text{Couple of moment } I\boldsymbol{\alpha}.$$

The plane through P^* normal to the axis cuts the axis at O. Since $\mathbf{a}^*$ lies in this plane and $\boldsymbol{\alpha}$ is normal to it, the vectors $m\mathbf{a}^*$ and $I\boldsymbol{\alpha}$ are perpendicular; hence the inertia forces may be reduced to a single force $\mathbf{R} = m\mathbf{a}^*$ (§ 71, Theorem 2).

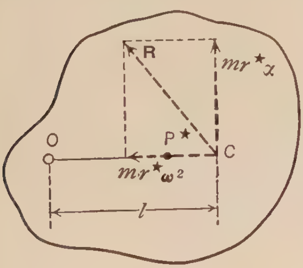


FIG. 199a.

To locate $\mathbf{R}$ we form a couple of moment $I\boldsymbol{\alpha}$ by adding forces $-m\mathbf{a}^*$ at O and $m\mathbf{a}^*$ at a point C on the line OP^*. The inertia forces are thus reduced to the force $\mathbf{R} = m\mathbf{a}^*$ at C (Fig. 199a). To find the distance $l = OC$ we equate the moment of $\mathbf{R}$ about the axis to $I\alpha$. Replacing $\mathbf{R}$ by its radial and transverse projections, of magnitude $mr^*\omega^2$ and $mr^*\alpha$ respectively, we thus find

$$l{\cdot}mr^*\alpha = mk^2\alpha, \quad l = \frac{k^2}{r^*}$$

where k is the radius of gyration about the axis. Thus C coincides with the center of oscillation of the body regarded as a physical pendulum (§ 197, 1). We state this result in the

THEOREM. *If the center of mass of the rotating body does not lie on the axis in Cases* 1 *or* 2 *and* $\alpha \neq 0$, *the inertia forces have a resultant* $m\mathbf{a}^*$ *which cuts the line* OP^* *produced at a distance* k^2/r^* *from* O. *If the body is regarded as a physical pendulum, this point is its center of oscillation.*

The point C is often called the *center of percussion.* To show the meaning of this term, consider a physical pendulum of weight W hanging in equilibrium from a horizontal axis O (Fig. 199*b*); the bearing will then exert an upward reaction $\mathbf{V} = -\mathbf{W}$. If the pendulum has a plane of symmetry normal to the axis and is given a blow represented by a horizontal force $\mathbf{F}$ in this plane, the bearing will in general exert a horizontal reaction $\mathbf{H}$. By D'Alembert's Principle the external forces $\mathbf{F}$ and $\mathbf{H}$ must have the same resultant as the inertia forces brought into play by the angular acceleration α of the pendulum ($\omega = 0$ at instant of striking). We have seen that the inertia forces have a resultant $\mathbf{R} = m\alpha\times\mathbf{r}^*$ acting through C; hence $\mathbf{F}$ and $\mathbf{H}$ must be equivalent to $\mathbf{R}$. If the line of $\mathbf{F}$ passes below C, $\mathbf{H}$ will have the same direction as $\mathbf{F}$: if $\mathbf{F}$ passes above C, $\mathbf{H}$ will be opposed to $\mathbf{F}$; but if $\mathbf{F}$ passes through C, $\mathbf{H} = 0$ and $\mathbf{F} \equiv \mathbf{R}$. *The horizontal reaction at the axis vanishes only when the line of the blow passes through the center of percussion.*

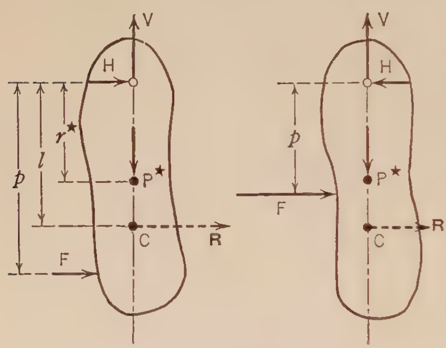


FIG. 199*b*.

In any case H may be found by taking moments about C: $F(p - l) - Hl = 0$.

Example. Bearing Reactions on a Physical Pendulum. Let Fig. 199*c* represent a physical pendulum which is symmetric with respect to a plane through P^* normal to the axis (the plane of the paper). Its inertia forces have a resultant $\mathbf{R} = m\mathbf{a}^*$ passing through its center of oscillation C. Resolve $\mathbf{R}$ into its tangential and normal projections

at C, and let **P**, **Q** be the corresponding projections of the bearing reaction.

To find P equate the moments about C of the external and inertia forces:

$$-Pl + W(l - b)\sin\theta = 0,$$
$$P = W \sin\theta(1 - b^2/k^2).$$

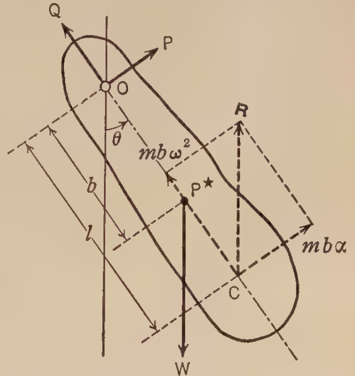


FIG. 199c.

To find Q equate components along OC:

$$Q - W\cos\theta = \frac{W}{g}b\omega^2,$$
$$Q = W(\cos\theta - b\omega^2/g).$$

Knowing the angle where the pendulum comes to rest, we may compute ω^2 from the equation of energy.

PROBLEMS

1. Find the center of percussion of a thin uniform rod of length h about an axis at one end.

2. A boy strikes a ball with a cylindrical bat 3 ft. long. Assuming that the bat turns about an axis 6 in. from the end, where must the ball be struck so as to produce no normal reaction on the hands?

200. Torsion Pendulum. Consider a body suspended by a vertical wire, clamped at the upper end. We assume that the body is attached so that the axis of the wire passes through the center of mass and is a principal axis of inertia at this point. If the body is displaced through an angle θ, the twisted wire will exert a restoring torque about its axis proportional to θ, say $-C\theta$. The negative sign indicates that the torque and the angle are opposite in sense. Since the inertia forces of the body reduce to the couple $I\alpha$ (§ 198).

$$I\frac{d^2\theta}{dt^2} = -C\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{C}{I}\theta.$$

Since the axis of the wire is a principal axis the reaction at the clamped end simply balances the weight of the body and does not tend to throw the wire out of plumb. The body will therefore rotate about the wire as a rigid vertical axis.

The equation above is the differential equation of a simple harmonic motion (§ 119, 1) with x replaced by θ and $n^2 = C/I$.

The complete torsional oscillations therefore have the period

$$(1) \qquad T = \frac{2\pi}{n} = 2\pi\sqrt{\frac{I}{C}}$$

which, moreover, is independent of the amplitude.

This result gives a second experimental method of finding moments of inertia. For this purpose we may use a torsion pendulum formed by two equal disks rigidly connected by vertical straps, the upper disk being attached to the wire at its center (Fig. 200). To determine I for the pendulum place a circular cylinder of known moment of inertia I_0 on the lower disk so that its axis coincides with the axis of the wire, and observe the period T_0 of the loaded pendulum. Since the moment of inertia is now $I + I_0$, we have from (1)

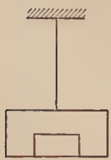


FIG. 200.

$$T^2C = 4\pi^2 I, \qquad T_0^2 C = 4\pi^2(I + I_0);$$

$$\frac{T^2}{T_0^2} = \frac{I}{I + I_0} \quad \text{and} \quad I = I_0\frac{T^2}{T_0^2 - T^2}.$$

We may now find the moment of inertia I_1 of a body having a plane of symmetry perpendicular to the axis of rotation. Place the body on the lower disk so that the wire is normal to this plane at the mass center; the wire is then a principal axis of inertia of the loaded pendulum (§ 198, Case 1). If the period is now T_1 we have as before

$$I = I_1\frac{T^2}{T_1^2 - T^2}; \quad \text{hence} \quad I_1 = I\frac{T_1^2 - T^2}{T^2}.$$

201. Uniform Rotation. We turn next to the important case of a body revolving about a fixed axis with constant angular velocity ω. If O is a fixed point on the axis, and $\mathbf{p}$ denotes the normal vector from the axis to a particle P of mass dm, the inertia forces $-\omega^2\mathbf{p}\,dm$ reduce to the

$$(1) \qquad \text{Force:} \quad -\int \omega^2\mathbf{p}\,dm = -m\omega^2\mathbf{p}^* \quad \text{at} \quad O, \quad \text{and}$$

$$(2) \qquad \text{Couple:} \quad -\int (\mathbf{k}z + \mathbf{p})\times\omega^2\mathbf{p}\,dm = -\omega^2\mathbf{k}\times\int \mathbf{p}z\,dm.$$

This couple equals $d\mathbf{H}_O/dt$ (§ 198) and may also be computed from (§ 198, 1).

Both force and couple are zero when, and only when, $\mathbf{p}^* = 0$ and $\int \mathbf{p}z\, dm = 0$. Then no bearing reactions are called into play by the motion; the bearing reactions then merely serve to balance the impressed forces, such as weight, on the rotating body. We therefore have the

THEOREM. *If a body rotates about a fixed axis with constant angular velocity, the bearing reactions due to its motion will vanish when, and only when,*

(*a*) *its center of mass lies on the axis, and*
(*b*) *the axis is a principal axis of inertia at one of its points.*

Suppose now that the axis does not pass through P^*, the center of mass: $\mathbf{p}^* \neq 0$. Then if $\int \mathbf{p}z\, dm = 0$ for a point O, the axis is a principal axis at O and the inertia forces reduce to a single "force" $-m\omega^2\mathbf{p}^*$ at O. If the direction of this resultant inertia force is reversed we obtain the so-called *centrifugal force* which at any instant must just balance the external forces.

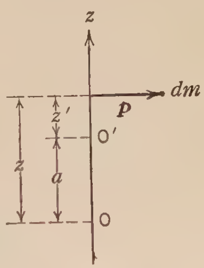


FIG. 201*a*.

If however $\int \mathbf{p}z\, dm \neq 0$, O is not a principal axis and the inertia forces reduce to the force *and couple* given by (1) and (2). We then inquire whether the axis is a principal axis of inertia at some other point O'. This will be the case if $\int \mathbf{p}z'\, dm = 0$ where z' is measured from O'. If $OO' = a$, $z' = z - a$ (Fig. 201*a*); then

$$(3) \qquad \int \mathbf{p}z'\, dm = \int \mathbf{p}z\, dm - a\int \mathbf{p}\, dm = \int \mathbf{p}z\, dm - am\mathbf{p}^*.$$

Hence if $\int \mathbf{p}z\, dm$ and $\mathbf{p}^*$ are parallel vectors, there is just one value of a that will make the right-hand member zero. Since $a = OO'$, this locates a point O' at which the axis is a principal axis of inertia, that is, $\int \mathbf{p}z'\, dm = 0$. But if $\int \mathbf{p}z\, dm$ and $\mathbf{p}^*$ are not parallel, the right-hand member of (3) will not vanish for any value of a. Therefore *an axis, not passing through the center*

of mass, will either be a principal axis of inertia at one point only or at none of its points.

If the axis passes through P^* (3) becomes

$$\int \mathbf{p}z'\,dm = \int \mathbf{p}z\,dm \quad \text{since} \quad \mathbf{p}^* = 0.$$

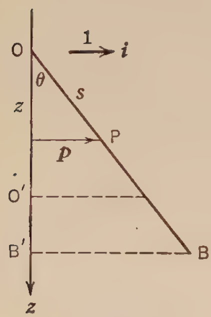


FIG. 201*b*.

Hence if $\int \mathbf{p}z\,dm = 0$, then $\int \mathbf{p}z'\,dm = 0$ for any choice of O'. Therefore *if an axis through the center of mass is a principal axis at one point, it is a principal axis at all of its points.*

Example 1. Consider a thin, uniform rod OB of length l revolving about the axis Oz (Fig. 201*b*). Then if $OP = s$,

$$\mathbf{p} = s \sin\theta\,\mathbf{i}, \quad z = s\cos\theta, \quad dm = \frac{m}{l}\,ds, \qquad \text{and}$$

$$\int \mathbf{p}z\,dm = \mathbf{i}\frac{m}{l}\sin\theta\cos\theta \int_0^l s^2\,ds = \tfrac{1}{3}\,ml^2\sin\theta\cos\theta\,\mathbf{i}.$$

Since $m\mathbf{p}^* = \frac{1}{2}\,ml\sin\theta\,\mathbf{i}$, we may make $\int \mathbf{p}z'\,dm = 0$ by choosing $a = \frac{2}{3}\,l\cos\theta$ in (3). Thus if $OO' = \frac{2}{3}\,OB'$, O' will be the only point on the axis at which it is a principal axis of inertia. The inertia forces of the rod therefore reduce to the single "force."

$$-m\mathbf{p}^*\omega^2 = -\tfrac{1}{2}\,ml\omega^2\sin\theta\,\mathbf{i} \quad \text{at } O'.$$

In any case we can always apply the basic Theorems I and II of § 179 in the solution of problems, choosing any convenient point as center of moments.

Example 2. A thin rod AB, hinged at A, revolves about a vertical axis z with constant angular velocity ω (Fig. 201*c*). Find the angle θ that it makes with the axis and the reaction at A.

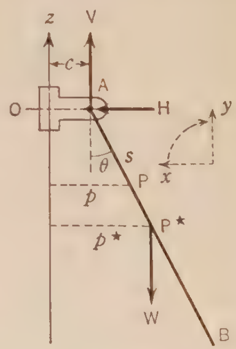


FIG. 201*c*.

The inertia forces $-\omega^2\mathbf{p}\,dm$ lie in the axial plane through the rod. With the positive directions indicated their sum is

$$\text{(i)} \qquad m\mathbf{a}^* = [m\omega^2 p^*, 0] = [m\omega^2(c + \tfrac{1}{2}\,l\sin\theta),\ 0] \qquad (§\ 181, 4);$$

and their moment-sum about A is

$$M_A = \int_0^l \omega^2 ps \cos\theta \, dm$$

where s is the distance AP along the rod. Since

$$p = c + s \sin\theta, \quad dm = \frac{m}{l} ds,$$

(ii) $$M_A = \frac{m}{l}\omega^2 \cos\theta \int_0^l (c + s\sin\theta)s\,ds = m\omega^2 l \cos\theta\,(\tfrac{1}{2}c + \tfrac{1}{3}l\sin\theta).$$

The external forces are the weight $[0, -W]$ and the reaction $[H, V]$ at A. Since these are equivalent to the inertia forces we have from (i)

(iii) $$H = \frac{W}{g}\omega^2(c + \tfrac{1}{2}l\sin\theta), \quad V - W = 0;$$

and from (ii)

$$\tfrac{1}{2}Wl\sin\theta = \frac{W}{g}\omega^2 l\cos\theta\,(\tfrac{1}{2}c + \tfrac{1}{3}l\sin\theta) \quad \text{or}$$

(iv) $$\omega^2 = \frac{g\tan\theta}{c + \tfrac{2}{3}l\sin\theta}.$$

When ω is given, the angle θ may be found from (iv); then H and V are given by (iii).

Thus if $l = 3$ ft., $c = 1$ ft., $\omega = 2\pi$ rad./sec., (iv) becomes

$$\frac{\tan\theta}{1 + 2\sin\theta} = \frac{\omega^2}{g} = \frac{4\pi^2}{32} = 1.234.$$

By trial we find that $\theta = 74° 30'$. Then from (iii)

$$H = 1.234\,(1 + 1.5 \times 0.9636)W = 3.02\,W, \quad V = W.$$

PROBLEMS

1. Using the result of Example 1, show that $\omega^2 = 3g/2l\cos\theta$. If $l = 6$ ft., $\omega = 4$ rad./sec., $W = 8$ lb., find the angle θ that the rod makes with the axis and the reaction at O normal to the axis.

2. In Example 2 locate a point on the axis at which it is a principal axis of inertia if $l = 6$ ft., $c = 1$ ft., $\theta = 30°$.

3. The lower end of a thin uniform rod of length l is carried round in a horizontal circle of radius c with constant angular velocity ω at an angle θ to the vertical. Show that

$$\omega^2 = \frac{g\tan\theta}{c - \tfrac{2}{3}l\sin\theta}.$$

Find ω when $l = 3$ ft., $c = 4$ in., $\theta = 5°$. If the rod weighs 1 lb., what is the horizontal reaction at the bottom?

202. Balance of Revolving Masses. In § 73, Example 2, we found that when a number of weights W_i with vector eccentricities $\mathbf{p}_i$ are fixed to a shaft, the condition for its *standing balance* is

$$(1) \qquad \sum W_i \mathbf{p}_i = 0.$$

Remembering that $\mathbf{p}_i$ is the normal vector from the axis of revolution to the center of gravity of W_i, this condition states that the center of gravity of all the weights lies on the axis (§ 77).

Suppose now that each weight has a plane of symmetry normal to the shaft-axis or a line a symmetry parallel to this axis. Then if the shaft and weights revolve with constant angular velocity ω the inertia forces of each weight W_i may be reduced to a single force

$$(2) \qquad m\mathbf{a}_i{}^* = -\frac{W_i}{g}\omega^2\,\mathbf{p}_i \quad \text{at} \quad O_i \qquad (\S\ 198).$$

The shaft is said to be in running balance when these resultant inertia forces reduce to zero. In this case the motion will not produce reactions at the bearings. The absence of such kinetic reactions is very important in high-speed machines; otherwise the bearings are subject to periodic stresses (proportional to ω^2) which cause undue wear and often violent vibration. The vibrations are especially severe when the period $2\pi/\omega$ of these stresses nearly coincides with the natural period of vibration of the supports (*resonance*, § 175).

By the Equivalence Theorem of § 74 the inertia forces (2) are equivalent to zero when, and only when

(*a*) their sum is zero, and
(*b*) the sum of their moments about any point is zero.

From (2), the sum of the inertia forces is a " force "

$$(3) \qquad \mathbf{F} = -\frac{\omega^2}{g}\sum W_i \mathbf{p}_i.$$

If O is an arbitrary origin on the axis of rotation and $\overrightarrow{OO_i} = z_i\mathbf{k}$ (Fig. 202*a*), the moment of the inertia forces about O is

$$(4) \qquad \mathbf{M}_O = -\frac{\omega^2}{g}\sum z_i\mathbf{k}\times W_i\mathbf{p}_i = -\frac{\omega^2}{g}\mathbf{k}\times\sum z_i W_i \mathbf{p}_i.$$

The necessary and sufficient conditions for running balance, $\mathbf{F} = 0$, $\mathbf{M}_O = 0$, thus become

$$(5),\ (6) \qquad \sum W_i\mathbf{p}_i = 0, \quad \sum z_i W_i \mathbf{p}_i = 0.$$

The first of these is precisely the condition (1) for standing balance.

THEOREM. *The weights W_i are mounted on a shaft so that their centers of gravity have the position vectors*

$$\mathbf{r}_i = z_i\mathbf{k} + \mathbf{p}_i \text{ relative to an origin on the axis.}$$

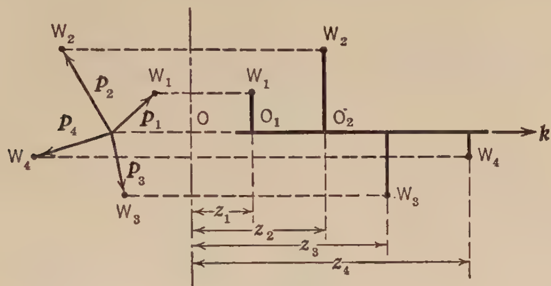


FIG. 202*a*.

Then the shaft is in running balance when, and only when,

(*a*) *the vectors $W_i\mathbf{p}_i$ form a closed polygon, and*
(*b*) *the vectors $z_iW_i\mathbf{p}_i$ form a closed polygon.*

We shall call the polygons $W_i\mathbf{p}_i$ *and* $z_iW_i\mathbf{p}_i$ the *force* and *moment* polygons respectively. The corresponding sides of these polygons are evidently parallel.

The center of gravity of each weight W_i lies in a definite plane normal to the axis and in a definite plane through the axis; these are the *normal* and *axial planes* of W_i.

When the weights all lie in the same normal plane, z_i is the same for all and the force and moment polygons are similar. The closing of force polygon therefore implies the closing the moment polygon. Hence *when all the weights lie in the same normal plane, their static balance also assures their running balance.*

Two weights can be in running balance only when they lie in the same normal and axial planes and on opposite sides of the axis. For both force and moment polygons consist of a line segment described twice, say aba, and ABA; hence $\mathbf{p}_1$ and $\mathbf{p}_2$ have opposite directions and $z_1 = AB/ab = BA/ba = z_2$.

Three weights can be in running balance only when they lie in the same normal plane or in the same axial plane. For the two polygons $abca$, $ABCA$ are either similar triangles or they form a

rectilinear segment described twice. In the former case

$$\frac{AB}{ab} = \frac{BC}{bc} = \frac{CA}{ca} \quad \text{so that} \quad z_1 = z_2 = z_3$$

and the weights lie in the same normal plane; in the latter $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are parallel and the weights lie in the same axial plane.

Example 1. A shaft carries two balanced flywheels whose planes are 1 ft. and 2 ft. respectively from the normal plane of a crank between them (Fig. 202*b*). If the crank has a radius of 9 in. and is equivalent to a mass of 100 lb. at its crank-pin, balance the shaft by means of two weights W_1, W_2 in the planes of the flywheels and at the same distance p from the axis.

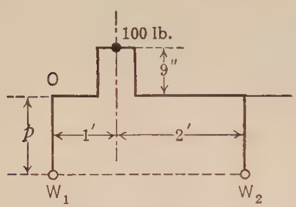


FIG. 202*b*.

As the flywheels alone are in running balance, we must balance three weights of 100, W_1, W_2 lb. Since these must lie in the same axial plane, we may take their eccentricities as 9 $\mathbf{j}$, $-p\mathbf{j}$, $-p\mathbf{j}$ in. where $\mathbf{j}$ is a unit vector. Then with O as origin we have from (5) and (6):

$$100 \times 9\,\mathbf{j} - W_1 p\mathbf{j} - W_2 p\mathbf{j} = 0, \quad 1 \times 100 \times 9\,\mathbf{j} + 0 - 3\,W_2 p\mathbf{j} = 0.$$

Hence

$$W_2 p = 300, \quad W_1 p = 600 \text{ lb.-in.}$$

If we take $p = 24$ in., $W_2 = 12.5$ lb., $W_1 = 25$ lb.

Example 2. A shaft has a number of cranks of weights equivalent to W_i at the crank-pin radius p_i. *If the force* ($W\mathbf{p}$) *polygon closes* but the moment ($zW\mathbf{p}$) polygon does not (Fig. 203*d*), let $\overrightarrow{AB} = \sum z_i W_i \mathbf{p}_i$. Then the inertia forces of the shaft at speed ω reduce to a couple of moment (4):

$$-\frac{\omega^2}{g}\mathbf{k} \times \sum z_i W_i \mathbf{p}_i = -\frac{\omega^2}{g}\mathbf{k} \times \overrightarrow{AB}.$$

Since the unit axial vector $\mathbf{k}$ is normal to $\overrightarrow{AB}$ the magnitude of this unbalanced torque is $(\omega^2/g) \cdot AB$. Note that AB will be the same for any center of moments O (§ 68, Theorem 2).

Thus for a shaft with three equal cranks at 120°, each equivalent to W lb. at a radius r ft. and spaced d ft. apart, we find

$$AB = 2\,dWr \cos 30° = \sqrt{3}\,dWr.$$

Hence the torque on the bearings amounts to

$$\sqrt{3}\,\frac{W}{g}\,\omega^2 rd \text{ lb.-ft.}$$

203. Balancing by Two Masses in Given Axial Planes. In order to balance a number of eccentric weights W_i having given eccentricities $\mathbf{p}_i$ and in given normal planes z_i, we construct the vector sums $\sum W_i\mathbf{p}_i$ and $\sum z_i W_i \mathbf{p}_i$, say

$$\overrightarrow{ab} + \overrightarrow{bc} + \overrightarrow{cd} = \overrightarrow{ad}, \quad \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}.$$

If ad is parallel to AD (which is always the case if the weights lie in a common normal plane), balance may be effected by adding one extra weight W chosen so that

$$W\mathbf{p} = \overrightarrow{da}, \quad zW\mathbf{p} = \overrightarrow{DA}.$$

The first equation gives $W\mathbf{p}$, the second z.

In general ad will not be parallel to AD. Then, since corresponding sides of the polygons must be parallel, balance can not be effected by adding one weight. In this case *we can always balance the given weights by adding two weights placed in different normal planes chosen at pleasure.* Let W_1, W_2 be the added weights, W_3, $W_4, \ldots, W_n$ the given weights. Then if the origin of z is taken in the normal plane of W_1, $z_1 = 0$ and the equations for balance are

$$(1), (2) \quad W_1\mathbf{p}_1 + W_2\mathbf{p}_2 + \sum_3^n W_i\mathbf{p}_i = 0, \quad z_2 W_2\mathbf{p}_2 + \sum_3^n z_i W_i \mathbf{p}_i = 0.$$

Since z_2 is given, the second equation determines $W_2\mathbf{p}_2$ and the first $W_1\mathbf{p}_1$.

The force equation (1) may be replaced by a second moment equation for an origin O' in the plane of W_2; then $z_2' = 0$ and the equations for balance are

$$(3), (4) \quad z_1' W_1\mathbf{p}_1 + \sum_3^n z_i' W_i\mathbf{p}_i = 0, \quad z_2 W_2\mathbf{p}_2 + \sum_3^n z_i W_i \mathbf{p}_i = 0.$$

These give $W_1\mathbf{p}_1$ and $W_2\mathbf{p}_2$ respectively. Equation (1) may now be used to check the solution.

To solve the problem analytically we replace the equations (1), (2) or (3), (4) by four scalar equations. Thus if the angle between a fixed axial plane and $\mathbf{p}_i$ is ϕ_i, we have on resolving parallel and normal to this plane

$$\mathbf{p}_i = [p_i \cos \phi_i, \quad p_i \sin \phi_i],$$

and each vector equation yields two scalar equations.

The corresponding graphic solutions are obvious. Thus in

the second method we draw two moment ($zW\mathbf{p}$) polygons: $ABCD$ for an origin O in the plane of W_1, $A'B'C'D'$ for an origin O' in the plane of W_2 (Fig. 203a). If the $+z$ directions are $\overrightarrow{OO'}$ and $\overrightarrow{O'O}$ respectively and the distance $OO' = a$,

$$z_1 = 0, \quad z_2 = a \text{ for } ABCD; \quad z_1' = a, \quad z_2' = 0 \text{ for } A'B'C'D'.$$

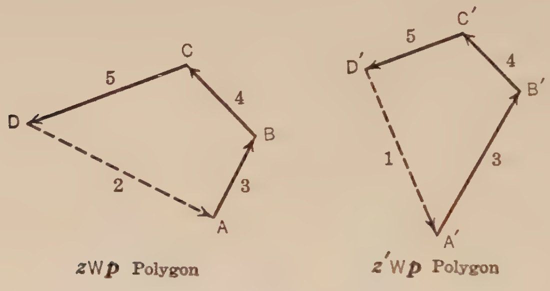


FIG. 203a.

The closing vectors

$$\overrightarrow{DA} = aW_2\mathbf{p}_2, \quad \overrightarrow{D'A'} = aW_1\mathbf{p}_1$$

determine $W_2\mathbf{p}_2$, $W_1\mathbf{p}_1$. We may now check the solution by the closure of the force ($W\mathbf{p}$) polygon.

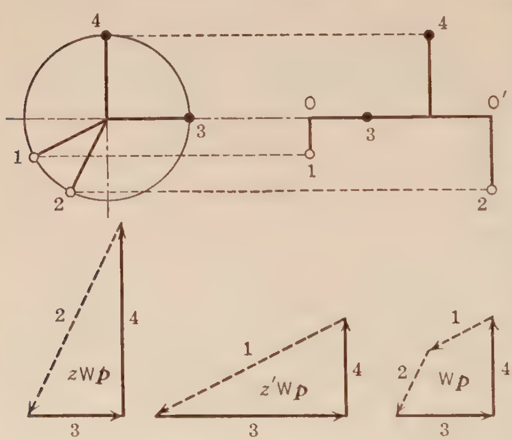


FIG. 203b.

Example 1. A crank-shaft has four equidistant cranks of the same radius. If the middle pair are in perpendicular axial planes and are equivalent to 120 lb. at the crank-pin, determine the weights and positions of the outside pair for running balance.

Number the cranks 1, 3, 4, 2 in order (Fig. 203b) and take the equal distances between them as unity. Denote

the equal crank radii by p and let $W = 120$ lb. We may now form the following table:

Crank	z	z'	W	p	ϕ	zWp	$z'Wp$	Wp
3	1	2	W	p	0°	Wp	$2\,Wp$	$120\,p$
4	2	1	W	p	90°	$2\,Wp$	Wp	$120\,p$
1	0	3	W_1	p	ϕ_1	0	$3\,W_1p$	$89.5\,p$
2	3	0	W_2	p	ϕ_2	$3\,W_2p$	0	$89.5\,p$

The moment polygons for the points O and O' are drawn as shown. Their closing sides give the directions of cranks 2 and 1 and also

$$3\,W_2p = \sqrt{5}\,Wp, \quad 3\,W_1p = \sqrt{5}\,Wp;$$

hence $$W_1 = W_2 = 40\sqrt{5} = 89.5 \text{ lb.}$$

Or we might have drawn the closing sides of the force (Wp) polygon parallel to the known directions of 1 and 2 and then found W_1 and W_2 by measurement. The angle between cranks 1 and 2 is about 37°.

Example 2. Three weights W_3, W_4, W_5 are mounted on a shaft as shown in Fig. 203*c* and in the table below. Find the value and position of the weights W_1, W_2 to be placed in normal planes through O and O' so that the shaft will be in running balance.

Crank	z	z'	W	p	ϕ	zWp	$z'Wp$	Wp
3	−1	4	100	1	30°	−100	400	100
4	1	2	100	1	150°	100	200	100
5	2	1	150	1	270°	300	150	150
1	0	3	W_1	p_1	ϕ_1	0	$3\,W_1p_1$	76
2	3	0	W_2	p_2	ϕ_2	$3\,W_2p_2$	0	115

The moment polygons for the points O and O' are drawn as shown. Their closing sides 2 and 1, give on measurement (from large-scale diagrams)

$$3\ W_2p_2 = 346, \quad \phi_2 = 60°: \qquad W_2p_2 = 115;$$
$$3\ W_1p_1 = 229, \quad \phi_1 = 221°: \qquad W_1p_1 = 76.$$

The force ($W\mathbf{p}$) polygon may now be drawn; its closure checks the solution.

If we take $p_1 = p_2 = 3$, $W_1 = 25.3$ lb., $W_2 = 38.3$ lb.

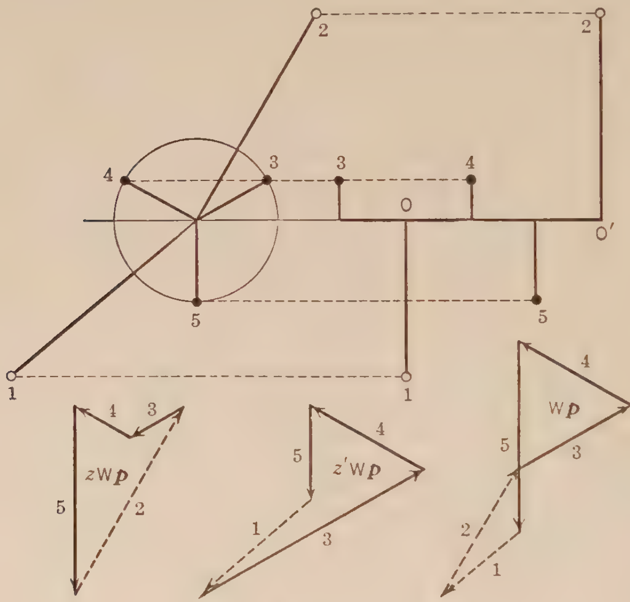


Fig. 203c.

Example 3. An unbalanced crank-shaft is to be balanced by two weights W_1, W_2. If the force ($W\mathbf{p}$) polygon of the unbalanced cranks closes, what can be said about the position of the balance weights?

Since the $W\mathbf{p}$ polygon for the cranks *and* the balance weights must close for running balance, the $W\mathbf{p}$ polygon for the two balance weights must close: i.e.

(i) $$W_1\mathbf{p}_1 + W_2\mathbf{p}_2 = 0.$$

Hence $\mathbf{p}_1$ and $\mathbf{p}_2$ have opposite directions, and W_1, W_2 must lie in the same axial plane.

Since the moment ($zW\mathbf{p}$) polygon for the cranks does not close, let

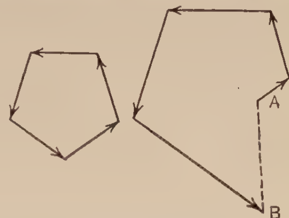
BA denote the closing side (Fig. 203d). If we construct the moment polygon for the cranks *and* balance weights relative to the same point O, it must close for running balance; that is

(ii) $$z_1W_1\mathbf{p}_1 + z_2W_2\mathbf{p}_2 = \overrightarrow{BA}.$$

Hence $\mathbf{p}_1$ and $\mathbf{p}_2$ are parallel to BA.

Let us choose, for example, $W_1 = W_2$. Then $\mathbf{p}_1 = -\mathbf{p}_2$ from (i); and from (ii)

$$(z_1 - z_2)W_1\mathbf{p}_1 = \overrightarrow{BA}.$$



Force and moment polygons for unbalanced shaft.

FIG. 203d.

Now z_1 and z_2 (which define the normal planes of W_1 and W_2) may be chosen at pleasure provided $z_1 \neq z_2$. The equation above then gives $W_1\mathbf{p}_1$.

PROBLEMS

1. A crank-shaft has four equidistant cranks of the same weight and radius. The inner cranks have the same direction, the outer cranks are opposed to these. Show that the shaft is in running balance.

2. A locomotive crank-axle has two perpendicular cranks of equal weight and radius. These are to be balanced by weights on the driving wheels (Fig. 203e). If the distance between cranks is d, between wheels l, show that the central angle θ subtended by the balance weights is $2 \tan^{-1} d/l$. Apply this result to Example 1 above.

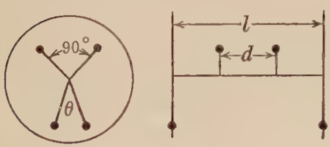


FIG. 203e.

3. A crank-shaft has three cranks of equal weight W and radius p at angles of 120° with each other. The middle crank is at the same distance d from the others. Show that the shaft may be balanced by two equal weights $W_1 = W_2$ in the axial plane perpendicular to the middle crank and that $\mathbf{p}_1 = -\mathbf{p}_2$. If the balance weights are on opposite sides of the middle crank and at the same distance l, show that $lW_1p_1 = dWp \cos 30°$.

[Draw the moment polygon relative to the middle crank. See Example 3.]

4. A crank-shaft has five cranks (*1, 2, 3, 4, 5*, taken in order from left to right) of equal radius, spaced at equal distances along the shaft. Cranks *2, 3, 4* are at 120° and are equivalent to 100 lb. at the crank-pin. Find the equivalent weights for cranks *1, 5* and the angles between *1, 2* and *4, 5* for running balance. [See Example 3.]

5. In Example 1 find the weights and positions of the outer cranks

1, *2* if the cranks *3*, *4* have weights equivalent to 50 and 75 lb. at the crank-pin.

6. A shaft 7 ft. long carries two flywheels at its ends and four cranks at distances of 2, 3, 4, 5 ft. from the left-hand wheel. Each crank is 90° in advance of the one on its left and is equivalent to 70 lb. at a radial distance of 10 in. Find the position and magnitude of two weights in the planes of the flywheels and 20 in. from their axis that will balance the shaft, if the flywheels themselves are balanced. Sketch the balanced shaft.

7. A shaft has four equal cranks at 90° spaced d ft. apart. The cranks are numbered *1*, *2*, *3*, *4* from left to right and each is equivalent to W lb. at a radial distance of r ft. Compute the torque on the bearings due to the inertia forces in each of the three arrangements shown (Fig. 203*f*). [See § 202, Ex. 2.]

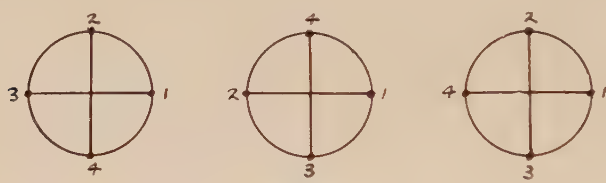


FIG. 203*f*.

204. Balance of Masses in S.H.M. We have seen in § 119 that s.h.m. is the projection of uniform circular motion on a diameter. If a particle Q revolves in a circle of radius p with the constant angular velocity ω, its acceleration is

$$\mathbf{a}_Q = -\omega^2\mathbf{p} \quad \text{where} \quad \mathbf{p} = \overrightarrow{OQ}.$$

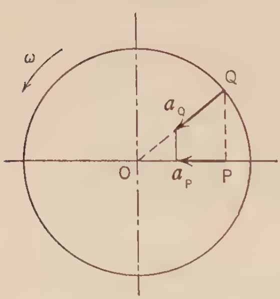


FIG. 204*a*.

Let P be the projection of Q on a diameter. Then the acceleration of P is the projection of $\mathbf{a}_Q$ on this diameter (§ 106).

If P is the center of gravity of a reciprocating body of weight W performing this s.h.m., its inertia forces have a resultant at P equal to

$$m\mathbf{a}_P = \frac{W}{g}\operatorname{proj}\mathbf{a}_Q = -\frac{\omega^2}{g}\operatorname{proj}W\mathbf{p}. \qquad (\S\ 190).$$

Consider, now, a number of weights W_i whose centers of gravity P_i describe simple harmonic motions of the same period T along

a series of coplanar parallel lines (Fig. 204*b*). Each s.h.m. of amplitude p_i, may be regarded as the projection of a uniform motion of a point Q_i in a circle of radius p_i with the angular velocity $\omega = 2\,\pi/T$ rad./sec. If $\mathbf{p}_i$ is the vector radius to Q_i,

$$\text{Inertia Force of } W_i = -\frac{\omega^2}{g}\,\text{proj } W_i\mathbf{p}_i.$$

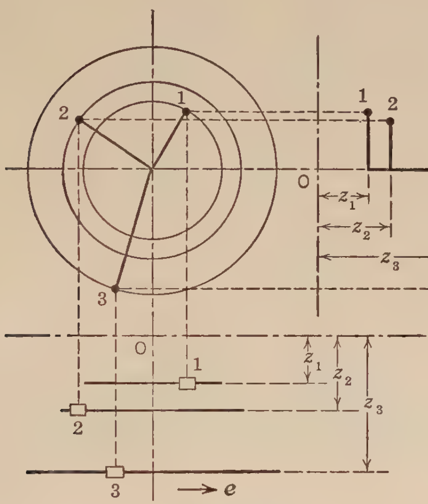


FIG. 204*b*.

When these inertia forces reduce to zero for all positions of the reciprocating masses, the masses are said to be in *running balance*. Hence if $\mathbf{e}$ is a unit vector along an axis of reciprocation, the conditions for balance are

$$\mathbf{e}\cdot\sum W_i\mathbf{p}_i = 0, \quad \mathbf{e}\cdot\sum z_iW_i\mathbf{p}_i = 0;$$

that is, the sum of the forces and of their moments about O vanish. These equations must hold for all positions of the vectors $\mathbf{p}_i$ as they make a complete revolution. Otherwise expressed, they must hold if the vectors $\mathbf{p}_i$ are held fast in one position and the unit vector $\mathbf{e}$ is revolved in their plane. But if the equations are true for all values of $\mathbf{e}$, we must have

(1) $$\sum W_i\mathbf{p}_i = 0, \quad \sum z_iW_i\mathbf{p}_i = 0.$$

These are precisely the conditions for the running balance of the weights W_i mounted on a shaft with eccentricities $\mathbf{p}_i$ (§ 202). In brief:

In order that a set of reciprocating masses describing parallel, coplanar simple harmonic motions of the same period be in running balance, it is necessary and sufficient that these masses, when transferred to the corresponding particles revolving about a common axis and in similarly spaced normal planes, be in running balance.

205. Balance of Reciprocating Masses. When the crank of an engine (slider-crank) mechanism is revolving at the constant rate of ω rad./sec., the acceleration of the crosshead and piston are given approximately by

$$a = -r\omega^2\left(\cos\phi + \frac{\cos 2\phi}{n}\right) \qquad (§\ 112,\ 9),$$

where $n = l/r$, the ratio of the length of the connecting-rod to that of the crank. If we write this as

$$a = -r\omega^2 \cos\phi - \frac{r}{4n}(2\omega)^2 \cos 2\phi,$$

each term on the right may be regarded as acceleration in s.h.m. The first corresponds to motion in a circle of radius r and angular speed ω, the second to motion in a circle of radius $r/4n$ and of angular speed 2ω.

For an infinite connecting-rod the second term is zero; the crosshead then has the s.h.m. corresponding to the first term. For a finite connecting-rod both terms contribute to the resultant inertia force of the reciprocating parts; if their total weight is W, these contributions are the

$$\text{Primary inertia force} = -\frac{W}{g} r\omega^2 \cos\phi,$$

$$\text{Secondary inertia force} = -\frac{W}{g}\frac{r}{4n}(2\omega)^2 \cos 2\phi.$$

Consider now an engine having several cylinders with parallel, coplanar axes and located on the same side of the crank-shaft. Its reciprocating parts (piston, piston-rod and cross-head) are said to be in *primary balance* when its primary inertia-forces reduce to zero, in *secondary balance* when its secondary inertia-forces reduce to zero. Now the primary and secondary inertia-forces are assignable to parallel, coplanar, simple harmoinc mo-

tions of the periods $2\pi/\omega$ and π/ω respectively; and the necessary and sufficient conditions that each set separately be in balance are given by the equations (§ 204, 1). Thus if the vectors $\mathbf{p}_i$ give the crank radii in any position of the crank-shaft, the conditions for primary balance are

$$(1), (2) \qquad \sum W_i \mathbf{p}_i = 0, \quad \sum z_i W_i \mathbf{p}_i = 0.$$

Let ϕ_i be the angle measured in a definite sense from a fixed axial plane to $\mathbf{p}_i$. Since the angular speed and radius corresponding to the secondary simple harmonic motions are 2ω and $r/4n_i$, the vectors $\mathbf{p}_i$ must be revolved through an additional angle ϕ_i to get the relative crank positions for these (imaginary) motions, and their lengths then multiplied by $1/4n_i$. Thus if the vectors $\mathbf{p}_i$ become $\mathbf{p}_i'$ when shifted ahead an angle ϕ_i, the conditions for secondary balance are

$$\sum W_i \frac{\mathbf{p}_i'}{4n_i} = 0, \quad \sum z_i W_i \frac{\mathbf{p}_i'}{4n_i} = 0.$$

If the lengths of the connecting-rods and cranks are the same for all cylinders, n_i is constant and a common factor of all the terms in these sums. The conditions for secondary balance then reduce to

$$(3), (4) \qquad \sum W_i \mathbf{p}_i' = 0, \quad \sum z_i W_i \mathbf{p}_i' = 0.$$

The distances z_i are measured from some convenient normal plane to the axes of the several cylinders (Fig. 204*b*).

The only difference between conditions (1), (2) and (3), (4) is that the vectors $\mathbf{p}_i'$ make twice the angle with the fixed axial plane as the vectors $\mathbf{p}_i$; that is, if ϕ_i is the angle of $\mathbf{p}_i$, $2\phi_i$ is the angle of $\mathbf{p}_i'$.

If the reciprocating masses are all equal the factors W_i may be dropped from equations (1), (2), (3), (4).

In the following examples the upper diagram shows the actual crank arrangement, the lower the corresponding arrangement for the secondary simple harmonic motions. The reciprocating masses, the crank-arms, and the lengths of the connecting-rods are assumed to be the same for all cylinders. Moreover the axes of the cylinders are spaced at equal distances. The letter O indicates the center of moments.

Example 1. In the three-crank engine of Fig. 205*a*, the primary and secondary force polygons both close, but the moment polygons do not. Thus both primary and secondary inertia forces reduce to couples.

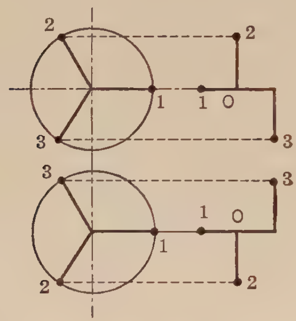


FIG. 205*a*.

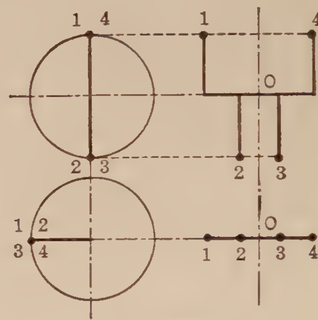


FIG. 205*b*.

Example 2. In the four-crank engine of Fig. 205*b*, the primary force polygon and the primary and secondary moment polygons close; the secondary force polygon does not. The secondary forces reduce to a single force at O. This resultant secondary force is equal to the projection of a vector of length $4\ Wr\omega^2/gn$ revolving at twice the rate of the crank.

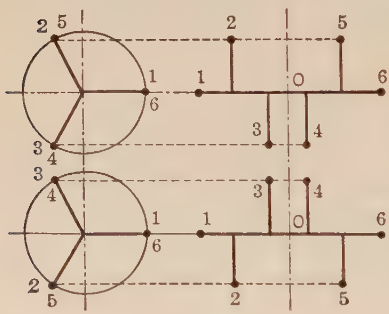


FIG. 205*c*.

Example 3. In the six-crank engine of Fig. 205*c* the primary and secondary force and moment polygons all close. It is therefore in perfect primary and secondary balance. This does not mean, however, that the inertia forces are perfectly balanced; for these forces were calculated from the *approximate* formula (§ 112, 9). Nevertheless this engine is one of the most perfectly balanced engines that can be constructed.

PROBLEMS

1. An engine has four cranks of equal radius: *1*, *2*, *3*, *4* from left to right (Fig. 205*d*). Cranks *2* and *3* are at 120° and lie in the same normal plane. Cranks *1* and *4* make angles of 120° with both *2* and *3*; they lie in the same axial plane and at distances d_1, d_4 from the

plane of *2, 3*. Show that engine (not practicable) has perfect primary and secondary balance when

$$W_2 = W_3 = W_1 + W_4, \quad W_1/W_4 = d_4/d_1.$$

2. An engine has five equidistant cranks of equal radius (*1, 2, 3, 4, 5* from left to right) having the angular positions shown in Fig. 205*e*. If the corresponding reciprocating masses are proportional to 1, 2, 3, 2, 1, show that the engine has perfect primary and secondary balance.

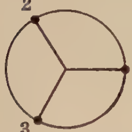


FIG. 205*d*.

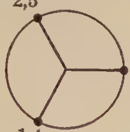


FIG. 205*e*.

FIG. 205*f*.

FIG. 205*g*.

3. Figs. 205*f*, *g* show the crank arrangement on two five-crank engines with cranks at 72°. The cranks, numbered *1, 2, 3, 4, 5* from left to right, are equidistant and of equal radius. If the reciprocating masses are all equal, show that primary and secondary force polygons close for both arrangements, but that the moment polygons do not. Which arrangement gives the smaller unbalanced primary couple? Secondary couple?

4. In the four-crank engine shown in Fig. 205*h*, $W_1 = W_4$, $W_2 = W_3$, and the ratio $r = a/b$ is given. Show that the angles α, β must be computed from the equations

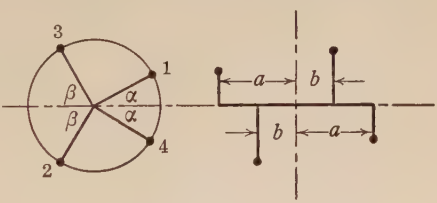


FIG. 205*h*.

$$4 \cos^4 \alpha + (r^2 - 1)\cos^2 \alpha - r^2 = 0, \quad \cos \alpha \cos \beta = \tfrac{1}{2},$$

in order that both primary polygons and the secondary force polygons close; then $W_2/W_1 = 2 \cos^2 \alpha$.

[From the closure of the three polygons we find $W_1 \cos \alpha = W_2 \cos \beta$, $rW_1 \sin \alpha = W_2 \sin \alpha$, $W_1 \cos 2\alpha = -W_2 \cos 2\beta$. Eliminate W_1, W_2 from these equations.]

5. When $a/b = 2$ in Problem 4 compute α, β and W_2/W_1.

206. Governors. A prime mover such as a steam or gas engine, a steam or hydraulic turbine, is in practice often subjected to a power demand which varies within certain limits. In order to adjust automatically the driving force to the load on the prime

mover it must be equipped with a *governor*; and the ideal of operation is to design the governor so that the prime mover works at about the same speed over its entire range of load. The governor is essentially a mass connected to the engine so that it revolves at the same or at a proportional rate. A decrease in the load on the engine increases the driving force available for the acceleration of its moving parts; the engine speed thus increases and with it the governor speed. The governor then changes its position and actuates a mechanism that decreases the supply or effectiveness of the working fluid. An increase in load produces the contrary effect.

The action of the governor may depend solely on a *change* $\Delta\omega$ in angular speed or on such a change coupled with the *rate of change* $d\omega/dt$. In the former case we have a *centrifugal governor*, in the latter, an *inertia governor*. Governors are also classified according to the arrangement of the revolving masses. If these form essentially a conical pendulum (§ 157, Ex. 4), the governor is called a *pendulum* or a *flyball governor*; as the speed changes the balls revolve in different normal planes. If the governing masses are pivoted to the flywheel or some other rotor keyed to the crankshaft, the governor is called a *shaft governor*; as the speed changes the masses are displaced in the same normal plane. Centrifugal governors may be either pendulum or shaft governors; inertia governors, however, are all of the shaft type.

We shall consider in the following articles the *steady states* of a governor, namely, the positions in which it can run at constant speed. In such positions the governor is said to be in *equilibrium*. The study of the dynamical stability of a governor and the oscillations about a steady state necessitates treating the governor and engine as a single dynamical system. This more difficult problem will not be considered.

207. Pendulum Governors. In dealing with the equilibrium of governors we must first consider the inertia forces generated by a *uniform* rotation ω. If we neglect the inertia of the links connecting the balls with the governor shaft, we need only consider the inertia forces of the balls and of the axial sleeve, which is usually weighted to secure certain desirable characteristics in governor action. Since the sleeve has always approximate axial symmetry, its inertia forces reduce to zero (§ 198, Theorem). The inertia forces of each ball, of mass m, reduce to the resultant

ma^* at its center (§ 198, Theorem). These resultants reversed are the *centrifugal forces* of the balls; each equals $C = m\omega^2 r$ and is directed away from the axis. When the governor is revolving steadily the centrifugal force on each ball must equilibrate all the external forces on the corresponding pendulum. These external forces are

(*a*) the weight W of the ball (neglecting the weight of the pendulum rod),

(*b*) the reaction R at the point of suspension,

(*c*) the stress S in the link joining the pendulum to the sleeve,

(*d*) and if springs are employed, any spring load T acting on the pendulum.

The equilibrium of pendulum governors is thus reduced to a problem in plane statics involving the equilibrium of C with the forces listed above. Such problems may be solved graphically or analytically by the methods of Chapter IV. In the analytical method the condition for equilibrium is obtained by taking moments about the point of suspension O. The equation so obtained relates the speed to the position of the governor.

A pendulum governor is actuated entirely by the normal inertia forces; for the tangential inertia forces due to angular acceleration are perpendicular to the plane of the governor and play no part in changing its position.

Pendulum governors are classed as *gravity* or *spring* governors according as the centrifugal force C is principally balanced by weights or by spring loads. Both types are considered below.

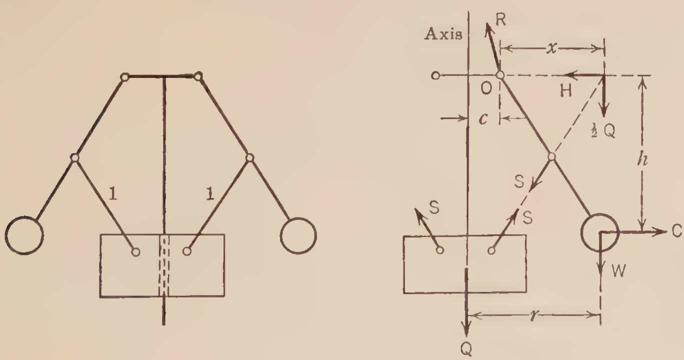


FIG. 207*a*.

Example 1. *Loaded Watt Governor.* In the governor of Fig. 207a the weight of the sleeve Q is balanced by the tensions S in the links 1; the vertical component of S is therefore equal to $\frac{1}{2}Q$. Each pendulum is in equilibrium under the centrifugal force $C = (W/g)\omega^2 r$ and the external forces W, S, R.

In taking moments about O it is convenient to shift S along its line of action to a point on the same level as O and there replace it by its vertical and horizontal projections, $\frac{1}{2}Q$, H. The moment of S about O then reduces to $\frac{1}{2}Qx$, since the moment of H is zero. The moment equation therefore reads $Ch - W(r - c) - \frac{1}{2}Qx = 0$ or

$$\frac{W}{g}\omega^2 rh = W(r - c) + \tfrac{1}{2}Qx. \tag{1}$$

From (1) we may compute ω for any governor position. In the right-hand member the distances $r - c$ and x are measured from the point of suspension O; hence a shift in the position of the governor axis does not alter their values. Equation (1) now shows that for any given position of the pendulum, $\omega^2 r$ has the same value for all positions of the governor axis. By shifting the axis to the right we decrease r and therefore increase the ω corresponding to the given position.

Suppose now that an increase $\Delta\omega$ in speed is required at a given position of the governor before it will rise. This means that the frictional resistance of the sleeve and linkage is equivalent to an additional weight ΔQ such that

$$\frac{W}{g}(\omega + \Delta\omega)^2 rh = W(r - c) + \tfrac{1}{2}(Q + \Delta Q)x.$$

If we subtract (1) from this equation and neglect the term in $(\Delta\omega)^2$ we obtain

$$2\frac{W}{g}\omega\,\Delta\omega rh = \tfrac{1}{2}x\,\Delta Q.$$

Divide this by (1) member for member; then

$$2\frac{\Delta\omega}{\omega} = \frac{\frac{1}{2}x\,\Delta Q}{W(r - c) + \frac{1}{2}Qx}, \qquad \text{or}$$

$$\tfrac{1}{2}\Delta Q = 2\frac{\Delta\omega}{\omega}\left(W\frac{r - c}{x} + \tfrac{1}{2}Q\right).$$

In order to overcome friction the governor must exert a force ΔQ on the sleeve as it begins to rise. Since this lift becomes zero in the new position of equilibrium, its mean value is $\frac{1}{2}\Delta Q$ during the motion. $\frac{1}{2}\Delta Q$ is sometimes called the *effort* of the governor. The equation above shows that the effort is increased by an increase in the sleeve load Q.

Example 2. *Spring Governor.* In the governor shown schematically in Fig. 207*b* the balls are attached to 90° bell-cranks. The rollers on their inner ends press against a plate fastened to a compressed spring fixed at its upper end. As the balls swing outwards the plate is forced up against the spring, and the sleeve connected to the regulating mechanism rises.

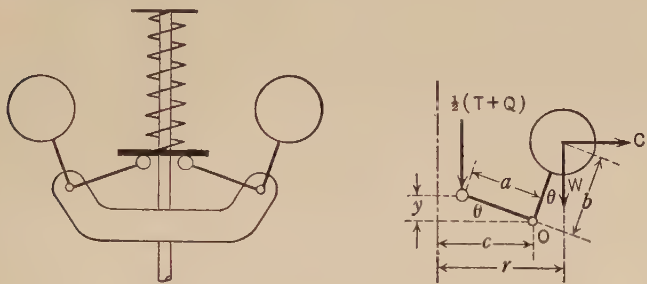


FIG. 207*b*.

If T is the force exerted by the spring and Q the weight of the sleeve, each roller carries the load $\frac{1}{2}(T + Q)$. The bell-crank is in equilibrium under the centrifugal force C, the loads W, $\frac{1}{2}(T + Q)$ and the reaction at O. On taking moments about O we have

$$\tfrac{1}{2}(T + Q)\, a \cos\theta = Cb\cos\theta + Wb\sin\theta,$$

or on division by $b \cos\theta$,

$$\frac{W}{g}\omega^2 r = \frac{1}{2}\frac{a}{b}(T + Q) - W\tan\theta. \tag{2}$$

In practice Q and W are small compared with T and the angle θ is small. Therefore the terms $Qa/2\,b - W\tan\theta$ on the right may usually be neglected, especially as they tend to cancel each other. The equation for governor equilibrium is therefore

$$\frac{W}{g}\omega^2 r = \frac{1}{2}\frac{a}{b}T \qquad \text{approximately.} \tag{3}$$

If the increase $\Delta\omega$ in speed is required to make the governor rise, the frictional resistance is equivalent to an additional force ΔT exerted by the spring, where

$$\frac{W}{g}(\omega + \Delta\omega)^2 r = \frac{1}{2}\frac{a}{b}(T + \Delta T).$$

If we subtract (3) from this equation and neglect the term the $(\Delta\omega)^2$ we obtain

$$2\frac{W}{g}\omega\Delta\omega\, r = \frac{1}{2}\frac{a}{b}\Delta T.$$

Divide this equation by (2) member for member; then

$$2\frac{\Delta\omega}{\omega} = \frac{\Delta T}{T}; \quad \text{and} \quad \tfrac{1}{2}\Delta T = \frac{\Delta\omega}{\omega} T$$

is the *effort* of the governor. Thus if a 1 per cent change in speed is required to make the governor rise when in a given position, its effort is 0.01 T.

208. Characteristic Curve of a Governor. The characteristics of a flyball governor are shown by its *C-r curve* obtained by plotting the centrifugal force C of the ball as ordinate against its radial distance r from the axis (Fig. 208*a*). For a given r the value of C may be computed from the equilibrium equation; or C may be found graphically so that it balances the external forces.

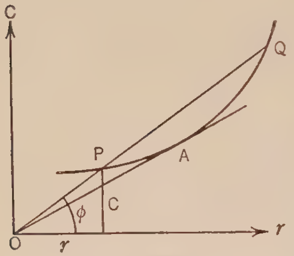


FIG. 208*a*.

Stability. A governor is said to be *stable* in a given position of equilibrium if it will return to this position when given a slight displacement while running at constant speed. To find the condition for stability let the moment Ch of C about the point of suspension be given as a function of r by the equilibrium equation, say $Ch = f(r)$. For the positions r_1 and r_2, where $r_2 > r_1$, we then have

$$m\omega_1{}^2 r_1 h_1 = f(r_1), \quad m\omega_2{}^2 r_2 h_2 = f(r_2).$$

Here $f(r_1)$ and $f(r_2)$ denote the balancing moments of external forces. Now if at speed ω_1 the governor is displaced to position r_2, the moment $m\omega_1{}^2 r_2 h_2$ tends to swing the ball further out, while the moment $f(r_2)$ tends to restore it to its former position. The latter moment will prevail if

$$m\omega_2{}^2 r_2{}^2 h_2 > m\omega_1{}^2 r_2 h_2, \quad \text{i.e., if} \quad \omega_2 > \omega_1.$$

Therefore *the governor is stable if ω increases with r*: $d\omega/dr > 0$.

This condition has a simple geometric meaning on the C-r curve. Draw a line OP from the origin to any point P of the curve. If OP makes an angle ϕ with the r-axis,

$$(1) \qquad \tan\phi = \frac{C}{r} = \frac{m\omega^2 r}{r} = m\omega^2.$$

Hence ω increases when ϕ increases. The governor is therefore stable at a point of the C-r curve provided ϕ increases with r at that point. Thus in Fig. 208a the governor is stable above A, unstable below A. The transition point A, where OA is tangent to the curve, is called an *astatic point.*

If the line OP cuts the C-r curve in a second point Q, the governor will run at the same speed ω at both P and Q. If, in particular, the curve is a straight line through the origin, the governor will run at the same speed in all positions. Such a governor is said to be *isochronous.* Although early efforts in governor design aimed at securing isochronous governors, a strictly isochronous governor would be of no practical utility. For if the speed increases above the normal the balls will fly out to their outermost position; the regulating mechanism then reduces the engine power, the speed eventually falls below normal and the balls will collapse. The repetition of this cycle gives rise to the objectionable oscillations in speed known as *hunting.*

Example. In the spring governor of § 207 the equation of equilibrium is

$$\text{(i)} \qquad \frac{W}{g}\,\omega^2 r = \tfrac{1}{2}\frac{a}{b}\,T \qquad \text{approximately.}$$

To investigate the stability of this governor let T_0 be the compressive force of the spring when the ball-arm is vertical. When this arm makes an angle θ with the vertical the spring is compressed an additional amount y; and from similar triangles

$$\frac{y}{a} = \frac{r - c}{b} \quad \text{or} \quad y = \frac{a}{b}(r - c).$$

If y is expressed in inches and σ is the *stiffness* of the spring (the number of pounds required to compress it one inch)

$$T = T_0 + \sigma y = T_0 + \sigma\frac{a}{b}(r - c).$$

Now from (i) we see that ω^2 is proportional to

$$\frac{T}{r} = \frac{a}{b}\sigma + \frac{T_0 - \dfrac{a}{b}c\sigma}{r}.$$

Hence ω will increase with r provided the last term is negative, that is

$$T_0 < \frac{a}{b}c\sigma.$$

This, then, is the condition for stability. The condition for isochronism is

$$T_0 = \frac{a}{b} c\sigma.$$

As a numerical example consider a governor of this type having 8-lb. balls and designed to make 240 r.p.m. when the ball-arms are vertical; also $a = 4$ in., $b = 5$ in., $c = 6$ in. In this position $r = \frac{1}{2}$ ft., $\omega = 8\pi$ rad./sec., and from (i)

$$T_0 = 2\frac{b}{a}\frac{W}{g}\omega^2 r = 2 \times \tfrac{5}{4} \times \tfrac{8}{32} \times 64\pi^2 \times \tfrac{1}{2} = 197 \text{ lb.}$$

If we neglect friction, the stability of this governor requires that the spring have a stiffness

$$\sigma > \frac{bT_0}{ac} \quad \text{or} \quad \sigma > 41 \text{ lb./in.}$$

PROBLEMS

1. Neglecting friction, find the angular speed at which the sleeve of the governor of Fig. 208*b* begins to rise.

2. Owing to friction the governor in Problem 1 does not rise until its angular speed is 1 per cent higher than the theoretical value. Find the effort of the governor.

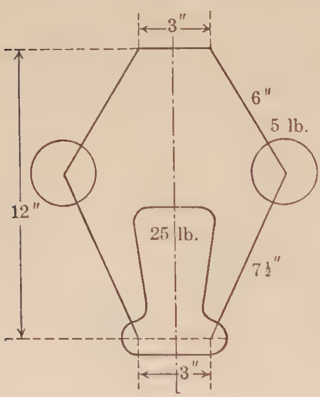


FIG. 208*b*.

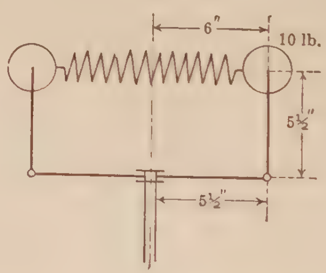


FIG. 208*c*.

3. In the spring governor of Fig. 207*b* the balls weigh 10 lb., $a = b = 5$ in., and $c = 6$ in. The compressive force on the spring when the ball-arms are vertical is 500 lb. Find the angular speed of the governor in this position.

If the stiffness of the spring is 100 lb./in., find the angular speed when the sleeve has risen 1 in.

4. The spring governor of Fig. 208*c* begins to act at 120 rev./min. when the ball-arms are vertical. If the stiffness of the spring is 8 lb./in. find the angular speed when the sleeve has risen $1\frac{1}{2}$ in.

5. In the governor of Fig. 207*a* show that the limiting ratio of the rise of the sleeve to the vertical rise of the ball is $x/(r - c)$. [Find the instantaneous center of the link *1*.]

6. Show that $d\omega/dr = 0$ at an astatic point from its defining property on the C-r curve.

7. Fig. 208*d* shows one-half of a loaded Watt governor in which the links AC and BD are of lengths l and $l + a$. Show that

(*a*) if the axis of rotation is to the *left* of AB the governor is stable in all positions;

(*b*) if the axis is at a distance c to the *right* of AB, the governor is stable only when

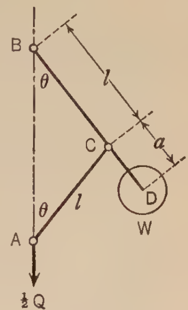


FIG. 208*d*.

$$\sin^3 \theta > \frac{c}{l + a}.$$

[Since r increases with θ we may take $d\omega^2/d\theta > 0$ as the condition of stability.]

209. Shaft Governors. A shaft governor is attached to the flywheel or some other body which revolves with the crank-shaft. The mass which causes the governing action moves in a plane normal to the shaft. We shall assume that this mass is symmetric with respect to a plane normal to the axis and regard the entire mass as concentrated in this plane.

If the mass is constrained to move along a radius of the flywheel (in a slot for example) only *normal* inertia forces $r\omega^2\, dm$ are available for governor action. This is the basic arrangement of the true *centrifugal* shaft governor. Such governors are rarely used.

If the mass is pivoted to the flywheel at its center of mass, its centrifugal force is balanced by the pin reaction and only *tangential* inertia forces $r\alpha\, dm$ are available for governor action. This arrangement would constitute a true *inertia* shaft governor. Such a governor, however, does not fix the speed ω at which the engine should run; for the tangential inertia forces depend only on the *rate* $\alpha = d\omega/dt$ at which the speed changes.

Practical shaft governors depend on both normal and tangential inertia forces for their action. According as the former or latter

predominate in the governing action, they are called *centrifugal* or *inertia* governors. It should be remembered, however, that all governing action is due to inertia forces and that the *direction* of these forces is at the root of the above classification.

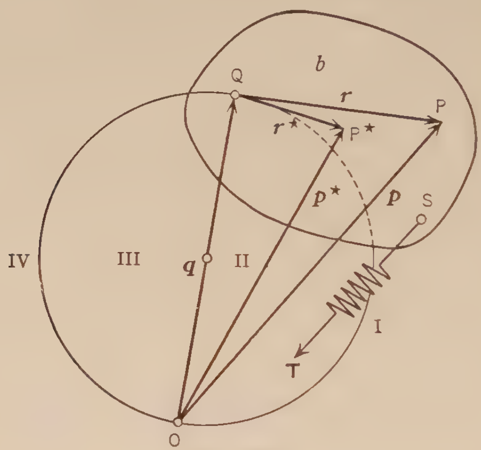


FIG. 209a.

Consider now a shaft governor in which the effective body b is pivoted at the point Q of the flywheel revolving about the axis O (Fig. 209a). P^* is the center of mass of b, and P one of its particles of mass dm. Now let

$\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ = angular velocity and acceleration of wheel,
$\boldsymbol{\omega}_r$, $\boldsymbol{\alpha}_r$ = angular velocity and acceleration of b relative to wheel,
$\mathbf{v}_r$, $\mathbf{a}_r$ = velocity and acceleration of P relative to wheel,
$\mathbf{a}$, $\mathbf{a}_b$ = absolute and body accelerations of P,
$\mathbf{p}$, $\mathbf{p}^*$ = position vectors of P and P^* referred to O,
$\mathbf{r}$, $\mathbf{r}^*$ = position vectors of P and P^* referred to Q.

From the Theorem of Coriolis (§ 134, 2)

$$\mathbf{a} = \mathbf{a}_b + 2\,\boldsymbol{\omega}\times\mathbf{v}_r + \mathbf{a}_r.$$

We proceed to find the moment of the inertia forces $\mathbf{a}\,dm$ about Q, taking in turn these three parts of $\mathbf{a}$.

Body Accelerations. The inertia forces $\mathbf{a}_b\,dm$ are due to $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, regarding b fixed relative to the wheel. By the Theorem proved in the next article, these forces are equivalent to a

$$\text{Force } m\mathbf{a}_b^* = -m\omega^2\mathbf{p}^* + m\boldsymbol{\alpha}\times\mathbf{p}^* \text{ at } P^*$$

and Couple $I^*\alpha$, where I^* is the moment of inertia of b about P^*. Their moment about Q is therefore

$$\mathbf{r}^*\times(-m\omega^2\mathbf{p}^* + m\alpha\times\mathbf{p}^*) + I^*\alpha = m\omega^2\mathbf{p}^*\times\mathbf{r}^* + m\alpha\,\mathbf{p}^*\cdot\mathbf{r}^* + I^*\boldsymbol{\alpha}.$$

Coriolis Accelerations. Since

$$2\,\boldsymbol{\omega}\times\mathbf{v}_r = 2\,\boldsymbol{\omega}\times(\boldsymbol{\omega}_r\times\mathbf{r}) = -2\,\boldsymbol{\omega}\cdot\boldsymbol{\omega}_r\,\mathbf{r}$$

the Coriolis accelerations all pass through Q, and the moment of the corresponding inertia forces about Q is zero.

Relative Accelerations. The inertia forces $\mathbf{a}_r\,dm$ are due to $\boldsymbol{\omega}_r$ and $\boldsymbol{\alpha}_r$, regarding b as a body revolving about the fixed point Q. By the Theorem of § 198, these may be reduced to a

$$\text{Force } m\mathbf{a}_r^* \text{ at } Q \quad \text{and} \quad \text{Couple } I\boldsymbol{\alpha}_r,$$

where I is the moment of inertia of b about Q. Their moment about Q is therefore $I\boldsymbol{\alpha}_r$.

Total Accelerations. The total moment of b's inertia forces about Q is therefore

$$\int \mathbf{r}\times\mathbf{a}\,dm = m\omega^2\mathbf{p}^*\times\mathbf{r}^* + m\boldsymbol{\alpha}\,\mathbf{p}^*\cdot\mathbf{r}^* + I^*\alpha + I\boldsymbol{\alpha}_r.$$

In a governor the external forces on b are its weight W at P^*, the reaction of the pin at Q, and the tension T of a spring attached to b and to the flywheel. If $\mathbf{M}$ denotes the moment of $\mathbf{T}$ about Q, we have from the basic Theorem II

$$(1) \qquad m\omega^2\mathbf{p}^*\times\mathbf{r}^* + m\boldsymbol{\alpha}\,\mathbf{p}^*\cdot\mathbf{r}^* + I^*\boldsymbol{\alpha} + I\boldsymbol{\alpha}_r = \mathbf{M} + \mathbf{r}^*\times\mathbf{W}.$$

Suppose, now, that the moment of the weight is small in comparison with M, so that it may be neglected. When the flywheel is running at constant speed, α and α_r are zero and

$$(2) \qquad m\omega^2\mathbf{p}^*\times\mathbf{r}^* = \mathbf{M},$$

that is, the moment of the spring tension is just equal to that of the normal inertia forces. As the wheel begins to accelerate the tangential inertia forces come into play, and from (1) and (2)

$$m\alpha\,\mathbf{p}^*\cdot\mathbf{r}^* + I^*\alpha + I\alpha_r = 0,$$

$$(3) \qquad \alpha_r = -\frac{\mathbf{p}^*\cdot\mathbf{r}^* + k^{*2}}{k^2}\,\alpha.$$

This gives the *initial* angular acceleration α_r; after ω has changed, α_r is given by (1).

The first three terms in (1) represent the moments of inertia forces about Q arising from the body accelerations, namely $-m\omega^2\mathbf{p}^*$ and $m\boldsymbol{\alpha}\times\mathbf{p}^*$ at P^*, and the couple $I^*\boldsymbol{\alpha}$. The sense of these moments depends on the location of P^*. Draw a circle on QO as diameter; this circle and the line QO produced divide the plane into four regions marked I, II, III, IV. Now $\mathbf{M}$ is balanced by the moments of the *reversed* inertia forces; the sense of these moments for different positions of P^* is given in the following table.

	Positive α			
Location of P^*	I	II	III	IV
$-m\omega^2\mathbf{p}^*\times\mathbf{r}^*$	+	+	−	−
$-m\boldsymbol{\alpha}\,\mathbf{p}^*\cdot\mathbf{r}^*$	−	+	+	−
$-I^*\boldsymbol{\alpha}$	−	−	−	−

When the wheel is running at constant speed the moment of the centrifugal force (first line of table) must be balanced by the moment of the spring tension. But when the wheel accelerates, the moments of the tangential forces (second and third lines) produce an unbalance which sets b in motion. These moments have the same sign when P^* is in I or IV; moreover in these regions the initial value of α_r, given by (3), is always opposed to α. The governing action is therefore most powerful when P^* is in I and IV.

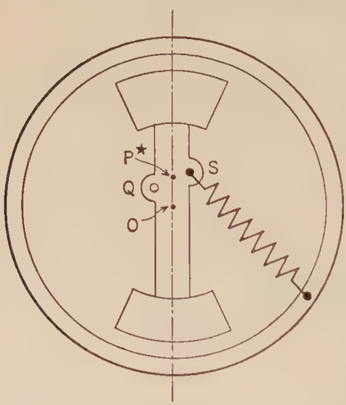


Fig. 209b.

Example. Rites Governor. In this inertia governor (Fig. 209b) the body b is a wide bar with heavy ends with its center of mass P^* near the shaft axis O. This design gives sufficient centrifugal force to fix the mean speed; and the large I^* due to the heavy ends makes the tangential inertia forces very effective in regulation about this speed. The moment $I^*\boldsymbol{\alpha}$ preponderates over $m\boldsymbol{\alpha}\,\mathbf{p}^*\cdot\mathbf{r}^*$ to such an extent that P^* may even enter the circle on OQ as diameter. The spring tension T at constant speed ω may be computed from (2) when its moment arm about Q is given. Since $\mathbf{p}^* = \overrightarrow{OP^*} = \overrightarrow{OQ} + \overrightarrow{QP^*}$, $\mathbf{r}^* = \overrightarrow{QP^*}$, this equation may be written

$$m\omega^2\overrightarrow{OQ}\times\overrightarrow{QP^*} = \overrightarrow{QS}\times\mathbf{T}.$$

210. Kinetics of Plane Motion. Consider a rigid body of mass m moving parallel to a plane, say the xy-plane. If O is a fixed point of the body, the body velocities may be compounded of an instantaneous translation $\mathbf{v}_O$ and an instantaneous rotation ω about an axis Oz through O (§ 124). Hence the motion relative to O (i.e. relative to a body having a translation $\mathbf{v}_O$) is a rotation about the axis Oz. The angular velocity ω and acceleration α of this rotation are defined in § 121; and if $\mathbf{k}$ is a unit vector in $+z$ direction, $\boldsymbol{\omega} = \omega\mathbf{k}$, $\boldsymbol{\alpha} = \alpha\mathbf{k}$.

Choose O so that

$$\mathbf{a}_O \times \overrightarrow{OP^*} = 0. \tag{1}$$

This condition is fulfilled only when

(*a*) $\mathbf{v}_O$ is constant, or
(*b*) O is the center of mass P^*, or
(*c*) $\mathbf{a}_O$ passes through P^*.

Then, according to § 184, the inertia forces of the body may be reduced to a

$$\text{Force } m\mathbf{a}^* \text{ at } O \quad \text{and} \quad \text{Couple of moment } \frac{d\mathbf{H}_O'}{dt},$$

where $\mathbf{H}_O'$ is the moment of relative momentum about O. Since the relative motion is a rotation about Oz

$$\mathbf{H}_O' = I\boldsymbol{\omega} - \omega \int \mathbf{p}z\, dm \tag{2}$$ (§ 198, 1)

where I is the moment of inertia of the body about Oz. If, in particular, Oz is a principal axis of inertia,

$$\mathbf{H}_O' = I\boldsymbol{\omega}, \quad \frac{d\mathbf{H}_O'}{dt} = I\boldsymbol{\alpha}.$$

We state these results in the

THEOREM. *If O is a point of a body in plane motion such that $\mathbf{a}_O \times \overrightarrow{OP^*} = 0$, and the axis Oz is a principal axis of inertia at O, then the inertia forces may be reduced to a*

Force $m\mathbf{a}^*$ at O and Couple of moment $I\boldsymbol{\alpha}$.

By D'Alembert's Principle the inertia forces are equivalent to the external forces. Hence if $\mathbf{F}$ and $\mathbf{M}_O$ represent the force-sum and moment-sum of all the external forces acting on the body, we

have the dynamical equations

$$(3), (4) \qquad m\mathbf{a}^* = \mathbf{F}, \quad I\boldsymbol{\alpha} = \mathbf{M}_O.$$

In practice O is usually taken at the center of mass; then

$$m\mathbf{a}^* = \mathbf{F}, \quad I^*\boldsymbol{\alpha} = \mathbf{M}^*.$$

Equation (4) is only valid when Oz is a principal axis of inertia at O. This is true in the important Cases 1 and 2 of § 198. But even when Oz is not a principal axis of inertia, we have the scalar equation

$$(5) \qquad I\alpha = M_z$$

to determine the rotation. For when (1) is satisfied,

$$\frac{d\mathbf{H}_O'}{dt} = \mathbf{M}_O \quad \text{and} \quad \mathbf{k}\cdot\frac{d\mathbf{H}_O'}{dt} = \mathbf{k}\cdot\mathbf{M}_O = M_z;$$

and from (2)

$$\mathbf{k}\cdot\mathbf{H}_O' = I\omega \quad \text{so that} \quad \mathbf{k}\cdot\frac{d\mathbf{H}_O'}{dt} = I\alpha.$$

To determine the motion of the body and the reactions upon it, the two vector equations (3), (4) are needed. These, of course, are equivalent to six scalar equations. But if the motion alone is required, three scalar equations will suffice:

$$(6) \qquad ma_x^* = F_x, \quad ma_y^* = F_y, \quad I\alpha = M_z.$$

The first two are formed by taking x- and y-components of (3), the third by taking z-components of (4). The last equation is the same as (5), and is therefore valid for any axis Oz on which condition (1) is satisfied — in particular the axis through the center of mass.

An excellent way of solving problems in plane kinetics is as follows:

Draw a free-body diagram showing all the external forces and the inertia forces replaced by the force $m\mathbf{a}^*$ *and the couple* $I^*\boldsymbol{\alpha}$ *at the center of mass. Then express the equivalence of the inertia and external forces by taking components in any direction or moments about any axis so that the resulting equations are as simple as possible.*

In the free-body diagrams that follow, the external and inertia forces will be shown as vectors drawn full or in dashes respectively. Couples are indicated by directed arcs,

Example 1. *Solid of Revolution Rolling Down an Inclined Plane.* Assume that the axis of the solid (through the center of gravity G) remains horizontal as it rolls down a plane inclined β to the horizontal. Let r be the radius of the circle of contact and k the radius of gyration about the axis. Since the solid is turning about the line of contact as an instantaneous axis, the velocity of G is $r\omega$ and its acceleration is

$$a = r\frac{d\omega}{dt} = r\alpha.$$

The inertia forces reduce to the force $m\mathbf{a}$ at G and the couple $I^*\boldsymbol{\alpha}$. The external forces are the weight W and the reaction of the plane having normal and tangential components N, F. The friction F must be adequate to prevent slipping.

By resolving parallel and perpendicular to the plane, and by taking moments about G, we obtain

$$W \sin\beta - F = \frac{W}{g}a, \quad N - W\cos\beta = 0, \quad Fr = \frac{W}{g}k^2\alpha.$$

Since $\alpha = a/r$,

$$F = \frac{W}{g}\frac{k^2}{r^2}a$$

and on substituting this in the first equation we find

$$a = \frac{g \sin\beta}{1 + k^2/r^2}. \tag{7}$$

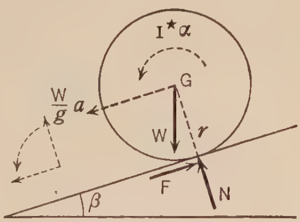


FIG. 210*a*.

Thus a is less than $g \sin\beta$, the acceleration in sliding down a *smooth* plane (§ 157, Ex. 3). For a

	Sphere	Cylinder	Cylindrical Shell
$k^2/r^2 =$	$\frac{2}{5}$	$\frac{1}{2}$	1
$a =$	$\frac{5}{7}\, g \sin\beta$	$\frac{2}{3}\, g \sin\beta$	$\frac{1}{2}\, g \sin\beta$.

With the above value of a

$$F = \frac{W \sin\beta}{1 + r^2/k^2}.$$

As $F > 0$, the friction has the direction shown in the figure, namely up the plane. In order that the motion be pure rolling (no slipping at the point of contact), $F \leqq \mu N$, that is

$$\frac{W \sin\beta}{1 + r^2/k^2} \leqq \mu W \cos\beta \quad \text{or} \quad \mu \geqq \frac{\tan\beta}{1 + r^2/k^2}.$$

Example 2. *Wheel Set in Motion by a Force.* In Fig. 210*b* the force P is applied at the axis of the wheel. By resolving horizontally, vertically, and by taking moments about the center G, we obtain

$$P - F = \frac{W}{g}a, \quad N - W = 0, \quad Fr = \frac{W}{g}k^2\alpha,$$

where k is the radius of gyration about the axis. If α is positive and the wheel (radius r) rolls without slipping, the acceleration of G is $a = r\alpha$ towards the left. Then $\alpha = a/r$,

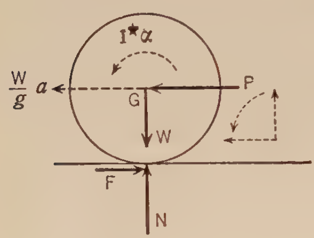


FIG. 210*b*.

$$F = \frac{W}{g}\frac{k^2}{r^2}a,$$

and from the first equation

$$a = \frac{P}{W}\frac{g}{1 + k^2/r^2}. \tag{8}$$

With this value of a we find

$$F = \frac{P}{1 + r^2/k^2}.$$

As $F > 0$ the friction has the direction shown in the diagram, namely opposed to P. In order that there be no slipping, $F \leqq \mu N$, that is

$$\frac{P}{1 + r^2/k^2} \leqq \mu W \quad \text{or} \quad \mu \geqq \frac{P/W}{1 + r^2/k^2}.$$

Example 3. *Wheel Set in Motion by a Couple.* In Fig. 210*c* the external couple C exerts a positive moment on the axle of the wheel. If C is powerful the wheel will tend to slip backward at the point of contact. If the motion is pure rolling, we suppose that this tendency is resisted by friction F acting to the left. We shall test this assumption after we have solved the problem.

By resolving horizontally, vertically, and by taking moments about G, we obtain

$$F = \frac{W}{g}a, \quad N - W = 0, \quad C - Fr = \frac{W}{g}k^2\alpha.$$

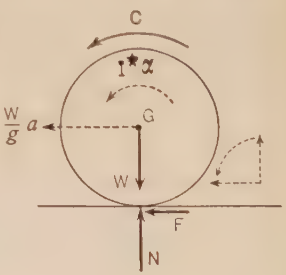


FIG. 210*c*.

If α is positive and the wheel rolls without slipping, the acceleration of G is $a = r\alpha$ towards the left. If we put $\alpha = a/r$ and eliminate F from the first and third equations we find

$$a = \frac{C}{W}\frac{g}{r + k^2/r}. \tag{9}$$

With this value of a the first equation gives

$$F = \frac{C}{r + k^2/r}.$$

As $F > 0$, the friction has the assumed direction, namely the direction in which G moves. In order that the wheel roll without skidding, $F \leqq \mu N$, that is

$$\frac{C}{r + k^2/r} \leqq \mu W \quad \text{or} \quad \mu \geqq \frac{C/W}{r + k^2/r}.$$

This problem shows that the external driving force on a self-propelled vehicle, such as locomotive or an automobile, is the forward friction of the track on the driving wheels. Although this friction is brought into action by the turning effort of engine, this force *alone, internal* to the vehicle as a whole, is powerless to set it in motion relative to track. Thus when the friction is too small the wheels will slip and no forward motion occurs; we then say that there is not enough *traction.*

On the other hand when a car is moved by an external pull, as that exerted at the coupling of a railway car, the friction between wheels and track acts as a resistance. This is clear from Example 2.

Example 4. *Reel with String about Axle.* Let the string, wound about the axle of the reel, be pulled horizontally from its lower side with a force P (Fig. 210*d*). Assume that α is positive (counterclock-wise) and that the friction F opposes P. If these directions are incorrect, α and F will be negative in our solution.

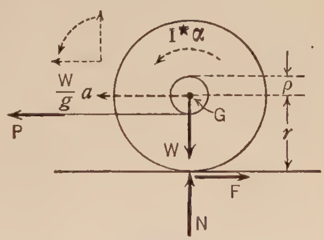


FIG. 210*d*.

The figure shows the inertia and the external forces. By resolving horizontally, vertically, and taking moments about the center, we obtain

$$P - F = \frac{W}{g}a, \quad N - W = 0, \quad Fr - P\rho = \frac{W}{g}k^2\alpha$$

where r, ρ are the radii of reel and axle. If the reel rolls without slipping $\alpha = a/r$; and on eliminating F from the first and third equations we find

$$a = \frac{P}{W}\frac{r(r - \rho)}{r^2 + k^2}g.$$

Since $r > \rho$, a is positive and the wheel will move to the left as assumed above. If the reader doubts this result, he may test it experimentally, using a string wound about a spool.

With this value of a the first equation gives

$$F = P\frac{k^2 + \rho r}{k^2 + r^2}.$$

Since F is positive its assumed direction is correct.

When $\rho = 0$, the results above agree with those of Example 2.

Example 5. *Rocking Pendulum.* A body, consisting in part of a circular cylinder of radius r which rolls without slipping on a horizontal support, has its center of gravity G below the axis O of the cylinder and at a distance b from O (Fig. 210e). If the body is slightly shifted from its position of equilibrium, it will vibrate to and fro under the action of gravity. We shall find the period of these *small* vibrations.

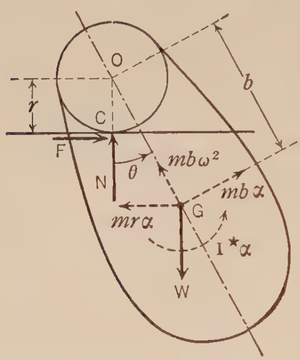


FIG. 210e.

When α is positive, the acceleration a_O of O is $r\alpha$ horizontally to the left. The acceleration of G is $\mathbf{a}^* = \mathbf{a}_O + \mathbf{a}_{GO}$, where $\mathbf{a}_{GO}$ is the acceleration of G in a rotation about O, namely $b\omega^2$ toward O and $b\alpha$ perpendicular to OG (§ 127). Hence $m\mathbf{a}^*$ is the sum of the three vectors drawn in dashes at G; their magnitudes are $mr\alpha$, $mb\omega^2$, $mb\alpha$.

Now the inertia forces of the pendulum reduce to the force $m\mathbf{a}^*$ at G and the couple $I^*\boldsymbol{\alpha} = mk^2\boldsymbol{\alpha}$, where k is the radius of gyration about an axis through G. The external forces are the weight $\mathbf{W}$ at G and the reaction $[F, N]$ at C.

By taking moments about C we obtain

$$mk^2\alpha - mr\alpha(b\cos\theta - r) + mb\alpha(b - r\cos\theta) + mb\omega^2 r\sin\theta = -Wb\sin\theta,$$

or on putting $W = mg$ and collecting terms

$$(k^2 - 2br\cos\theta + r^2 + b^2)\alpha + br\omega^2\sin\theta = -gb\sin\theta.$$

As θ remains small, we have $\cos\theta = 1$, $\sin\theta = \theta$ approximately; and since ω is small, the term in $\omega^2\sin\theta$ is small to the third order and may be neglected. With these approximations

$$[k^2 + (b - r)^2]\alpha = -gb\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{gb}{k^2 + (b - r)^2}\theta.$$

This is the approximate differential equation of a simple pendulum of length

$$l = \frac{k^2 + (b - r)^2}{b} \qquad (§ 171).$$

The period of the pendulum for small vibrations is therefore $2\pi\sqrt{g/l}$ approximately.

Example 6. Rod Sliding in a Vertical Plane between a Smooth Wall and Floor. Let the rod AB be of length $2b$; then the square of its radius of gyration about G is $\frac{1}{3}b^2$. The reactions P, Q at the smooth supports are horizontal and vertical respectively.

As long as the rod is in contact with the wall, $OG = b$; hence G moves in circle of radius b about O as center. Moreover ω and α for OG are numerically the same as for the rod. The inertia forces thus reduce to the force $[mb\omega^2, mb\alpha]$ shown at G and the couple $I^*\boldsymbol{\alpha}$.

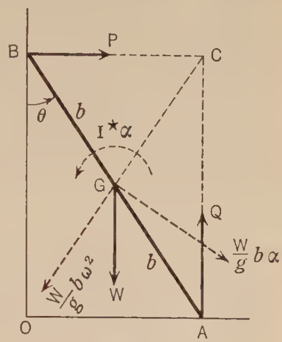


FIG. 210*f*.

On taking moments about C we have

$$Wb \sin\theta = \frac{W}{g}\frac{b^2}{3}\alpha + \frac{W}{g}b^2\alpha, \qquad \text{or}$$

$$\text{(i)} \qquad \alpha = \tfrac{3}{4}\frac{g}{b}\sin\theta.$$

This is precisely the equation of motion of the rod when the wall is removed and the point A fixed (physical pendulum); for the *reduced length* of the rod is

$$l = b + \frac{\frac{1}{3}b^2}{b} = \tfrac{4}{3}b \qquad (§\ 197,\ 2).$$

The rod will leave the wall when $P = 0$. To find out if this occurs we resolve horizontally; thus

$$P = \frac{W}{g}b(\alpha\cos\theta - \omega^2\sin\theta).$$

Let θ_0 be the initial value of θ. Then on writing $\alpha = \omega\, d\omega/d\theta$ in (i) and integrating from θ_0 to θ we obtain

$$\text{(ii)} \qquad \omega^2 = \tfrac{3}{2}\frac{g}{b}(\cos\theta_0 - \cos\theta).$$

In view of (i) and (ii) we obtain

$$P = \tfrac{3}{4}W\sin\theta(3\cos\theta - 2\cos\theta_0).$$

Hence $P = 0$ when $\cos\theta = \frac{2}{3}\cos\theta_0$. P moreover changes sign as θ passes through the angle $\cos^{-1}(\frac{2}{3}\cos\theta_0)$, showing that the rod leaves the wall at this point. Thereafter the motion of the rod is of a different character.

PROBLEMS

1. A solid cylinder weighing 120 lb. rolls up a plane inclined at an angle of 30° to the horizontal. A cord, attached to the axis of the cylinder and parallel to the plane, runs over a smooth peg at the top, and has its other end fastened to a weight of 140 lb. hanging freely. Find the acceleration of the cylinder and the tension of the cord.

2. A cord, attached to a hook in the ceiling, runs vertically downward for a distance and is then wrapped about the curved surface of a solid disk of weight W and radius r. If the cylinder is allowed to fall, show that its acceleration and the tension in the cord are $a = \frac{2}{3}g$ and $T = \frac{1}{3}W$.

3. If in Fig. 210*d* the cord is wrapped about the axle in the reverse direction and pulled horizontally from its *upper* side with a force P, find the acceleration a and the friction F if no slipping occurs.

When $P = 5$ lb., find a and F if the reel consists of two cylindrical disks, each 2 ft. in diameter and weighing 10 lb., joined by an axle $\frac{1}{2}$ ft. in diameter and weighing 20 lb. (between disks). What is the direction of F?

4. A steel pipe lies on a flat-car with its axis perpendicular to the track. When the car is given an acceleration a, show that the pipe will have the acceleration $\frac{1}{2}a$ relative to the track if it rolls without slipping. (Treat the pipe as a thin cylindrical shell.)

5. A cylinder and sphere, having equal weights and radii, roll in contact down a plane inclined at an angle 30° to the horizontal. Which body must lead? If the coefficient of friction between the bodies is $\mu = 0.2$, find their common acceleration and the normal pressure between them.

6. In Fig. 210*g* the reel is the same as that described in Problem 3. The cord, unwound from below, runs parallel to the plane and is fastened above.

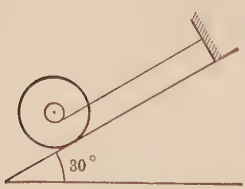


FIG. 210*g*.

(*a*) If the reel cannot slip on the plane, show that it will remain at rest. Compute the tension T of the cord.

(*b*) If the reel slips and $\mu = 0.1$ between reel and plane, compute α, ω, and T

7. In Fig. 210*g* let the radii of reel and axle be r, ρ and the inclination of the plane β. Show that the reel will slip if

$$\mu < \rho \tan \beta/(r - \rho).$$

8. A car having two pairs of wheels is driven by an electric motor exerting a torque on the front axle. The weight of the whole car is W; each pair of wheels is of weight w, radius r, and radius of gyration k. If the motor exerts the torque C in starting, prove that the

acceleration of the car and the friction in its front wheels are

$$a = \frac{Cr}{Wr^2 + 2\,wk^2}\,g, \quad F = \frac{C}{r}\,\frac{Wr^2 + wk^2}{Wr^2 + 2\,wk^2}.$$

211. Energy Equation in Plane Motion. The kinetic energy of a body in plane motion is equal to $\frac{1}{2}\,mv^{*2}$ plus the kinetic energy of its motion relative to P^* (§ 185). This relative motion is a rotation and has the kinetic energy $\frac{1}{2}\,I^*\omega^2$ (§ 191, 2), where I^* is the moment of inertia of the body about the axis P^*z. Hence the total

(1) $$\text{Kinetic Energy} = \tfrac{1}{2}\,mv^{*2} + \tfrac{1}{2}\,I^*\omega^2.$$

The Principle of Work and Energy for a rigid body now states that the change in kinetic energy between the instants t_1 and t_2 is equal to the work done by the external forces in this interval:

(2) $$\tfrac{1}{2}\,mv_2^{*2} + \tfrac{1}{2}\,I^*\omega_2^2 - \tfrac{1}{2}\,mv_1^{*2} - \tfrac{1}{2}\,I^*\omega_1^2 = W_{12}.$$

If the exte nal forces $\mathbf{F}_i$ act on the body at the points P_i having position vectors $\mathbf{r}_i$ relative to a fixed point O of the body, the velocity of P_i is $\mathbf{v}_O + \boldsymbol{\omega}\times\mathbf{r}_i$ (§ 123, 2) and the total work

$$W_{12} = \sum\int_{t_1}^{t_2} \mathbf{F}_i\cdot(\mathbf{v}_O + \boldsymbol{\omega}\times\mathbf{r}_i)\,dt \qquad (§ 163, 1)$$

$$= \sum\int_{t_1}^{t_2} \mathbf{F}_i\cdot\mathbf{v}_O\,dt + \sum\int_{t_1}^{t_2} \mathbf{r}_i\times\mathbf{F}_i\cdot\boldsymbol{\omega}\,dt.$$

$$= \int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}_O\,dt + \int_{t_1}^{t_2} M_z\,\omega\,dt$$

where $\mathbf{F}$ and M_z are the force-sum and moment-sum about the axis Oz. We may put

$$\mathbf{v}_O\,dt = \mathbf{T}_O\,ds, \quad \omega\,dt = d\theta$$

in these integrals if the forces $\mathbf{F}_i$ depend only upon the position:

(3) $$W_{12} = \int_{s_1}^{s_2} \mathbf{F}\cdot\mathbf{T}_O\,ds + \int_{\theta_1}^{\theta_2} M_z\,d\theta.$$

In the first integral the tangential component of $\mathbf{F}$ is integrated over the path of O.

The values of the integrals in (3) depend on the choice of O; of course their *sum* is independent of this choice. When O is taken at the center of mass the first integral gives the change

$$\tfrac{1}{2}\,mv_2^{*2} - \tfrac{1}{2}\,mv_1^{*2}$$

in translational energy, the second the change

$$\tfrac{1}{2} I^*\omega_2^2 - \tfrac{1}{2} I^*\omega_1^2$$

in rotational energy. This is readily proved by multiplying the equations

$$m\mathbf{a}^* = \mathbf{F}, \quad I^*\alpha = \mathbf{M}^*$$

by $\mathbf{v}^*\cdot$ and $\boldsymbol{\omega}\cdot$ and integrating.

In applying the energy equation (2) it is usually simplest to compute the work of each force separately and add the results to obtain W_{12}.

We shall now apply the energy equation to the examples of § 210 in turn.

Example 1. *Solid of Revolution Rolling Down an Inclined Plane.* When the center G travels a distance x down the plane it falls a vertical distance $x \sin \beta$ and the work of gravity is $Wx \sin \beta$ (§ 189). The reaction $\mathbf{R} = [F, N]$ of the plane acts at the instantaneous center C and hence the

$$\text{Work of } \mathbf{R} = \int \mathbf{R}\cdot\mathbf{v}_c \, dt = 0.$$

Or we may reason as follows. N obviously does no work; and if we introduce a pair of forces $\mathbf{F}$, $-\mathbf{F}$ at G, the force $\mathbf{F}$ at C is equivalent to a force $\mathbf{F}$ at G and a couple of moment Fr. The work of $\mathbf{F}$ at G is $-Fx$; and as the wheel turns through x/r radians as G moves a distance x, the work of the couple is $Fr\cdot x/r = Fx$ (§ 191, 4). Since the work of force and couple cancel, the friction does no work.

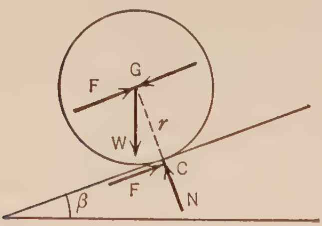


FIG. 211.

If the body starts from rest, the energy equation (2) becomes

$$\tfrac{1}{2}\frac{W}{g}k^2\omega^2 + \tfrac{1}{2}\frac{W}{g}v^2 = Wx \sin \beta$$

where $v = r\omega$ is the velocity of G. Hence

$$\left(\frac{k^2}{r^2} + 1\right)v^2 = 2\,gx \sin \beta,$$

and on differentiating with respect to t,

$$\left(\frac{k^2}{r^2} + 1\right) 2\,va = 2\,gv \sin \beta.$$

This gives the equation (§ 210. 7) for a

To obtain the change in rotational energy alone, we note that the only external moment about G is Fr; hence

$$\tfrac{1}{2}\frac{W}{g}k^2\left(\frac{v}{r}\right)^2 = Fr\cdot\frac{x}{r} = Fx,$$

and on differentiating with respect to t,

$$\frac{W}{g}\frac{k^2}{r^2}a = F$$

in agreement with § 210, Example 1.

Example 2. Wheel Set in Motion by a Force (Fig. 210*b*). As G moves a distance x from rest, the force P does the work Px. The energy equation is therefore

$$\tfrac{1}{2}\frac{W}{g}k^2\omega^2 + \tfrac{1}{2}\frac{W}{g}v^2 = Px, \qquad \text{or}$$

$$\left(\frac{k^2}{r^2} + 1\right)v^2 = 2\frac{P}{W}gx.$$

On differentiating with respect to t we get (§ 210, 8).

Example 3. Wheel Set in Motion by a Couple (Fig. 210*c*). As G moves a distance x the wheel turns through x/r radians and the work of the couple C is Cx/r. The equation of energy is therefore

$$\tfrac{1}{2}\frac{W}{g}k^2\omega^2 + \tfrac{1}{2}\frac{W}{g}v^2 = C\frac{x}{r}, \qquad \text{or}$$

$$\left(\frac{k^2}{r^2} + 1\right)v^2 = 2\frac{C}{W}\frac{gx}{r}.$$

On differentiating with respect to t we get (§ 210, 9).

Example 4. Reel with String about Axle (Fig. 210*d*). To find the work done by **P** introduce the forces **P**, $-$**P** at G; we then have a force **P** at G and couple of moment $-P\rho$. As G moves a distance x from rest, the work done is $Px - P\rho\cdot x/r$ and the equation of energy is

$$\tfrac{1}{2}\frac{W}{g}k^2\omega^2 + \tfrac{1}{2}\frac{W}{g}v^2 = Px\left(1 - \frac{\rho}{r}\right).$$

On differentiating this with respect to t we obtain the equation for a of § 210, Example 4.

Example 5. Rocking Pendulum (Fig. 210*e*). As the pendulum falls from the angle θ_0 to θ the work of gravity is $Wb\,(\cos\theta - \cos\theta_0)$. The speed of G is $\omega\cdot CG$ since C is the instantaneous center; and if θ is small CG is nearly equal to $b - r$. With this degree of approximation the equation of energy is

$$\tfrac{1}{2}\frac{W}{g}k^2\omega^2 + \tfrac{1}{2}\frac{W}{g}(b - r)^2\omega^2 = Wb(\cos\theta - \cos\theta_0), \qquad \text{or}$$

$$[k^2 + (b - r)^2]\omega^2 = 2\,gb(\cos\theta - \cos\theta_0).$$

On differentiating this with respect to t we get the equation of motion obtained in § 210, Example 5.

Example 6. *Rod Sliding in a Vertical Plane between a Smooth Wall and Floor* (Fig. 210*f*). The energy equation is

$$\frac{1}{2}\frac{W}{g}\frac{b^2}{3}\omega^2 + \frac{1}{2}\frac{W}{g}(b\omega)^2 = Wb(\cos\theta_0 - \cos\theta).$$

This gives equation (ii) of § 210, Example 6.

PROBLEMS

1. Find the acceleration of the cylinder in Problem 1, § 210 by means of the energy equation.

2. Find the acceleration of the disk in Problem 2, § 210 by means of the energy equation.

3. Find the acceleration of the reel in Problem 3, § 210 by means of the energy equation.

4. In Problem 4, § 210, the friction between car and pipe is $F = m(a - r\alpha)$, where m is the mass of the pipe, α its angular acceleration. Making use of this fact apply the energy equation to find α. [The work done by F in t sec. is $\int_0^t Fv\,dt$, where v is the velocity of the car.]

5. By means of the energy equation find the common acceleration of the cylinder and sphere in Problem 5, § 210, when both are regarded as *smooth*.

6. Find the acceleration of the car in Problem 8, § 210, by means of the energy equation.

212. Rolling Resistance. Let a wheel roll along a level roadway under the action of no forces except its weight W and the reaction R of the road. If C is the point of contact with the road we have from (§ 210, 4)

$$(1) \qquad I_C\alpha = M_C;$$

for since the acceleration of C passes through the center of the wheel (§ 128, Ex. 2), C may be used as center of moments. Now $M_C = 0$ since both W and R pass through C; hence $\alpha = 0$ and ω is constant. Thus the wheel should keep on rolling with undiminished speed. We know however that the speed of the wheel will gradually decrease until it comes to rest. This discrepancy between theory and experiment is due to the fact that neither wheel nor roadway is rigid. Both are deformed in the vicinity of their contact and moreover the contact is not along a line as assumed above, but over an area. As a result of these deforma-

tions it may be shown that some slipping always accompanies "rolling." The kinetic energy of the wheel is thus diminished by the work dissipated in this sliding friction.

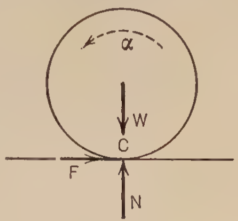


FIG. 212.

It is simplest, however, to adhere to the assumption that the wheel is in line contact with the roadway at C and take the rolling resistance into account by introducing a resisting moment about the line of contact. If N is the normal reaction of the roadway on the wheel, this moment may be written as fN where f has the dimensions of length. We may thus simulate the effect of rolling resistance by shifting N forward in the direction of motion so that it has the lever-arm f about C. The length f, the so-called "*coefficient*" *of rolling friction*, is usually assumed to be independent of the radius of the wheel, although this is not borne out by experiment. For ordinary steel wheels rolling on steel rails f is about 0.02 inch.

Example. A pair of wheels of weight W and radius r rolls down a plane inclined at angle β to the horizontal. If we take rolling resistance into account, the moment equation (1) about C becomes

$$I_C\alpha = Wr \sin \beta - Nf = W(r \sin \beta - f \cos \beta).$$

The wheels will roll at constant speed when $\alpha = 0$, that is, when

$$r \sin \beta - f \cos \beta = 0 \quad \text{or} \quad \tan \beta = \frac{f}{r}.$$

This gives an experimental method of finding f.

213. Kinematics of a Rigid Body. We shall now investigate the most general motion of a free rigid body by means of the following

PRINCIPLE. *The velocities of any two points of a rigid body have equal projections on the line joining the points.*

Proof. If O_1 is an origin fixed in space, and A, B points of the body, $\overrightarrow{AB} = \overrightarrow{O_1B} - \overrightarrow{O_1A}$; hence

$$\frac{d}{dt}\overrightarrow{AB} = \mathbf{v}_B - \mathbf{v}_A.$$

Since the length AB is constant, the derivative of $\overrightarrow{AB}$ is perpendicular to AB (§ 84); hence on taking projections of both members on AB,

$$0 = \text{proj}_{AB}\mathbf{v}_B - \text{proj}_{AB}\mathbf{v}_A.$$

THEOREM 1. *The velocities of all points of a rigid body are determined by the velocities of any three of its points not in a straight line.*

Proof. Let A, B, C be the three non-collinear points of the body whose velocities are known. Then if P is any point of the body not in the plane ABC, $\mathbf{v}_P$ is uniquely determined by its known projections on AP, BP, CP:

$$\text{proj}_{AP}\mathbf{v}_P = \text{proj}_{AP}\mathbf{v}_A, \text{ etc.}$$

Knowing $\mathbf{v}_P$ we may find the velocity of any point in the plane ABC by the same method.

From this theorem we conclude that if $\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C$ are the same in any two motions of a rigid body, the two motions are identical.

THEOREM 2. *If, at any instant, three points of a rigid body, not in a straight line, have the same velocity, the motion is an instantaneous translation.*

Proof. The three points have the same velocities as in a translation of the body. The motion therefore *is* an instantaneous translation by Theorem 1.

If, in particular, the three points have zero velocity, the body is at rest for the instant. This case is excluded in the following Theorems.

THEOREM 3. *If, at any instant, two points of a rigid body, have zero velocity, the motion is an instantaneous rotation about an axis through the points.*

Proof. Let $\mathbf{v}_A = 0$, $\mathbf{v}_B = 0$. Then if C is any point not on the line AB, the projection of $\mathbf{v}_C$ on both AC and BC is zero, and $\mathbf{v}_C$ must be normal to the plane ABC (excluding $\mathbf{v}_C = 0$). The points A, B, C then have the same velocities as in a certain rotation of the body about AB. Hence, by Theorem 1, the velocities of all points are the same as in this rotation.

THEOREM 4. *If, at any instant, a point of a rigid body has zero velocity, the motion is an instantaneous rotation about an axis through this point.*

Proof. Let $\mathbf{v}_A = 0$. If B is a second point such that $\mathbf{v}_B \neq 0$,

$$\text{proj}_{AB}\mathbf{v}_B = 0 \quad \text{and} \quad \mathbf{v}_B \perp AB.$$

Pass a plane through AB normal to $\mathbf{v}_B$ (Fig. 213). Take a point C not in this plane such that $\mathbf{v}_C \neq 0$; then

$$\text{proj}_{AC}\mathbf{v}_C = 0 \quad \text{and} \quad \mathbf{v}_C \perp AC.$$

Pass a plane through AC normal to $\mathbf{v}_C$ and let it cut the first plane in the line AD. Then the projections of $\mathbf{v}_D$ on the non-coplanar lines AD, BD, CD are all zero and consequently $\mathbf{v}_D = 0$. Hence, by Theorem 3, the motion is an instantaneous rotation about the axis AD.

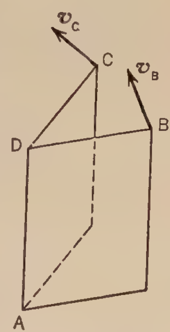


FIG. 213.

If a rigid body has one point O fixed, this theorem states that the instantaneous velocities of its points P are the same as in a certain rotation about an axis through O. Hence from (§ 109, 4) there is a vector $\boldsymbol{\omega}$ at every instant such that

(1) $$\mathbf{v}_P = \boldsymbol{\omega} \times \overrightarrow{OP} \qquad \text{for all points } P.$$

As the motion proceeds $\boldsymbol{\omega}$ varies, in general, both in magnitude *and direction*. With a fixed axis of rotation

$$\boldsymbol{\omega} = \frac{d\theta}{dt}\mathbf{e} = \frac{d}{dt}(\theta\mathbf{e})$$

where $\mathbf{e}$ is a unit vector along the axis; then $\boldsymbol{\omega}$ may be regarded as the time rate of change of the vector angle $\theta\mathbf{e}$. But with a variable axis of rotation $\boldsymbol{\omega}$ can no longer be expressed as the time derivative of a vector angle. Nevertheless $\boldsymbol{\omega}$ is still called an " angular velocity " in this case since (1) and (§ 109, 4) are identical in form.

THEOREM 5. *If O is any point of a free rigid body, the velocities of its points are the same as if they were compounded of an instantaneous translation* $\mathbf{v}_0$ *and an instantaneous rotation* $\boldsymbol{\omega}$ *about an axis through O; and* $\boldsymbol{\omega}$ *is the same for any choice of O.*

Proof. The motion of the body relative to a frame of reference having a translation of velocity $\mathbf{v}_O$ is an instantaneous rotation about an axis through O (Theorem 4); for O has zero velocity relative to this frame. The absolute velocity of any point P of the body is therefore the sum of $\mathbf{v}_O$ and its velocity in the relative rotation (§ 110).

If $\boldsymbol{\omega}$ is the angular velocity of the instantaneous rotation we have

(2) $$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega} \times \overrightarrow{OP}.$$

Moreover for any other point Q

$$\mathbf{v}_Q = \mathbf{v}_O + \boldsymbol{\omega} \times \overrightarrow{OQ}.$$

On subtracting this from (2) we get

$$\mathbf{v}_P - \mathbf{v}_Q = \boldsymbol{\omega}\times(\overrightarrow{OP} - \overrightarrow{OQ}), \qquad \text{or}$$

(3) $$\mathbf{v}_P = \mathbf{v}_Q + \boldsymbol{\omega}\times\overrightarrow{QP}.$$

Thus $\mathbf{v}_P$ may also be compounded of a translation $\mathbf{v}_Q$ and a rotation of the *same angular velocity* $\boldsymbol{\omega}$ about an axis through Q.

If $\mathbf{w} = \overrightarrow{PQ}$ is a vector fixed in a rigid body b having the angular velocity $\boldsymbol{\omega}$, and O_1 is an origin fixed in space,

$$\mathbf{w} = \overrightarrow{O_1Q} - \overrightarrow{O_1P}, \quad \frac{d\mathbf{w}}{dt} = \mathbf{v}_Q - \mathbf{v}_P = \boldsymbol{\omega}\times\overrightarrow{PQ}$$

from (3); hence

(4) $$\frac{d\mathbf{w}}{dt} = \boldsymbol{\omega}\times\mathbf{w}.$$

On referring to § 133 we see that if $\mathbf{u}$ is any variable vector and $\partial\mathbf{u}/\partial t$ is its rate of change relative to b, then its absolute rate of change is

(5) $$\frac{d\mathbf{u}}{dt} = \boldsymbol{\omega}\times\mathbf{u} + \frac{\partial\mathbf{u}}{\partial t}.$$

The Theorem of Coriolis (§ 134, 2), proved before when b had plane motion, may now be extended to the case when b has any motion whatever; for, by virtue of (5), our former proof applies word for word to the more general case.

214. Kinetics of a Rigid Body with One Point Fixed. We shall next study the kinetics of a homogeneous solid of revolution having one point O on its axis of symmetry (z-axis) fixed. This axis, which necessarily passes through the center of mass P^*, is a principal axis of inertia at O (§ 198, Case 3). If x, y are any two axes perpendicular to z and to each other, x and y are also principal axes of inertia at O; for the body is symmetric with respect to both xz and yz planes (§ 198, Case 1). Thus x, y, z form a set of three principal axes of inertia mutually perpendicular to each other.*

Since the body has the point O fixed, its motion at any instant is a rotation about some axis through O with the angular velocity ω (§ 213, Theorem 4). If we express ω as the sum of its projections on the axes,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_x + \boldsymbol{\omega}_y + \boldsymbol{\omega}_z,$$

* It can be shown that there are always three mutually perpendicular axes of inertia at any point of a body.

the moment of momentum about O is

$$\mathbf{H} = \int \mathbf{r}\times\mathbf{v}\, dm = \int \mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r})\, dm = \sum_{x,y,z} \int \mathbf{r}\times(\boldsymbol{\omega}_x\times\mathbf{r})\, dm.$$

Thus $\mathbf{H}$ is expressed as the sum of the angular momenta due to $\boldsymbol{\omega}_x$, $\boldsymbol{\omega}_y$, $\boldsymbol{\omega}_z$. Since x, y, z are *principal* axes at O, these partial angular momenta are given by (§ 198, 2). Therefore

(1) $$\mathbf{H} = A\boldsymbol{\omega}_x + B\boldsymbol{\omega}_y + C\boldsymbol{\omega}_z,$$

where A, B, C are the moments of inertia of the body about the axes x, y, z respectively.

Since the point O is at rest, the inertia forces of the body may be reduced to a

$$\text{Force } m\mathbf{a}^* \text{ at } O \quad \text{and} \quad \text{Couple of moment } \frac{d\mathbf{H}}{dt} \qquad (\S\ 183).$$

If the external forces on the body have the force-sum $\mathbf{F}$ and the moment-sum $\mathbf{M}$ about O, D'Alembert's Principle gives the dynamical equations:

(2), (3) $$m\mathbf{a}^* = \mathbf{F}, \quad \frac{d\mathbf{H}}{dt} = \mathbf{M}.$$

We now choose the x and y axes so that they remain principal axes for which the moments of inertia have the *constant* values A, B as the motion proceeds. This will always be the case when the axes are fixed in the body. But in the case of a solid of revolution, $A = B$, and the condition is fulfilled for any set of axes x, y perpendicular to the z-axis of symmetry. In either case the motion of the trihedral xyz, regarded as a rigid body with the point O fixed, is an instantaneous rotation about some axis through O with the angular velocity $\boldsymbol{\Omega}$. Of course when the axes are fixed in the body, $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

Since

(1) $$\mathbf{H} = [A\omega_x, B\omega_y, C\omega_z],$$

the rate of change of $\mathbf{H}$ relative to the moving trihedral xyz is

$$\frac{\partial \mathbf{H}}{\partial t} = \left[A\frac{d\omega_x}{dt}, B\frac{d\omega_y}{dt}, C\frac{d\omega_z}{dt}\right]$$

and its absolute rate of change is

(4) $$\frac{d\mathbf{H}}{dt} = \frac{\partial \mathbf{H}}{\partial t} + \boldsymbol{\Omega}\times\mathbf{H} \qquad (\S\ 213,\ 5).$$

Hence (3) may be written

$$(5) \qquad \frac{\partial \mathbf{H}}{\partial t} + \Omega \times \mathbf{H} = \mathbf{M}.$$

If we choose P^* as center of moments (instead of the fixed point O) the dynamical equations are

$$m\mathbf{a}^* = \mathbf{F}, \quad \frac{d\mathbf{H}^*}{dt} = \mathbf{M}^*,$$

where $\mathbf{H}^*$ and $\mathbf{M}^*$ both refer to the center of mass P^*(§ 183). If A^*, B^*, C are the principal moments of inertia at P^*,

$$\mathbf{H}^* = [A^*\omega_x, \quad B^*\omega_y, \quad C\omega_z].$$

215. Equation of Energy. Since the velocity of any point of the body is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, the kinetic energy is

$$T = \tfrac{1}{2} \int \mathbf{v}\cdot\mathbf{v}\, dm = \tfrac{1}{2} \int (\boldsymbol{\omega}\times\mathbf{r})\cdot\mathbf{v}\, dm = \tfrac{1}{2}\, \boldsymbol{\omega}\cdot \int \mathbf{r}\times\mathbf{v}\, dm,$$

and $$\frac{dT}{dt} = \int \mathbf{v}\cdot\mathbf{a}\, dm = \int (\boldsymbol{\omega}\times\mathbf{r})\cdot\mathbf{a}\, dm = \boldsymbol{\omega}\cdot \int \mathbf{r}\times\mathbf{a}\, dm.$$

But by definition

$$\mathbf{H} = \int \mathbf{r}\times\mathbf{v}\, dm$$

and hence $$\frac{d\mathbf{H}}{dt} = \int (\mathbf{r}\times\mathbf{a} + \mathbf{v}\times\mathbf{v})\, dm = \int \mathbf{r}\times\mathbf{a}\, dm.$$

Therefore

$$(1) \qquad T = \tfrac{1}{2}\, \boldsymbol{\omega}\cdot\mathbf{H} = \tfrac{1}{2}\, (A\omega_x^2 + B\omega_y^2 + C\omega_z^2) \qquad (\S\ 214,\ 1),$$

$$(2) \qquad \frac{dT}{dt} = \boldsymbol{\omega}\cdot\frac{d\mathbf{H}}{dt} = \boldsymbol{\omega}\cdot\mathbf{M} \qquad (\S\ 214,\ 3).$$

On integrating (2) between the instants t_1 and t_2 we obtain the equation of energy:

$$T_2 - T_1 = \int_{t_1}^{t_2} \mathbf{M}\cdot\boldsymbol{\omega}\, dt.$$

Since $T_2 - T_1$ is the change in kinetic energy, the integral on the right gives the work done by the external forces on the body in this interval.

216. Composition of Angular Velocities. A body revolves about the axis Oz of the system $Oxyz$ with the angular velocity ω' relative to the system. If the axes themselves revolve about

O with the angular velocity $\boldsymbol{\Omega}$, let us find the absolute angular velocity $\boldsymbol{\omega}$ of the body.

A point P of the body has the transferred velocity $\boldsymbol{\Omega}\times\mathbf{r}$ due to the rotation of the axes and the velocity $\boldsymbol{\omega}'\times\mathbf{r}$ relative to the axes. The velocity of P is therefore the sum of these velocities (§ 110):

$$\mathbf{v} = \boldsymbol{\Omega}\times\mathbf{r} + \boldsymbol{\omega}'\times\mathbf{r} = (\boldsymbol{\Omega} + \boldsymbol{\omega}')\times\mathbf{r}.$$

Hence the points of the body have the same velocity as if the body were revolving about O with the absolute angular velocity

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}'.$$

In brief: *Angular velocities about axes through the same point may be compounded by vector addition.*

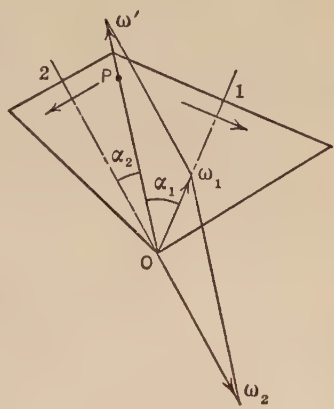


FIG. 216*a*.

Example. Rolling Cones. Fig. 216*a* represents two circular cones *1*, *2* with axes intersecting at O. If cone *1* has the angular velocity $\boldsymbol{\omega}_1$, cone *2* will have an angular velocity $\boldsymbol{\omega}_2$ determined by the condition for pure rolling. Thus if P is a point on the element of contact OP, P will have the same velocity whether regarded as a point of *1* or *2*:

(i) $$\boldsymbol{\omega}_1\times\overrightarrow{OP} = \boldsymbol{\omega}_2\times\overrightarrow{OP}.$$

Hence, as far as magnitudes are concerned,

(ii) $$\omega_1 \sin\alpha_1 = \omega_2 \sin\alpha_2 \quad \text{or} \quad \frac{\omega_1}{\omega_2} = \frac{\sin\alpha_2}{\sin\alpha_1}.$$

The angular velocity of cone *2* relative to cone *1* is $\boldsymbol{\omega}' = \boldsymbol{\omega}_2 - \boldsymbol{\omega}_1$, and since

$$(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1)\times\overrightarrow{OP} = 0 \quad \text{from (i)},$$

$\boldsymbol{\omega}'$ is a vector along the element of contact. Thus the motion of cone *2* relative to *1* is an instantaneous rotation about the element of contact.

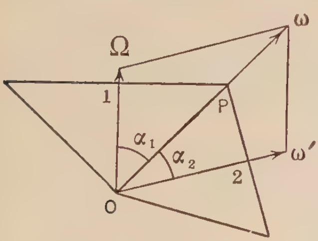


FIG. 216*b*.

Suppose now (Fig. 216*b*) that the cone *1* is fixed and cone *2* is rolling over it with the angular velocity $\boldsymbol{\omega}$. Then $\boldsymbol{\omega}$ is a vector along the element of contact OP. Now the rotation $\boldsymbol{\omega}$ of cone *2* may be compounded

of the rotation $\boldsymbol{\Omega}$ of its axis *02* about *01* and the rotation $\boldsymbol{\omega}'$ of cone *2* relative to its axis; hence

(iii) $$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}'.$$

On multiplying (ii) in turn by $\boldsymbol{\Omega}\times$ and $\boldsymbol{\omega}'\times$ we find

(iv) $$\omega \sin \alpha_1 = \omega' \sin (\alpha_1 + \alpha_2),$$

(v) $$\omega \sin \alpha_2 = \Omega \sin (\alpha_1 + \alpha_2).$$

If the rotation is uniform, cone *2* will make a complete revolution about its axis in the time $2\pi/\omega'$ while its axis will make a complete revolution about *01* in the time $2\pi/\Omega$.

Let the student now draw the figures and set up the corresponding equations for rolling cones in internal contact.

As a numerical example, let $\alpha_1 = 45°$, $\alpha_2 = 30°$ in Fig. 216*b*, and suppose that cone *2* travels around the fixed cone *1* once every second. Then $\Omega = 2\pi$, and from (v)

$$\omega = 2\pi \frac{\sin 75°}{\sin 30°} = 12.13 \text{ rad./sec.};$$

and from (iv)

$$\omega' = 2\pi \frac{\sin 45°}{\sin 30°} = 2\pi\sqrt{2} = 8.89 \text{ rad./sec.}$$

The cone will revolve about its own axis once every $2\pi/\omega' = 0.707$ sec.

217. Gyroscope. A disk or wheel capable of revolving about an axis Oz which is free to move about the fixed point O is called a *gyroscope.*

The simplest problem in gyroscopic motion is as follows. A solid of revolution is spinning with constant angular velocity ω' about its axis Oz, supported at O. *If Oz is horizontal, what external forces acting on the body will cause Oz to revolve in a horizontal plane with the constant angular velocity Ω.*

Draw Oy vertically upward and Ox horizontal so that $Oxyz$ form a r-h system of axes. Since the body is a solid of revolution these axes fulfill the condition of § 214; and

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}', \quad \mathbf{H} = A\boldsymbol{\Omega} + C\boldsymbol{\omega}'.$$

Relative to $Oxyz$, $\mathbf{H}$ is constant; hence

$$\frac{d\mathbf{H}}{dt} = \boldsymbol{\Omega}\times\mathbf{H} = C\boldsymbol{\Omega}\times\boldsymbol{\omega}' \qquad (\S\ 214,\ 4).$$

The dynamical equations (§ 214, 2, 3) are therefore

$$(1),\ (2) \qquad -\frac{W}{g}\Omega^2\mathbf{r}^* = \mathbf{F}, \quad C\Omega\times\omega' = \mathbf{M}.$$

The prescribed motion thus demands that the external forces have the force-sum and moment-sum given by (1) and (2).

The motion is possible when the only external forces are the weight $\mathbf{W}$ of the body and the reaction $\mathbf{R}$ at the support O. For (1) and (2) then become

$$-\frac{W}{g}\Omega^2\mathbf{r}^* = \mathbf{W} + \mathbf{R}, \quad C\Omega\times\omega' = \mathbf{r}^*\times\mathbf{W}.$$

On putting $\mathbf{r}^* = b\mathbf{k}$ and taking components along the axes. we obtain

$$0 = R_x, \quad 0 = R_y - W, \quad -\frac{W}{g}\Omega^2 b = R_z;$$
$$C\Omega\omega' = Wb.$$

The last equation gives Ω; the others then determine $\mathbf{R}$.

The revolution of Oz about the vertical is called a *precession* of angular velocity Ω. Since Ω is constant the precession is said to be *steady*. Equation (2) shows that $\Omega\times\omega'$ and $\mathbf{M}$ have the same direction ($+x$ in the figure); hence

The precession turns the axis of spin toward the torque axis. The directions of Ω, ω', $\mathbf{M}$ are all determined by the rule of the r-h screw.

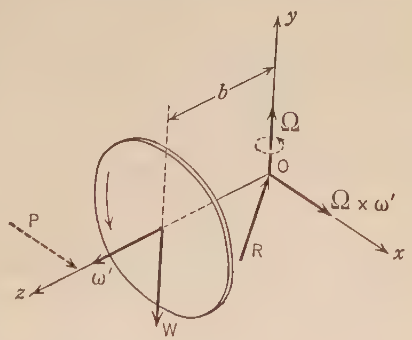


Fig. 217*a*.

This tendency is shown in many gyroscopic phenomena. For the moment equation $d\mathbf{H}/dt = \mathbf{M}$ shows that $d\mathbf{H}$ has the same direction as $\mathbf{M}$. Hence in all cases in which $\mathbf{H}$ has nearly the same direction as the axis of spin, the latter tends to turn toward the torque axis.

Thus in Fig. 217*a* let a horizontal force P be applied to the axis Oz in the direction of precession. This produces a moment about O with the axis Oy; hence Oz will tip upward. If P is reversed Oz will tip downward. A downward force P applied to Oz will

produce a moment about O with the axis Ox; hence Oz will swerve towards Ox.

Example 1. *Gyroscopic Couple in an Airplane.* Let the disk in Fig. 217*a* represent the propeller of an airplane advancing in the direction Oz. If the pilot O wishes to make a left turn (toward Ox) he must set the steering vanes so that a torque about Oy will be applied. Then Oz will tend to approach Oy, that is, the nose will tip upward. In a right turn the nose will tip downward.

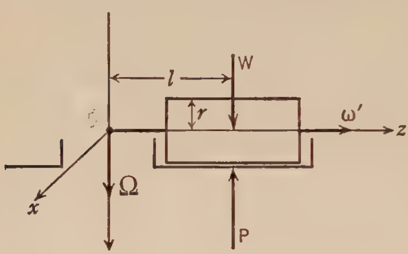


Fig. 217*b*.

Example 2. *Grinding Mill.* In Fig. 217*b* a heavy cylindrical roller of weight W, mounted on a horizontal shaft Oz, is rolled over a base plate by revolving Oz about a vertical axis Oy. When the spin $\boldsymbol{\omega}'$ of the roller is directed outwards, the precession is clockwise when viewed from above and Ω points downward. The inertia forces of the roller have the moment

$$C\boldsymbol{\Omega}\times\boldsymbol{\omega}' = C\Omega\omega'\mathbf{i} \quad \text{at } O.$$

If we assume that pure rolling occurs along the central section of the roller, $\omega' r = \Omega l$; hence the inertia forces have the moment $C\Omega^2 l/r$ about Ox. Setting this equal to the moment of the external forces about Ox we have

$$(P - W)l = C\Omega^2 \frac{l}{r} \quad \text{and} \quad P = W + \frac{C\Omega^2}{r}$$

Thus the pressure due to weight is increased an amount $C\Omega^2/r$ by gyroscopic action. If k is the radius of gyration of the roller about its axis, the pressure

$$P = W\left(1 + \frac{k^2}{gr}\Omega^2\right)$$

and is independent of the distance of the roller from the vertical axis.

Thus if the roller is a hollow cylinder of radii 16 and 12 in., weighing 2000 lb., and $\Omega = 1$ rev./sec.,

$$P = 2000\left(1 + \frac{\frac{1}{2}\left(\frac{16}{9} + 1\right)}{32 \times \frac{4}{3}}\, 4\pi^2\right) = 4560 \text{ lb.}$$

Of this 2560 lb. is due to gyroscopic action.

218. Steady Precession. Let us next determine the conditions for steady precession when the axis of spin $\boldsymbol{\omega}'$ makes a constant angle θ with the axis of precession Ω (Fig. 218a). Take Oz along the axis of spin and Ox horizontal. The body has the spin ω' about Oz relative to $Oxyz$, and the axes revolve about the vertical with the angular velocity

$$\boldsymbol{\Omega} = \Omega \sin \theta \, \mathbf{j} + \Omega \cos \theta \, \mathbf{k}.$$

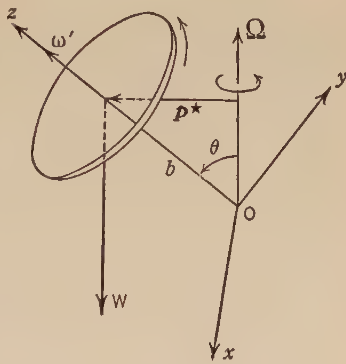


FIG. 218a.

The angular velocity of the body is therefore

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}' = \Omega \sin \theta \, \mathbf{j} + (\Omega \cos \theta + \omega')\mathbf{k}, \quad \text{and}$$

$$\mathbf{H} = A\Omega \sin \theta \, \mathbf{j} + C(\Omega \cos \theta + \omega')\mathbf{k}.$$

Since Ω, ω' and θ are constant, $\mathbf{H}$ does not vary relative to $Oxyz$. The dynamical equations (§ 214, 2, 5) are therefore

$$-\frac{W}{g}\Omega^2 \mathbf{p}^* = \mathbf{F}, \quad \boldsymbol{\Omega} \times \mathbf{H} = \mathbf{M}.$$

From the above values of $\boldsymbol{\Omega}$ and $\mathbf{H}$,

$$\boldsymbol{\Omega} \times \mathbf{H} = \{C(\Omega \cos \theta + \omega')\Omega \sin \theta - A\Omega^2 \sin \theta \cos \theta\}\mathbf{i};$$

hence

(1) $$\Omega \sin \theta (C\omega' + (C - A)\Omega \cos \theta)\mathbf{i} = \mathbf{M},$$

or since $$\Omega\omega' \sin \theta \, \mathbf{i} = \boldsymbol{\Omega} \times \boldsymbol{\omega}',$$

(2) $$\left(C + (C - A)\frac{\Omega}{\omega'}\cos \theta\right)\boldsymbol{\Omega} \times \boldsymbol{\omega}' = \mathbf{M}.$$

This is a necessary condition for a steady precession in which *θ is the angle between the axes of precession and spin.* When the spin ω' is very large in comparison with Ω, (2) may be replaced by the approximate equation

(3) $$C\boldsymbol{\Omega} \times \boldsymbol{\omega}' = \mathbf{M}$$

which is rigorously true when $\theta = 90°$ (§ 217, 2) or $A = C$.

Equation (2) *is the key to most of the technical applications of the gyroscope.* Its left member gives the moment of the inertia

forces about O. The negative of this moment,

$$\mathbf{G} = \left(C + (C - A)\frac{\Omega}{\omega'}\cos\theta\right)\boldsymbol{\omega}'\times\boldsymbol{\Omega}, \tag{4}$$

is called the *gyroscopic couple.* Since $\mathbf{M} + \mathbf{G} = 0$, the gyroscopic couple is exactly balanced by the moment of the external forces.*

Example 1. When the only external forces are the weight W and the reaction at O, $\mathbf{M} = Wb \sin\theta\,\mathbf{i}$ (Fig. 218a).

If $\sin\theta = 0$ both members of (1) are zero and the condition is fulfilled; the body then spins about a vertical axis, erect when $\theta = 0$, hanging down when $\theta = \pi$.

If $\sin\theta \neq 0$ we may cancel $\sin\theta$ in (1). Then

$$\text{(i)}\qquad (A - C)\cos\theta\,\Omega^2 - C\omega'\Omega + Wb = 0 \qquad \text{and}$$

$$\text{(ii)}\qquad \Omega = \frac{C\omega' \pm \sqrt{C^2\omega'^2 - 4\,Wb(A - C)\cos\theta}}{2\,(A - C)\cos\theta}.$$

Thus there are two, one, or no real values of Ω according as the radicand is positive, zero, or negative. In the case of two roots Ω_1, Ω_2, we have from (i)

$$\Omega_1\Omega_2 = \frac{Wb}{(A - C)\cos\theta}.$$

Hence if two different precessions are possible, they have the same or opposite directions according as

$$(A - C)\cos\theta \gtrless 0.$$

If the center of mass is at O, $b = 0$ and from (i)

$$\Omega_1 = 0, \quad \Omega_2 = \frac{C\omega'}{(A - C)\cos\theta}.$$

Finally if $\theta = \frac{1}{2}\pi$ or $A = C$ the solutions (ii) do not apply. In these cases (i) gives $\Omega = Wb/C\omega'$.

When the body has no spin ($\omega' = 0$) it is called a *spherical pendulum;* in this case (i) gives

$$\Omega = \pm\sqrt{\frac{Wb}{(C - A)\cos\theta}}.$$

Thus a spherical pendulum can have a conical rotation of angle

$$\theta < \tfrac{1}{2}\pi \quad \text{when} \quad C > A \quad \text{or} \quad \theta > \tfrac{1}{2}\pi \quad \text{when} \quad C < A.$$

The former motion is noteworthy because the center of mass is above the point of support.

* The gyroscopic couple is analogous to the centrifugal force of a particle revolving in a circle. For the centrifugal force is the negative of the inertia force on the particle and is just balanced by the external forces.

Example 2. *Pendulum Mill.* In the grinding mill of Fig. 218*b* the "pendulum" is revolved about a vertical axis with the angular velocity Ω. When Ω is large enough the conical grinder will roll around the side wall of the hopper pressing against it with a force P.

Let G be the center of mass of the grinder, r its radius at G, and $l = OG$. If we assume pure rolling along the section of radius r,

(i) $\omega' r = \Omega b = \Omega(l \sin \beta + r \cos \beta).$

The rolling motion requires that the angle θ between Ω and ω' be obtuse. From (2) we see that the moment of the inertia forces about O has the direction of $\Omega \times \omega'$ (the x-direction in the figure); its turning effect is therefore clockwise. Putting $\theta = \pi - \beta$ in (1) we find that this moment has the magnitude

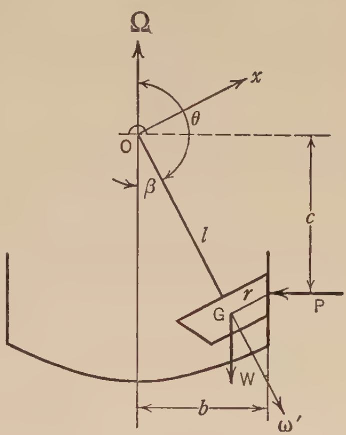


FIG. 218*b*.

$$\Omega \sin \beta \, (C\omega' - (C - A) \cos \beta) = \Omega^2 \sin \beta \left(C \frac{l}{r} \sin \beta + A \cos \beta \right)$$

in view of (i). On setting this equal to the moment of the external forces about Ox we have

(ii) $$Pc + Wl \sin \beta = \Omega^2 \sin \beta \left(C \frac{l}{r} \sin \beta + A \cos \beta \right).$$

Clearly no pressure P will develop unless the right-hand member is greater than $Wl \sin \beta$, i.e.

(iii) $$\Omega^2 > \frac{Wl}{C \dfrac{l}{r} \sin \beta + A \cos \beta}.$$

The friction along the hopper wall has no moment about Ox; its moment about O is equal to the driving torque on the vertical axis.

Consider, for example, a grinder for which $W = 200$ lb., $C = 0.9$ slug-ft.^{2}, $l = 3$ ft., $r = 0.5$ ft. To find its moment of inertia A about Ox we may, with sufficient accuracy, consider its mass concentrated at G; then

$$A = \tfrac{200}{32} \times 3^2 = 56.3 \text{ slug-ft.}^2$$

With $\beta = 35°$ we now find from (iii) that Ω must exceed 3.5 rad./sec. (33.4 rev./min.) before the grinder will press against the hopper. To find P for larger values of Ω we make use of (ii), noting that

$$c = l \cos \beta - r \sin \beta = 2.17 \text{ ft.}$$

Thus when $\Omega = 2\pi$ rad./sec., $P = 355$ lb.

PROBLEMS

1. A river steamer, driven by paddle wheels, is struck by a roller that gives it a list to starboard. How will the steamer react?

2. Rotary engines, in which cylinder and crank-case revolve about a longitudinal axis, were formerly used as airplane motors. In such an engine the revolving parts weigh 300 lb. and have a radius of gyration of 15 in. If the engine is making 1200 rev./min., what is the gyroscopic couple when the plane makes a turn of 300-ft. radius at 90 mi./hr.?

3. A solid disk of weight W and radius r is keyed to a shaft revolving Ω rad./sec. If the axis of the disk makes an angle θ with the shaft, show that the gyroscopic couple on the bearings is

$$\tfrac{1}{8}\frac{W}{g}r^2\Omega^2 \sin 2\,\theta.$$

4. A car of weight W has wheels of total weight w, radius r, and radius of gyration k about their axes. As it rounds a curve of radius R with the speed v, show that the gyroscopic couple on the car is approximately equal to

$$\frac{w}{g}k^2v^2 \cos \beta / Rr,$$

where β is the inclination of the track to the horizontal. Show also that the gyroscopic couple increases the tipping moment of the centrifugal force on the car and that it may be taken into account by *raising* the point of application of the centrifugal force a distance wk^2/Wr above the track.

5. A top with a blunt peg is given the spin ω' about its axis. If it performs a steady precession Ω on a *smooth* horizontal plane, prove

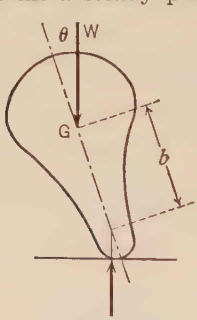


FIG. 218c.

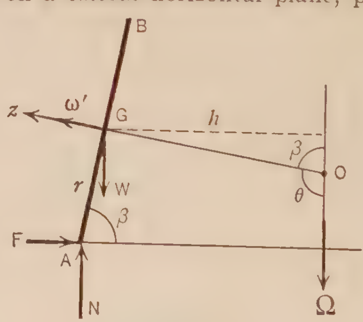


FIG. 218d.

that the inclination θ of its axis to the vertical is given by equation (i) of Example 1 provided that C, A denote the principal moments of inertia at the center of mass G and that b is the distance along the axis from G to the line of the normal reaction (Fig. 218c).

6. In Fig. 218d the wheel AB of radius r remains inclined at an angle β to the horizontal while rolling in a circle. If its center G revolves at the rate Ω in a circle of radius h, show that

$$\Omega^2\left(C\frac{h}{r} + A\cos\beta + \frac{W}{g}rh\right) = Wr\cot\beta.$$

If AB is a uniform disk and $r = 2$ ft., $h = 4$ ft., $\beta = 60°$, compute Ω and ω'.

219. Motion under No Forces. Let a rigid body be supported at its center of mass O so that it may have any motion in which O remains at rest. Since $\mathbf{a}^* = 0$ the inertia forces reduce to a couple of moment $d\mathbf{H}/dt$ (§ 214). If the only external forces are the weight $\mathbf{W}$ and reaction $\mathbf{R}$ at the support, we have by D'Alembert's Principle

(1), (2) $$0 = \mathbf{W} + \mathbf{R}, \qquad \frac{d\mathbf{H}}{dt} = 0;$$

for $\mathbf{M} = 0$ as both $\mathbf{W}$ and $\mathbf{R}$ act through O.

From (1) it appears that $\mathbf{W}$ and $\mathbf{R}$ are a pair of opposed forces at O; *the external forces thus reduce to zero.* The motion is therefore called *motion under no forces.*

From (2) we see that the $\mathbf{H}$ remains constant in magnitude and direction during the motion.

Moreover as $\mathbf{W}$ and $\mathbf{R}$ do no work during the motion the kinetic energy T remains constant. Since

(3) $$\boldsymbol{\omega}\cdot\mathbf{H} = 2\,T \qquad (\S\ 215, 1)$$

the projection of $\boldsymbol{\omega}$ on $\mathbf{H}$ is constant. If $\boldsymbol{\omega}$ is drawn from O, its end-point P must always lie on a fixed plane p (the *invariable plane*) normal to the constant vector $\mathbf{H}$.

If x, y, z are principal axes of inertia at O, fixed in the body,

(4) $$A\omega_x^2 + B\omega_y^2 + C\omega_z^2 = 2\,T \qquad (\S\ 215, 1);$$

hence the locus of P, i.e. the point $(\omega_x, \omega_y, \omega_z)$, in the body is an ellipsoid (the *energy ellipsoid*), with center at O and having the Cartesian equation (4). Thus as the motion proceeds P moves on the invariable plane fixed in space and on the energy ellipsoid fixed in the body.

If f denotes the function forming the left member of (4), the normal to the ellipsoid at P has the direction of

$$\left[\frac{\partial f}{\partial \omega_x}, \frac{\partial f}{\partial \omega_y}, \frac{\partial f}{\partial \omega_z}\right] = 2\,[A\omega_x, B\omega_y, C\omega_z] = 2\,\mathbf{H}$$

and is therefore also normal to the invariable plane p, that is, *the energy ellipsoid is always tangent to the invariable plane at P.*

Now the point of the body which momentarily coincides with P has zero velocity since it lies on the instantaneous axis of rotation. Consequently the ellipsoid rolls over the invariable plane *without slipping.*

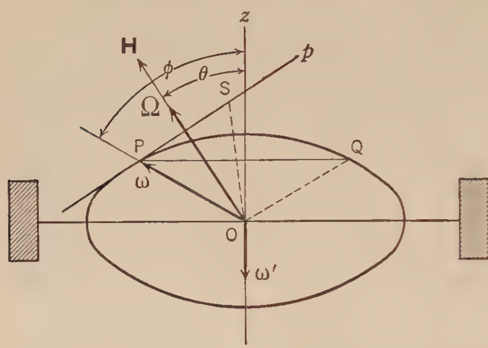


FIG. 219.

The above results thus give a mental picture of the motion which is embodied in

POINSOT'S THEOREM. *When a rigid body moves about its fixed center of mass under no forces, the energy ellipsoid, fixed in the body, rolls without slipping on a plane fixed in space; and the vector from the fixed point to the point of contact represents the instantaneous angular velocity of the body in direction and magnitude.*

Example. A wheel is mounted so that it may turn freely about its fixed center of mass, that is, its geometrical center O. Its axis of symmetry Oz and any two axes Ox, Oy, perpendicular to Oz and to each other, are principal axes of inertia at O. Its moments of inertia about Ox, Oy, Oz are A, $B = A$, C.

Suppose, now, that the wheel is given an initial angular velocity ω_0 about an axis OP making an angle ϕ with Oz (Fig. 219). Then since

$$\mathbf{H} = [A\omega_x,\ A\omega_y,\ C\omega_z], \quad \mathbf{H}\cdot\boldsymbol{\omega} = 2\,T,$$

the constant values of $\mathbf{H}$ and T are known. Since $A = B$ the energy ellipsoid (4) is now an ellipsoid of revolution about Oz; and the motion of the wheel is such that this ellipsoid rolls on the invariable plane p normal to $\mathbf{H}$. The normal to the ellipsoid at its point of contact P must always be parallel to $\mathbf{H}$; all such points of contact lie on a circle PQ whose plane is normal to Oz. As the motion proceeds the point P travels around this circle and $\overrightarrow{OP} = \boldsymbol{\omega}$ describes a circular cone POQ of semi-angle ϕ fixed in the body. Hence $\boldsymbol{\omega}$ has the constant magnitude ω, and its component $\omega_z = \omega\cos\phi$ is likewise constant. Again since OP is constant and P must lie on the plane p,

OP describes a circular cone POS in space having $\mathbf{H}$ as axis. At any instant the body cone POQ and the fixed cone POS are in contact along the element $\overrightarrow{OP} = \omega$; this element moreover is the instantaneous axis of the wheel and all of its points have zero velocity. *The motion of the wheel therefore rolls the body cone over the fixed cone* (§ 216, Ex.).

The angular velocity $\boldsymbol{\omega}$ of the body cone POQ may be compounded of the angular velocity $\boldsymbol{\Omega}$ of its axis Oz about $\mathbf{H}$ and its angular velocity $\boldsymbol{\omega}'$ relative to its axis: $\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}'$. On multiplying this equation by $\mathbf{k}\times$ and $\boldsymbol{\Omega}\times$ in turn we find

(i) $$\omega \sin \phi = \Omega \sin \theta,$$

(ii) $$\omega \sin (\phi - \theta) = \omega' \sin \theta.$$

Now $$\sin \theta = \frac{|\mathbf{k}\times\mathbf{H}|}{H} = \frac{A\sqrt{\omega_x^2 + \omega_y^2}}{H} = \frac{A\omega \sin \phi}{H},$$

$$\cos \theta = \frac{\mathbf{k}\cdot\mathbf{H}}{H} = \frac{C\omega_z}{H} = \frac{C\omega \cos \phi}{H}$$

$$\sin (\phi - \theta) = \frac{(C - A)\omega \sin \phi \cos \phi}{H}.$$

Hence from (i) and (ii)

(iii), (iv) $$\Omega = \frac{H}{A}, \qquad \omega' = \frac{C - A}{A}\,\omega \cos \phi.$$

Ω and ω' are respectively the angular velocity of precession and spin of the wheel. We may also express Ω in terms of ω' by noting that $H_z = C\omega_z$ or

$$H \cos \theta = C\omega \cos \phi = \frac{AC\omega'}{C - A}$$

from (iv). Hence from (iii)

$$\Omega = \frac{C\omega'}{(C - A) \cos \theta}.$$

This is in complete agreement with § 218, Example 1 when $b = 0$, provided that we replace ω' by $-\omega'$; for in the present problem $\boldsymbol{\omega}'$ has the direction of $-\mathbf{k}$.

220. Statics of a Rigid Body. Dynamics has been developed from three principles:

Principle I: Force and Acceleration,
Principle II: Vector Addition of Forces,
Principle III: Action and Reaction,

together with the assumption that the summations extended over systems of particles may be replaced by the corresponding integrals in the case of *bodies*, i.e. continuous mass distributions.

Now the inertia forces of any body of mass m may be reduced to a

$$\text{Force } m\mathbf{a}^* \text{ at } P^* \quad \text{and} \quad \text{Couple of moment } \frac{d\mathbf{H}^{*\prime}}{dt}$$

(§ 184), where $\mathbf{H}^{*\prime}$ is the moment of relative momentum about the center of mass P^*. Hence if $\mathbf{F}$ and $\mathbf{M}^*$ denote the force-sum and moment-sum about P^* of all the external forces on the body, we have by D'Alembert's Principle

$$(1),\ (2) \qquad m\frac{d\mathbf{v}^*}{dt} = \mathbf{F}, \quad \frac{d\mathbf{H}^{*\prime}}{dt} = \mathbf{M}^*.$$

These differential equations determine $\mathbf{v}^*$ and $\mathbf{H}^{*\prime}$ for any body, rigid or deformable. Now $\mathbf{v}^*$ gives the motion of P^* in any case; but the motion relative to P^* is in general not determined by $\mathbf{H}^{*\prime}$ except in the case of *rigid* bodies. Thus (1) and (2) completely determine the motion of a rigid body. The important rôle that rigid bodies play in mechanics is due to this fact.

To prove that $\mathbf{H}^{*\prime}$ determines the motion of a rigid body relative to P^* we recall that this relative motion is always an instantaneous rotation $\boldsymbol{\omega}$ about an axis through P^* (§ 213, Theorem 5). Since P^* is at rest in the motion relative to P^*,

$$(3) \qquad \mathbf{H}^{*\prime} = [A\omega_x,\ B\omega_y,\ C\omega_z] \qquad (\S\ 214,\ 1).$$

Hence when the principal moments of inertia A, B, C are known, $\mathbf{H}^{*\prime}$ determines $\boldsymbol{\omega} = [\omega_x, \omega_y, \omega_z]$.

Now $\mathbf{F}$ and $\mathbf{M}^*$ in (1) and (2) are completely determined when the magnitude, direction, and line of action of each external force is known. We conclude, therefore, that the motion (or state of rest) of a rigid body is not affected by shifting the external forces along their lines of action. Such a shift in the internal forces obviously does not impair the validity of (1) and (2). We have thus *proved* Principle B of Statics — the transmissibility of a force acting on a *rigid* body.

It still remains to prove Principle C of Statics. This states the basic criterion for equilibrium:

If the forces acting on a particle or a rigid body, initially at rest, can be reduced to zero by means of Principles A and B (vector addition and transmissibility of forces), the particle or body will remain at rest.

Proof. Since the application of Principles A and B does not

change the force-sum $\mathbf{F}$ and moment-sum $\mathbf{M}^*$, we must have $\mathbf{F} = 0$ and $\mathbf{M}^* = 0$ when the forces reduce to zero. Equations (1) and (2) then become

$$\frac{d\mathbf{v}^*}{dt}, \quad \frac{d\mathbf{H}^{*\prime}}{dt} = 0;$$

hence $\mathbf{v}^*$ and $\mathbf{H}^{*\prime}$ must remain constant, namely $\mathbf{v}^* = 0$, $\mathbf{H}^{*\prime} = 0$, since the body by hypothesis was initially at rest. Now $\mathbf{H}^{*\prime} = 0$ implies that $\boldsymbol{\omega} = 0$ in view of (3). Thus $\mathbf{v}^*$ and $\boldsymbol{\omega}$ are both identically zero, that is, the body remains at rest. In the case of a particle, $\mathbf{v}^* = \mathbf{v} = 0$ alone establishes the equilibrium.

We have now fulfilled the promise made in § 149, namely, to prove that the Principles *B* and *C* of Statics, not included in Principles I, II, and III, are consequences of the latter. Thus

Principles I, II and III from the basis of the whole of Dynamics — Statics and Kinetics.

221. Summary, Chapter XIV. For a rigid body of mass m, the

$$\text{Momentum} = \int \mathbf{v}\, dm = m\mathbf{v}^*,$$

$$\text{Moment of Momentum } \mathbf{H} = \int \mathbf{r}\times\mathbf{v}\, dm,$$

$$\text{Kinetic Energy} = \tfrac{1}{2}\int v^2\, dm.$$

The change in kinetic energy of a *rigid* body in any interval is equal to the work done by the *external* forces acting on it.

If a rigid body has a MOTION OF TRANSLATION, its inertia forces have the resultant $m\mathbf{a}$ at the center of mass. The external forces acting on the body are equivalent to this vector.

The moment of inertia I of a body about an axis is defined as $\int p^2\, dm$: the "sum" of the masses of its particles multiplied by the squares of their distances p from the axis. The equation $I = mk^2$ defines its *radius of gyration* k about this axis. If I^* is the moment of inertia about a parallel axis through the center of mass P^*,

$$I = I^* + md^2 \qquad \text{(Transfer Theorem)},$$

where d is the distance between the axes.

If a rigid body ROTATES ABOUT A FIXED AXIS Oz which is a principal axis of inertia at O (§ 198), the inertia forces reduce to a

$$\text{Force } m\mathbf{a}^* \text{ at } O \quad \text{and} \quad \text{Couple of moment } I\boldsymbol{\alpha}.$$

The dynamical equations are then

$$m\mathbf{a}^* = \mathbf{F}, \quad I\boldsymbol{\alpha} = \mathbf{M}_O,$$

where $\mathbf{F}$ and $\mathbf{M}_O$ denote the force-sum and moment-sum of all the external forces acting on the body. The scalar equation

$$I\alpha = M_z$$

holds for any axis Oz, whether principal or not. It suffices to determine the motion. The kinetic energy of the body is $\frac{1}{2} I\omega^2$. As the body turns through θ radians, the change in kinetic energy equals the

$$\text{Work done by the Torque } M_z = \int_0^\theta M_z \, d\theta.$$

If a plane through P^* normal to the axis of rotation cuts the axis at O, the axis is a principal axis at O when the mass distribution is symmetric with respect to this plane or to any line parallel to the axis. If the axis of rotation is itself a line of symmetry, it is a principal axis at *all* of its points.

If a body revolves uniformly about a fixed axis, the bearing reactions due to its motion are zero when, and only when,

(*a*) its center of mass lies on the axis, *and*
(*b*) the axis is a principal axis of inertia at one of its points.

If O is a point of a body in PLANE MOTION such that

(1) $\mathbf{v}_O$ is constant, or O is the center of mass P^*, or $\mathbf{a}_O$ passes through P^*, and

(2) the axis Oz, normal to the motion, is a principal axis at O, the inertia forces reduce to a

$$\text{Force } m\mathbf{a}^* \text{ at } O \quad \text{and} \quad \text{Couple of moment } I\boldsymbol{\alpha}.$$

The dynamical equations are then

$$m\mathbf{a}^* = \mathbf{F}, \quad I\boldsymbol{\alpha} = \mathbf{M}_O.$$

The scalar equations

$$ma_x^* = F_x, \quad ma_y^* = F_y, \quad I\alpha = M_z$$

hold whenever condition (1) is fulfilled. These suffice to determine the motion.

Let O be any point of a rigid body in motion. Then the velocities of its points P at any instant may be compounded of a translation $\mathbf{v}_O$ and a rotation $\boldsymbol{\omega}$ about an axis through O:

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega}\times\overrightarrow{OP}.$$

The angular velocity $\boldsymbol{\omega}$ is the same for any choice of O.

For a rigid body in MOTION WITH ONE POINT O FIXED, $\mathbf{v}_P = \boldsymbol{\omega}\times\overrightarrow{OP}$. Its dynamical equations are

$$m\mathbf{a}^* = \mathbf{F}, \quad \frac{d\mathbf{H}}{dt} = \mathbf{M}_O$$

where $\mathbf{H}$ is the angular momentum about O. If x, y, z are an orthogonal set of principal axes of inertia at O, and A, B, C the corresponding principal moments of inertia

$$\mathbf{H} = [A\omega_x, B\omega_y, C\omega_z].$$

If the axes xyz revolve about O with the angular velocity $\boldsymbol{\Omega}$,

$$\frac{d\mathbf{H}}{dt} = \frac{\partial\mathbf{H}}{\partial t} + \boldsymbol{\Omega}\times\mathbf{H},$$

where $\partial\mathbf{H}/\partial t$ denotes a rate of change relative to the moving axes.

For a more complete discussion of gyroscopic theory the reader is referred to

Theoretical Mechanics, Ames and Murnaghan, Ginn, 1929.

PROBLEMS

1. A 10-lb. weight hangs from a cord wound about a horizontal axle 1 ft. in diameter. The axle carries a brake drum 2.5 ft. in diameter. After the weight falls 4 ft. from rest (turning the axle and drum), a normal pressure of 48 lb. is applied to the drum by the brake. If $\mu = \frac{1}{4}$ at the brake, how much further will the weight fall?

2. An elevator car weighing 3600 lb., carries a load of 3000 lb. The hoisting cable, connecting the car with the counterweight of 4800 lb., is compensated by a second cable, of equal length and weight, whose ends are attached to the bottoms of the car and counterweight. Hoisting and compensating cables have a total weight of 2300 lb.

The normal running speed of 600 ft./min. follows a period of uniform acceleration lasting 6 sec. Find the horsepower exerted by the

hoisting motor t (<6) sec. after the car begins to rise, if the inertia of the drum and armature, and the friction losses, are neglected. [Express the work W as a function of t; then the power equals dW/dt (§ 163).]

What is the energy absorbed by the braking devices when the car is stopped in 35 ft. going downward?

3. A traveling crane, weighing 20 tons complete, is uniformly accelerated from rest 4 ft./sec.^{2} The armature of the driving motor weighs 1000 lb., measures 16 in. in diameter, and is practically a solid cylinder. The track wheels are 10 in. in diameter; and the gear ratio from motor to wheels is 2 to 1. Find the horsepower exerted by the motor, neglecting friction and the rotational inertia of wheels and gears, when the crane is moving 80 ft./min.

4. An electric hoist in a North Butte mine operates two cages in a vertical shaft 4000 ft. deep. The cages weigh 14,000 lb. apiece and are connected by 4000 ft. of free cable, weighing 3.75 lb./ft., so as to balance each other. The driving motor is directly connected to a drum 10 ft. in diameter, over which the cable runs; their moments of inertia are respectively 47,000 and 125,000 slug-ft.^{2}

The ascending cage, with a load of 15,000 lb., is uniformly accelerated for $t_1 = 2$ sec. to a speed of 2700 ft./sec., rises with this speed for t_2 sec., and is then uniformly retarded by the brakes for $t_3 = 12$ sec., so that it comes to rest at the top of the shaft. Compute

(*a*) the interval t_2 of uniform speed;
(*b*) the distances x_1, x_2, x_3 ascended during the intervals t_1, t_2, t_3;
(*c*) the work done by the motor in the intervals t_1 and t_2;
(*d*) the work absorbed by the braking mechanism in the interval t_3.

5. A sphere 8 in. in diameter is rolled with an initial speed of 8 ft./sec. on a horizontal floor. What distance will it go before coming to rest if the coefficient of rolling friction is 0.02 in.?

6. A truck has a solid rectangular door d ft. wide on the side (hinge-line forward). The door is standing open at right angles when the truck starts with an acceleration of a ft./sec.^{2} If friction and air resistance are neglected, show that the door will close with the angular velocity of $\sqrt{3\,a/d}$ rad./sec.

7. The rim of a flywheel weighs 1600 lb. and is 4 ft. in mean diameter. If the wheel is keyed to the shaft so that its plane is 1° out of true, find the torque on the bearings when it is making 240 r.p.m.

8. Using the notation of § 190, Example 5, find the greatest possible acceleration of the car with a front-wheel drive. In accelerating a car, which produces the greater *maximum* traction, rear-wheel or front-wheel drive?

9. An 80-lb. flywheel, with a radius of gyration of 7 in., revolves on a shaft transverse to a motor-car. With the engine making 900 r.p.m. the car rounds a curve of 40 ft. radius at 20 mi./hr. Compute the gyroscopic couple exerted on the car.

Show that when $\boldsymbol{\omega}$ for the flywheel points to the right of the car, the gyroscopic couple tends to resist overturning on curves.

CHAPTER XV

IMPACT

222. Fundamental Equations of Impact. When two " rigid " bodies collide, their velocities are in general abruptly changed. In order to produce these apparently instantaneous changes of velocity the forces generated by the impact must be very large. Now a force **F** acting on a particle during the interval Δt, produces a change of momentum equal to its impulse (§ 161):

$$\Delta(m\mathbf{v}) = \int_0^{\Delta t} \mathbf{F}\, dt.$$

Since we do not know the duration Δt of the impact nor the precise variation of **F** in this interval, we idealize the impact by taking it to be instantaneous, that is, $\Delta t = 0$. Then in order that the impulse above may still equal a finite change in momentum, we must suppose that in the actual impact F becomes infinite in such a way that

$$\lim_{\Delta t \to 0} \int_0^{\Delta t} \mathbf{F}\, dt = \mathbf{P}$$

is a *finite* vector. **P** is called an *instantaneous impulse* or *percussion.*

If other *finite* forces, such as gravity, act on the colliding bodies, their impulses disappear in the limit $\Delta t \to 0$. Such forces may therefore be neglected in the impact equation

$$\Delta(m\mathbf{v}) = \mathbf{P}.$$

Friction may also be neglected unless the normal force between the surfaces involved has the character of a sudden blow.

Finally, since the impact is regarded as instantaneous, the colliding bodies are assumed to be stationary during impact and the position vectors of their partic es constant.

Consider now a system of particles involved in an impact. The percussions acting on the particles are due to forces which are either external or internal to the system. By applying the Principle of Action and Reaction as in § 179 we may show that

In any system of particles the vector sum of the internal percussions, and of their moments about any point, is zero.

Now for any particle of mass m belonging to the system and subject to the external and internal percussions $\mathbf{P}$, $\mathbf{P}'$ we have

$$\Delta(m\mathbf{v}) = \sum \mathbf{P} + \sum \mathbf{P}'.$$

Moreover if the particle has the position vector $\mathbf{r}$ relative to any point, moving or fixed,

$$\mathbf{r}\times\Delta(m\mathbf{v}) = \sum \mathbf{r}\times\mathbf{P} + \sum \mathbf{r}\times\mathbf{P}',$$

or since $\mathbf{r}$ is constant during impact,

$$\Delta(\mathbf{r}\times m\mathbf{v}) = \sum \mathbf{r}\times\mathbf{P} + \sum \mathbf{r}\times\mathbf{P}'.$$

Now form these equations for each particle of the system and add the equations of each set. Then since both $\sum\sum \mathbf{P}' = 0$ and $\sum\sum \mathbf{r}\times\mathbf{P}' = 0$, we obtain

(1) $$\Delta \sum m\mathbf{v} = \sum\sum \mathbf{P},$$

(2) $$\Delta \sum \mathbf{r}\times m\mathbf{v} = \sum\sum \mathbf{r}\times\mathbf{P}.$$

These fundamental equations for impact may be stated as follows:

THEOREM I. *In an impact, the change in the momentum of a system of particles is equal to the sum of the external percussions.*

THEOREM II. *In an impact, the change in the moment of momentum about any point is equal to the sum of the moments of the external percussions about the point.*

Since $\sum m\mathbf{v} = (\sum m)\mathbf{v}^*$ (§ 181, 3), the change in momentum in (1) is equal to the change in $(\sum m)\mathbf{v}^*$.

We see also from (1) and (2) that:

If $\sum\sum \mathbf{P} = 0$, $\sum m\mathbf{v}$ or $\mathbf{v}^*$ is constant;
If $\sum\sum \mathbf{r}\times\mathbf{P} = 0$, $\sum \mathbf{r}\times m\mathbf{v}$ is constant.

The momentum is conserved when the sum of the external percussions is zero. The moment of momentum about any point is conserved when the sum of the moments of the external percussions about this point is zero.

Instead of taking (vector) moments about a point as above, it is usually simpler to take (scalar) moments about an axis. This is permissible since Theorem II also holds when *point* is replaced

by *axis* (§ 66). In this connection we note that a rigid body, revolving with the angular speed ω about a fixed axis z, has the moment of momentum $H_z = I\omega$ about the axis; for a particle dm at a distance p from the axis contributes $p \cdot \omega p \; dm$ to H_z and hence

$$H_z = \omega \int p^2 \, dm = I\omega.$$

This, of course, is the z-component of (§ 198, 1).

In the examples and problems of this article the bodies are supposed to adhere after impact. Impact of this character is said to be *inelastic.*

Example 1. A freight car weighing 30 tons runs into a 20-ton car standing on the track and both move on with the speed of 3 ft./sec. Find the speed of the first car at striking and the kinetic energy lost in the impact.

Since there is no percussion external to the system formed by the two cars, their momentum is conserved. Hence if v is the striking speed,

$$30\, v = 50 \times 3, \quad v = 5 \text{ ft./sec.}$$

The loss in kinetic energy is

$$\tfrac{1}{2}\frac{60{,}000}{32}25 - \tfrac{1}{2}\frac{100{,}000}{32}9 = 9375 \text{ ft.-lb.}$$

Example 2. *Ballistic Pendulum.* The speed of a bullet may be determined by firing it horizontally into a block of wood or box of sand, mounted to swing as a physical pendulum, and measuring the angle β through which it rises (Fig. 222*a*). If friction at the axis O is neglected, the only percussion external to the system bullet-pendulum is the impulsive reaction at O. Hence the moment of momentum of the system about O is the same before and after impact:

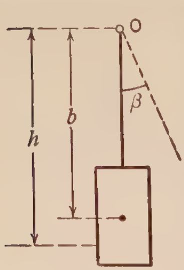


FIG. 222*a*.

$$\frac{w}{g}vh = \left(\frac{W}{g}k^2 + \frac{w}{g}h^2\right)\omega.$$

Here w, W are the weights of bullet and pendulum, k is the radius of gyration of the latter about the axis, and ω is the angular velocity imparted by the impact. Thus we have

(i) $$v = \left(\frac{W}{w}\frac{k^2}{h} + h\right)\omega.$$

If the line of fire passes through the center of percussion of the pendulum there will be no horizontal reaction at O (§ 199). In this

case there are no external percussions on the system bullet-pendulum and the momentum will be conserved. Hence if b and l denote the distances of the centers of mass and percussion from O,

$$\frac{w}{g}v = \frac{W}{g}b\omega + \frac{w}{g}l\omega, \qquad \text{or}$$

$$\text{(ii)} \qquad v = \left(\frac{W}{w}b + l\right)\omega.$$

On putting $h = l$ in (i) and comparing with (ii) we see that

$$\frac{k^2}{l} = b \quad \text{or} \quad l = \frac{k^2}{b}$$

in agreement with § 199.

By observing the angle β we may find ω by use of the energy equation:

$$\tfrac{1}{2}\left(\frac{W}{g}k^2 + \frac{w}{g}h^2\right)\omega^2 = (Wb + wh)\,(1 - \cos\beta).$$

Example 3. A rod of length l falls from a vertical position about a hinged end O (Fig. 222b). When horizontal it strikes a fixed obstacle O' at a distance d from O. Find the percussions P, P' at O and O' if the rod does not rebound.

From the energy equation for the rod we may find its angular velocity ω just before impact:

$$\tfrac{1}{2}\frac{W}{g}\frac{l^2}{3}\omega^2 = \tfrac{1}{2}Wl, \qquad \omega = \sqrt{\frac{3g}{l}}.$$

We now apply Theorems I and II, taking moments about O in the latter. With the positive directions indicated in Fig. 222b, the momentum and moment of momentum of the bar just before impact are

$$mv^* = -\frac{W}{g}\omega\frac{l}{2}, \qquad -I\omega = -\frac{W}{g}\frac{l^2}{3}\omega.$$

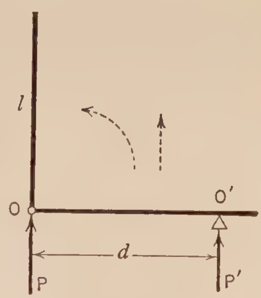


FIG. 222b.

Since the bar remains at rest after impact,

$$P + P' = \tfrac{1}{2}\frac{W}{g}\omega l, \qquad P'd = \tfrac{1}{3}\frac{W}{g}\omega l^2.$$

From these equations we may find P and P'.

Example 4. A stamp of mass m is operated by a toothed wheel as shown in Fig. 222c. Let ω, Ω be the angular velocities of the wheel

just before and after impact and 0, V the corresponding velocities of the stamp. If the wheel and stamp adhere after impact, $V = r\Omega$. We shall neglect friction at the axle of the wheel and at the guides of the stamp.

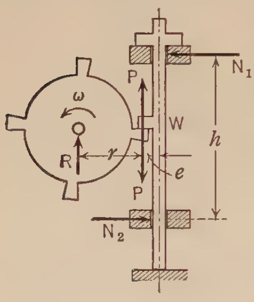


FIG. 222c.

Let **P** be the percussion of the wheel on the stamp. There will also be normal percussions $\mathbf{N}_1$, $\mathbf{N}_2$ at the guides as shown in the figure. Now apply Theorem I to the stamp: on resolving vertically and horizontally we find

$$m\Omega r - 0 = P, \quad 0 = N_2 - N_1.$$

Also since the momentum of the stamp is a vector through its center of gravity G we have, on taking moments about G (Theorem II),

$$0 = N_1 h - Pe,^* \quad \text{whence} \quad N_1 = N_2 = \frac{e}{h} P.$$

The percussion $-\mathbf{P}$ acts downward on the wheel at the point of impact (Action and Reaction). There is also a percussion **R** normal to the axle.† On taking moments about the center we have from Theorem II,

$$I\Omega - I\omega = -Pr$$

where I is the moment of inertia of the wheel. Note that axle friction is neglected.

On putting $P = m\Omega r$ in the last equation we find

$$\Omega = \frac{I\omega}{I + mr^2}.$$

From this result we may compute the loss of energy due to impact:

$$\tfrac{1}{2} I\omega^2 - \tfrac{1}{2} I\Omega^2 - \tfrac{1}{2} m\Omega^2 r^2 = \tfrac{1}{2} \frac{Im\omega^2 r^2}{I + mr^2}.$$

Example 5. *Piledriving.* Let the ram of the piledriver, of weight W, be dropped on a pile of weight w from a height h above the top of the pile. The striking velocity is then $v = \sqrt{2gh}$.

* Since $\mathbf{N}_2 = -\mathbf{N}_1$, these percussions form a couple of moment $N_1 h$.

† If the center of gravity of the wheel is at its geometrical center, the momentum of the wheel is zero before and after impact. Then, from Theorem I, **R** must be an upward percussion numerically equal to P.

We shall assume that the ram and pile adhere after impact and move with the common velocity V. If we neglect the resistance of the ground on the pile during the impact, there will be no external percussion on the system ram-pile and the momentum will be conserved. Hence

$$\frac{W}{g}v = \frac{W+w}{g}V \quad \text{or} \quad V = \frac{W}{W+w}v.$$

The kinetic energies of the system before and after impact are respectively

$$\tfrac{1}{2}\frac{W}{g}v^2 = Wh,$$

$$\tfrac{1}{2}\frac{W+w}{g}V^2 = \frac{W^2}{W+w}\frac{v^2}{2g} = \frac{W^2h}{W+w}.$$

Hence the loss in kinetic energy due to impact is

$$Wh - \frac{W^2h}{W+w} = \frac{w}{W+w}Wh.$$

Suppose, now, that the system ram-pile, endowed with the kinetic energy $W^2h/(W+w)$, descends a distance d before being brought to rest by the resistance of the ground. If the space average of this resistance is R, the work done on the system is $(W+w-R)d$. This work equals the change in kinetic energy:

$$(W+w-R)d = 0 - \frac{W^2h}{W+w}; \qquad \text{hence}$$

$$R = \frac{h}{d}\frac{W^2}{W+w} + W + w.$$

Thus if a 2-ton ram falling 10 ft. drives a $\frac{1}{2}$-ton pile $\frac{1}{2}$ inch into the ground,

$$R = \frac{10 \times 12}{\frac{1}{2}}\frac{4}{2\frac{1}{2}} + 2\tfrac{1}{2} = 386\tfrac{1}{2} \text{ tons.}$$

According to this theory the pile could support a dead weight of 386 tons without yielding. In practice the pile would be loaded with only a small fraction (say $\frac{1}{10}$) of this amount in order to be on the safe side. This is the more necessary in view of the uncertain assumptions on which the theory is based.

We have seen that the energy lost in impact is the fraction $w/(W+w)$ of the original energy. This fraction is nearly 0 or 1 according as W is large or small in comparison with w. As the energy lost is largely spent in deforming the pile and diminishes the supply available for driving it, it is clearly advantageous to use a heavy ram in piledriving.

If, on the other hand, a hammering process is used to deform or

shape an object, as in riveting or certain blacksmithing operations, the work of deformation should be kept large by repeated blows of a relatively light hammer.

PROBLEMS

1. A stone weighing 2 lb. is dropped on soft ground from a height of 4 ft. Show that the percussion is 1 lb.-sec.

2. A 1-oz. bullet going 1200 ft./sec. strikes a block of wood weighing 4 lb., becomes embedded in it, and carries it off without rotation. Find the velocity of the block after impact and the loss in kinetic energy.

3. In Atwood's Machine (Fig. 159*a*) $W = 3$ lb., $W' = 1$ lb. The weights start from rest and in $\frac{1}{2}$ sec. W strikes the floor. W' rises, comes to rest, and falls again. With what velocity will W begin to move when the string becomes taut?

4. A particle weighing $\frac{1}{4}$ lb., moving with the speed of 4 ft./sec., strikes the end of a 3-lb. rod 2 ft. long pivoted at the middle. If pivot friction is neglected and the impact is inelastic, find the angular velocity of the rod after impact.

5. A 7-lb. bar, 2 ft. 8 in. long, pivoted at one end, drops from rest when horizontal. When vertical its end strikes a particle weighing $\frac{1}{3}$ lb. resting on a horizontal plane. If pivot friction is neglected and the impact is inelastic, find the initial speed of the particle.

If the coefficient of friction between particle and plane is $\frac{1}{4}$, how far will the particle go before coming to rest?

6. A cube of side b slides with the velocity v on a horizontal plane until it meets a low obstacle transverse to its motion. Prove that its center will have the velocity $\frac{3}{8}\sqrt{2}\,v$ after inelastic impact.

Show that the cube will tip over if

$$v^2 > \tfrac{8}{3}(\sqrt{2} - 1)gb.$$

223. Direct Impact of Spheres. When two spheres, whose centers are moving in the same straight line, collide, their velocities are suddenly changed from $\mathbf{v}_1$, $\mathbf{v}_2$ to $\mathbf{V}_1$, $\mathbf{V}_2$. In the system composed of the spheres, the mutual percussions at the point of contact cancel (action and reaction); and as there are no external percussions the total momentum remains constant:

$$m_1\mathbf{V}_1 + m_2\mathbf{V}_2 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2.$$

If we choose a positive direction on the line of centers this equation may be replaced by the scalar equation

$$(1) \qquad m_1 V_1 + m_2 V_2 = m_1 v_1 + m_2 v_2$$

in which the velocities (speeds) are given the sign corresponding to the direction of motion. A second equation is now needed in order to determine V_1 and V_2.

Since there are no ideally rigid bodies, the impact of two bodies produces a deformation which reaches a certain maximum. When this deformation is permanent and the bodies adhere after impact, they are said to be *inelastic*. If the bodies completely recover their original shape, they are said to be *perfectly elastic*. Between these extremes all extremes of partial restitution, with separation after impact, may occur; in such cases the bodies are said to be *imperfectly elastic*.

Inelastic Impact. Since the spheres move as a whole after impact, $V_1 = V_2$, and from (1)

$$V_1 = V_2 = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.$$

Perfect Elastic Impact. When the deformation persists, wholly or in part, some of the kinetic energy of the bodies is expended in the work of deformation. In any case some energy is probably dissipated as heat generated by the impact. For perfectly elastic bodies, however, this loss is but a small part of the total kinetic energy. If we assume that *no* energy is lost, the constancy of kinetic energy gives a second equation:

$$m_1 V_1^2 + m_2 V_2^2 = m_1 v_1^2 + m_2 v_2^2. \tag{2}$$

If we write (1) and (2) as

$$m_2(V_2 - v_2) = -m_1(V_1 - v_1),$$
$$m_2(V_2^2 - v_2^2) = -m_1(V_1^2 - v_1^2),$$

and divide them member for member, we obtain

$$V_2 + v_2 = V_1 + v_1, \quad \text{or}$$
$$V_2 - V_1 = -(v_2 - v_1). \tag{3}$$

This equation states that *the velocity of either sphere relative to the other is simply reversed in direction*. Experiment shows that this relation is very nearly exact in the case of steel or ivory balls.

Equations (1) and (3) determine V_1 and V_2. When the spheres have equal masses the results are especially simple. The equations are then

$$V_1 + V_2 = v_1 + v_2, \quad V_2 - V_1 = v_1 - v_2,$$

whence $V_2 = v_1$, $V_1 = v_2$; that is, *the spheres exchange velocities.* In particular, if one sphere is at rest before impact, the other will be at rest after. This conclusion is readily verified by experiment.

Imperfect Elastic Impact. In this case some kinetic energy is lost and (2) and (3) are no longer true. After carrying out a series of experiments on impact Newton found that the relative velocity is reversed by impact and diminished in a ratio e which is constant for the given bodies; that is

$$V_2 - V_1 = -e(v_2 - v_1) \tag{4}$$

where e, the *coefficient of restitution,* is a positive number less than unity. Equations (1) and (4) now determine V_1, V_2 when v_1, v_2 are given.

Equation (4) applies also to inelastic and perfectly elastic impact if we give e the limiting values 0 and 1 respectively.

Since only internal forces act on the system of two spheres, the velocity v^* of their center of mass P^* is unaltered by the impact; thus

$$v^* = \frac{m_1 V_1 + m_2 V_2}{m_1 + m_2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

in agreement with (1). The kinetic energy of the system is equal to $\frac{1}{2}(m_1 + m_2)v^{*2}$ plus the kinetic energy of the motion relative to P^* (§ 185). Any change in kinetic energy must therefore be due to a change in the latter part. Now the velocities relative to P^* before impact are

$$v_1 - v^* = \frac{m_2}{m_1 + m_2}(v_1 - v_2), \quad v_2 - v^* = \frac{m_1}{m_1 + m_2}(v_2 - v_1);$$

the kinetic energy relative to P^* is therefore

$$\tfrac{1}{2} m_1(v_1 - v^*)^2 + \tfrac{1}{2} m_2(v_2 - v^*)^2 = \tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(v_2 - v_1)^2.$$

Similarly after the impact the relative energy is

$$\tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(V_2 - V_1)^2 \quad \text{or} \quad \tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}e^2(v_2 - v_1)^2$$

in view of (4). Hence

$$\text{Energy Loss} = \tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(1 - e^2)(v_2 - v_1)^2. \tag{5}$$

With inelastic spheres ($e = 0$) the entire relative energy before impact is lost. With perfectly elastic spheres ($e = 1$) there is no loss.

Example 1. A 4-lb. ball going 4 ft./sec. strikes a 5-lb. ball going 2 ft./sec. in the opposite direction. Find the velocities after impact and the loss in kinetic energy of $e = \frac{1}{2}$.

From (1) and (4)

$$4\ V_1 + 5\ V_2 = 4 \times 4 + 5 \times (-2) = 6,$$
$$V_2 - V_1 = -\tfrac{1}{2}(-2 - 4) = 3;$$

hence $$V_1 = -1, \quad V_2 = 2 \text{ ft./sec.}$$

The loss in kinetic energy is

$$\tfrac{1}{2}\tfrac{4}{32}(4^2 - 1^2) + \tfrac{1}{2}\tfrac{5}{32}(2^2 - 2^2) = \tfrac{15}{16} \text{ ft.-lb.}$$

This direct computation may be checked by use of (5).

Example 2. The direct impact of a sphere on an immovable plane (regarded as an infinite sphere) may be dealt with by putting $v_2 = V_2 = 0$ in (4). Then

(6) $$V_1 = -ev_1.$$

If a ball falls from a height h on a horizontal plane and rebounds to a height H,

$$v_1 = \sqrt{2\,gh}, \quad V_1 = -\sqrt{2\,gH},$$

and from (6), $e = \sqrt{H/h}$. The measurement of H and h thus gives an experimental method of finding e.

PROBLEMS

1. Two elastic spheres, *1* and *2*, move in opposite directions along the same line with equal speeds. On impact sphere *1* is reduced to rest. Show that $m_1 = 3\ m_2$.

2. A 2- and a 3-lb. sphere move along the same line with velocities of 3 and -2 ft./sec. Show that they will be reduced to rest if the impact is inelastic.

If $e = 0.8$, find their velocities after impact. What is the energy loss?

If the spheres are perfectly elastic what are their final velocities?

3. A ball dropped to a floor reaches one-half its original height on the second rebound. Find e.

4. A ball strikes another directly which is at rest. If the balls have the masses m_1, m_2 and are perfectly elastic, find their velocities after impact.

5. Two billiard balls B, C are in contact. A third A, moving along their line of centers, strikes B with the velocity v. Show that A, B, C have the velocities 0, 0, v after impact if $e = 1$.

6. Two billiard balls, A, B, moving together with the velocity v, strike a third C at rest on their line of centers. Show that A, B, C have the velocities 0, v, v after impact if $e = 1$.

224. The Restitution Equation. In order to deal with problems in impact less simple than the direct impact of spheres we shall assume that Newton's empirical equation

$$(1) \qquad V_2 - V_1 = -e(v_2 - v_1)$$

applies in any case of impact at a point provided $v_2 - v_1$ and $V_2 - V_1$ denote *the components of relative velocity along the common normal to the impinging surfaces at their point of contact.* This generalization should be regarded as a working hypothesis rather than a well established law.

Example 1. *Oblique Impact of Smooth Spheres.* The impact is said to be *oblique* when the spheres are not both moving along the line of centers at the moment of impact. Since friction is absent there are no tangential percussions at the point of contact. The component of the momentum of each sphere perpendicular to the line of centers is therefore unaltered by the impact. Consequently *the tangential velocity components of each sphere remain constant.* The normal velocity components are subject to the equations (§ 223, 1, 4) as in direct impact.

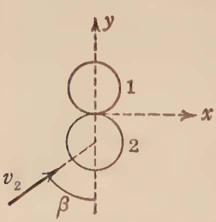


FIG. 224*a*.

Consider, for example, the impact of two billiard balls (Fig. 224*a*), one at rest ($v_1 = 0$), the other having a velocity v_2 which, at impact, makes an angle β with the line of centers. If the balls are perfectly elastic, they will simply exchange their normal y-components of velocity while the tangential x-components remain unaltered. Thus after impact the balls will have the velocities

$$\mathbf{V}_1 = [0, \; v_2 \cos \beta], \quad \mathbf{V}_2 = [v_2 \sin \beta, \, 0].$$

Thus ball *1* moves off in the $+y$ direction, ball *2* in the $+x$ direction. The reader may verify that no kinetic energy is lost.

Example 2. *Impact of a Sphere on a Fixed Plane.*

1. *Smooth Contact.* Let the sphere approach and leave the plane at the angles β and γ to the normal (Fig. 224*b*). The normal y-com-

ponents of velocity before and after impact are $-v \cos \beta$, $V \cos \gamma$; hence from the restitution equation

$$V \cos \gamma - 0 = -e(-v \cos \beta - 0), \quad \text{or}$$

(i) $$V \cos \gamma = ev \cos \beta.$$

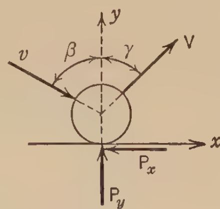


FIG. 224b.

When friction is absent the tangential x-components of velocity remain unaltered:

$$V \sin \gamma = v \sin \beta.$$

Hence on division

(ii) $$\tan \gamma = \frac{1}{e} \tan \beta.$$

If the bodies are perfectly elastic $e = 1$ and $\gamma = \beta$; *the angle of reflection equals the angle of incidence.*

If the bodies are imperfectly elastic $e < 1$ and $\gamma > \beta$; *the angle of reflection is greater than the angle of incidence.*

2. *Rough Contact.* Equation (i) holds as before. Let $[P_x, P_y]$ denote the percussion of the plane on the sphere. Now P_y is equal to change in the y-component of momentum: hence

$$P_y = m(V \cos \gamma + v \cos \beta) = m(1 + e)v \cos \beta$$

in view of (i). Now $P_x = -\mu P_y$ where μ is the coefficient of friction. Since P_x equals the change in the x-component of momentum,

$$P_x = -\mu m(1 + e)v \cos \beta = m(V \sin \gamma - v \sin \beta).$$

On substituting the value of V from (i) we obtain

$$-\mu(1 + e) \cos \beta = e \frac{\cos \beta}{\cos \gamma} \sin \gamma - \sin \beta$$

or

$$e \tan \gamma = \tan \beta - \mu(1 + e).$$

On comparing this equation with (ii) we see that the friction causes the ball to rebound nearer to the normal.

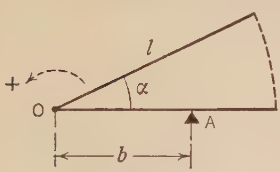


FIG. 224c.

Example 3. A rod of length l, pivoted at O, falls through an angle α and hits the obstacle A at a distance b from O (Fig. 224c). If it rebounds through an angle β, find the coefficient of restitution e.

Let ω, Ω be the angular velocities of the rod just before and just after impact. Then from the energy equation

$$\tfrac{1}{2}\frac{W}{g}\frac{l^2}{3}\omega^2 = W \tfrac{1}{2} l \sin \alpha, \quad \omega = -\sqrt{\frac{3 g \sin \alpha}{l}};$$

$$\tfrac{1}{2}\frac{W}{g}\frac{l^2}{3}\Omega^2 = W \tfrac{1}{2} l \sin \beta, \quad \Omega = \sqrt{\frac{3 g \sin \beta}{l}}.$$

From the restitution equation

$$b\Omega - 0 = -e(b\omega - 0) \quad \text{or} \quad \Omega = -e\omega.$$

Hence from the above values of ω and Ω, $e = \sqrt{\sin\beta/\sin\alpha}$.

225. Impact in Cases of Plane Motion. Consider a body of mass m in plane motion subjected to certain external percussions $\mathbf{P}$.

If $\mathbf{v}^*$ is the velocity of its center of mass G, its momentum is $m\mathbf{v}^*$ and from Theorem I.

$$m\Delta v^* = \sum \mathbf{P}. \tag{1}$$

The moment of momentum about G is

$$\mathbf{H}_G = \int \mathbf{r}\times\mathbf{v}\, dm = \int \mathbf{r}\times(\mathbf{v}^* + \mathbf{v}')\, dm$$

where $\mathbf{v}'$ is the velocity of any particle of the body relative to G. Now

$$\int \mathbf{r}\times\mathbf{v}^*\, dm = \left(\int \mathbf{r}\, dm\right)\times\mathbf{v}^* = m\mathbf{r}^*\times\mathbf{v}^* = 0$$

since $\mathbf{r}^* = 0$ (G is center of moments). Hence

$$\mathbf{H}_G = \int \mathbf{r}\times\mathbf{v}'\, dm = \mathbf{H}_G'.$$

About the center of mass the moment of momentum is equal to the moment of relative momentum. For plane motion, the motion relative to G is a rotation about the axis Gz perpendicular to the plane; and if Gz is a principal axis of inertia,

$$\mathbf{H}_G = \mathbf{H}_G' = I^*\boldsymbol{\omega} \qquad (\S\ 210)$$

where I^* is the moment of inertia of the body about Gz. Hence from Theorem II,

$$I^*\Delta\boldsymbol{\omega} = \sum \mathbf{r}\times\mathbf{P}. \tag{2}$$

Example 1. *Center of Percussion.* A rigid body at rest is subjected to a percussion $\mathbf{P}$ acting in a plane of symmetry through the center of mass G. Then Gz is a principal axis of inertia; and from (1) and (2)

$$m(\mathbf{v}^* - 0) = \mathbf{P}, \quad mk^{*2}(\omega - 0) = Pb'$$

where $b' = GC$, the perpendicular from G on $\mathbf{P}$ (Fig. 225*a*). By symmetry the initial motion of the body will be plane; G will move in the direction of $\mathbf{P}$ and the body will revolve with the initial angular speed ω about an instantaneous axis Iz. I will lie on the line GC

(§ 125). Hence from the first equation

$$P = mv^* = m\omega b$$

where $b = IG$. On substituting this value of P in the second we find

$$k^{*2} = bb' = GI \cdot GC.$$

This relation locates I. It shows, moreover, that I and C are related just as the centers of suspension and oscillation of a physical pendulum (§ 197, 3). Or, in view of § 199, we may say that C is the center of percussion with respect to I. If the axis Iz of the body were fixed, the percussion would not produce a reaction there.

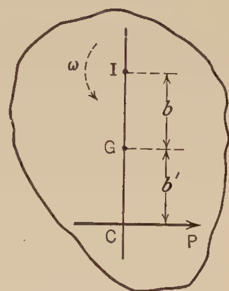


FIG. 225*a*.

Example 2. Let a particle of mass m_1 impinge directly on a rod of mass m at rest on a smooth plane (Fig. 225*b*). On collision percussions of equal magnitude and opposite direction will be developed at the point of contact. Since there are no *external* percussions on the system rod-particle, the momentum and the angular momentum of the system will be conserved. Thus if the velocity of the particle is v_1 just before, V_1 just after impact, and v^* is the initial velocity of G, the mass-center of the rod, we have by the conservation of momentum

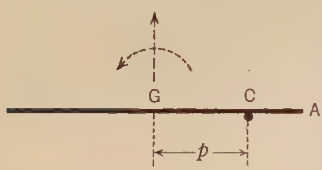


FIG. 225*b*.

(i) $$m_1 V_1 + mv^* = m_1 v_1.$$

Also, on taking G as center of moments, we have by the conservation of moment of momentum

(ii) $$m_1 V_1 p + mk^{*2}\omega = m_1 v_1 p.$$

The point C of the rod begins to move with the velocity $v^* + \omega p$ (§ 124); hence by the restitution equation

(iii) $$v^* + \omega p - V_1 = -e(0 - v_1) = ev_1.$$

Thus if v_1 is known, we have three linear equations to determine V_1, v^* and ω. The positive directions are indicated in the figure.

If we multiply (i) by p and subtract from (ii), we find

$$m(k^{*2}\omega - pv^*) = 0 \quad \text{or} \quad v^* = \frac{k^{*2}}{p}\omega.$$

This shows that the rod begins to turn about a point I (the center of percussion) at a distance k^{*2}/p from G. For a uniform rod of length l struck at one end A,

$$GI = \frac{k^{*2}}{p} = \frac{\frac{1}{12}l^2}{\frac{1}{2}l} = \tfrac{1}{6}l, \quad AI = \tfrac{2}{3}l.$$

PROBLEMS

1. A 1-lb. ball going 15 ft./sec. strikes a 2-lb. ball going 10 ft./sec. in a direction perpendicular to the first. At the instant of striking the second ball is moving perpendicular to the line of centers. If the balls are smooth and perfectly elastic, find their velocities after impact.

2. Show that a billiard ball of radius r, rebounding normally from the side of the table, will roll without slipping provided the cushion strikes it at a distance $7r/5$ above the table [cf. Ex. 1].

3. A $\frac{1}{2}$-lb. sphere going 4 ft./sec. strikes the end of a rod at rest on a horizontal plane. The rod is 3 ft. long, weighs 2 lb., and is struck at right angles to its length. Find the velocity of the sphere and the angular velocity of the rod after impact if $e = \frac{1}{2}$. What is the energy loss? [Locate, first, the instantaneous center of the rod.]

4. A ball having the velocity components [20, −64] ft./sec. strikes a smooth horizontal plane. If $e = \frac{2}{3}$ find the horizontal distance travelled in the first bounce. Show that the total horizontal distance covered in all the bounces is 160 ft.

5. A rod 4 ft. long and weighing 2 lb. is pivoted at the middle. A $\frac{1}{2}$-lb. ball, having a velocity of 12 ft./sec. at an angle of 60° with the rod, strikes it at a point 1 ft. from its center. If $e = \frac{1}{2}$ find the angular velocity of the rod after impact.

6. A uniform rod 4 ft. long and weighing 2 lb. has a translatory motion of 4 ft./sec. The rod impinges directly on a fixed obstacle 1 ft. from its center. If $e = \frac{1}{2}$, find the velocity of its center and the angular velocity just after impact. Locate the instantaneous center of the rod.

7. Solve Problem 5, § 222, when $e = \frac{1}{2}$.

226. Reduced Masses. In the direct impact of two spheres the equations (§ 223, 1, 4)

$$m_1V_1 + m_2V_2 = m_1v_1 + m_2v_2, \tag{1}$$

$$V_2 - V_1 = -e(v_2 - v_1), \tag{2}$$

determine V_1, V_2 in terms of v_1, v_2. Also, from (§ 223, 5), the

$$\text{Energy Loss} = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(1 - e^2)(v_2 - v_1)^2. \tag{3}$$

In more complicated cases of impact the restitution equation has the form (2) if v and V denote the components of velocity along the common normal to the impinging surfaces at their point of contact (§ 224). Moreover it is often possible to deduce from Theorems I and II an equation between these *normal velocity*

components of the form (1), namely

(1)′ $$M_1V_1 + M_2V_2 = M_1v_1 + M_2v_2,$$

where M_1, M_2 are certain constants, having the dimensions of mass, such that the kinetic energies of the bodies may be expressed in the form $\frac{1}{2}Mv^2$. When this is the case M_1, M_2 are called the *reduced masses* of the bodies. On solving (1)′ and (2) for V_1, V_2 we obtain

$$V_1 = \frac{(M_1 - eM_2)v_1 + (1 + e)M_2v_2}{M_1 + M_2},$$

$$V_2 = \frac{(M_2 - eM_1)v_2 + (1 + e)M_1v_1}{M_1 + M_2}.$$

Moreover equation (3) above will give the energy loss when m_1, m_2 are replaced by the reduced masses M_1, M_2. For every step in the calculation of the loss in § 223 has its exact counterpart in the case under consideration, the reduced masses taking the place of the actual masses of the impinging spheres. The method will be clear from the following example.

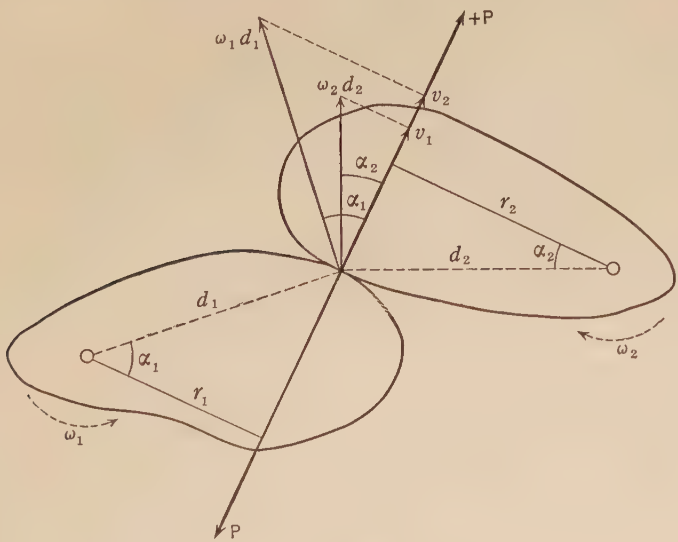


FIG. 226*a*.

Example. Impact of Bodies Revolving about Parallel Axes. Consider two bodies revolving about parallel axes with the angular velocities ω_1, ω_2 (Fig. 226*a*). When they impinge let the mutual percussions

of magnitude P change ω_1, ω_2 to Ω_1, Ω_2. If the angular velocities are taken positive in the senses shown in the figure, the normal velocity components will be positive in direction of $+P$.

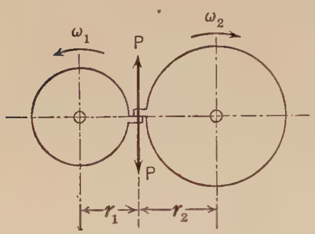


FIG. 226b.

If I_1, I_2 denote moments of inertia about the axes of rotation, Theorem II gives the equations

$$I_1(\Omega_1 - \omega_1) = -Pr_1, \quad I_2(\Omega_2 - \omega_2) = Pr_2;$$

hence, on eliminating P,

$$\text{(i)} \quad \frac{I_1}{r_1}(\Omega_1 - \omega_1) + \frac{I_2}{r_2}(\Omega_2 - \omega_2) = 0.$$

We now introduce the normal velocity components v, V into this equation. Since

$$v_1 = \omega_1 d_1 \cos \alpha_1 = \omega_1 r_1, \quad v_2 = \omega_2 d_2 \cos \alpha_2 = \omega_2 r_2,$$

and similarly

$$V_1 = \Omega_1 r_1, \quad V_2 = \Omega_2 r_2,$$

(i) may be written

$$\frac{I_1}{r_1^2}(V_1 - v_1) + \frac{I_2}{r_2^2}(V_2 - v_2) = 0,$$

which has the form of (1)′ if we put

$$(4) \qquad M_1 = \frac{I_1}{r_1^2}, \quad M_2 = \frac{I_2}{r_2^2}.$$

These constants clearly have the dimensions of mass. Moreover the kinetic energy of either body has the required form:

$$\tfrac{1}{2} I\omega^2 = \tfrac{1}{2}\frac{I}{r^2}(\omega r)^2 = \tfrac{1}{2} Mv^2.$$

The expressions (4) are therefore the reduced masses of the revolving bodies. *Each reduced mass equals the moment of inertia about the axis divided by the square of the perpendicular dropped from the axis on the common normal.*

As a numerical example suppose, in Fig. 226b, that

$$I_1 = 80 \text{ slug-ft.}^2, \quad r_1 = 2 \text{ ft.}, \quad \omega_1 = 2 \text{ rad./sec.};$$
$$I_2 = 400 \text{ slug-ft.}^2, \quad r_2 = 3 \text{ ft.}, \quad \omega_2 = 0.$$

Then the reduced masses are

$$M_1 = \tfrac{80}{4} = 20, \quad M_2 = \tfrac{400}{9};$$

and $v_1 = 4$ ft./sec., $v_2 = 0$. Taking $e = \frac{1}{2}$ we now find from (1)′ and (2):

$$V_1 = \frac{M_1 - eM_2}{M_1 + M_2} v_1 = -0.14, \quad V_2 = \frac{(1 + e)M_1}{M_1 + M_2} v_1 = 1.86 \text{ ft./sec.}$$

From (3) we find 82.7 ft.-lb. as the energy loss.

PROBLEMS

1. A rod of weight W_1, pivoted at one end, falls through an angle of 90° and strikes a second rod of weight W_2, pivoted at the middle (Fig. 226*c*). If the pivots are smooth and $W_1 = 20$ lb., $W_2 = 10$ lb., $l_1 = 6$ ft., $l_2{}^2 = 2$ ft., $e = \frac{1}{2}$, find the angular velocities of the rods just after impact and the energy loss.

Show, in general, that the energy lost is the fraction $(1 - e^2) W_2/(W_1 + W_2)$ of the energy just before impact.

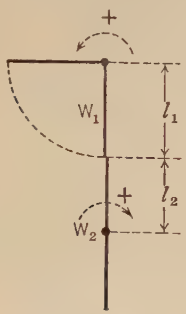


FIG. 226*c*.

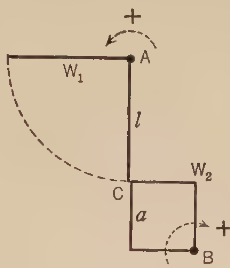


FIG. 226*d*.

2. The bar, pivoted at A, falls through an angle of 90° from rest and strikes the square block pivoted at B (Fig. 226*d*). Find the velocities at C just after impact and the energy loss, given that $W_1 = 4$ lb., $W_2 = 6$ lb., $l = 2$ ft., 8 in., $a = 1$ ft., $e = \frac{1}{2}$.

3. Find the energy loss in Example 4, § 222, for any value of e. Verify the former result by putting $e = 0$.

4. Show that the reduced masses of sphere and rod in Problem 3, § 225, are both $1/2\,g$; solve the problem.

5. Show that the reduced masses of particle and rod in Example 2, § 225, are

$$M_1 = m_1, \quad M_2 = \frac{k^{*2}}{k^{*2} + p^2}\, m.$$

Compute the energy loss if a 1-lb. ball going 20 ft./sec. strikes a uniform rod 4 ft. long and weighing 7 lb. at a point 1 ft. from its center when

(*a*) $e = \frac{2}{3}$, (*b*) $e = 0$.

6. Solve Problem 5, § 222, by the method of reduced masses when $e = \frac{1}{2}$. What is the energy loss?

227. Summary, Chapter XV. *In an impact on a system of particles, the changes in momentum and moment of momentum*

about any point are equal respectively to the sum of the external percussions and of their moments about that point.

Newton's Experimental Law: The relative velocity of two spheres is reversed by direct impact and diminished in a ratio e which is constant for the given bodies:

$$V_2 - V_1 = -e(v_2 - v_1).$$

The *coefficient of restitution* e is a positive number $\leqq 1$.

In *inelastic impact* $e = 0$; the bodies then adhere after impact In *perfect elastic impact* $e = 1$; the velocity of either sphere relative to the other is simply reversed in direction. *Imperfect elastic impact* occurs when e lies between 0 and 1. The kinetic energy of the spheres is conserved in perfect elastic impact; in all other cases some kinetic energy is lost.

In dealing with the impact of bodies in general, the above restitution equation is assumed to apply to the components of velocity along the common normal to the impinging surfaces at the point of contact.

INDEX

The numbers refer to pages. A starred number locates the definition of the term in question. The letter f after a number means "and following pages."

In the case of terms under a key or heading word, if a dash is used, this key word is to be read in *before* the term. If no dash is used, the key word is to be read in *after* the term (unless there is a word like "for" or "in" which indicates that the key word precedes it).

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66-1
67-1
71-1
73-1